Spherical orthogonal polynomials and symbolic-numeric Gaussian cubature formulas

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Abstract. It is well-known that the classical univariate orthogonal polynomials give rise to highly efficient Gaussian quadrature rules. We show how these classical families of polynomials can be generalized to a multivariate setting and how this generalization leads to truly Gaussian cubature rules for specific families of multivariate polynomials.
The multivariate homogeneous orthogonal functions that we discuss here satisfy a unique slice projection property: they project to univariate orthogonal polynomials on every one-dimensional subspace spanned by a vector from the unit hypersphere. We therefore call them spherical orthogonal polynomials.

1 Spherical orthogonal polynomials

The orthogonal polynomials under discussion were first introduced in [1] in a different form and later in [3] in the current form. Originally they were not termed spherical orthogonal polynomials because of a lack of insight into the mechanism behind the definition. In this paper we give several examples of these families of spherical orthogonal polynomials, present graphical illustrations of the bivariate case, compare them to radial orthogonal polynomials which are a special case of radial basis functions, and discuss some Gaussian cubature formulas which can be derived from the spherical orthogonal polynomials.

In dealing with multivariate polynomials and functions we shall often switch between the cartesian and the spherical coordinate system. The cartesian coordinates $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ are then replaced by $X = (x_1, \ldots, x_n) = (\xi_1 z, \ldots, \xi_n z)$ with $\xi_k, z \in \mathbb{R}$ where the directional vector $\xi = (\xi_1, \ldots, \xi_n)$ belongs to the unit sphere $S_n = \{\xi : ||\xi||_p = 1\}$. Here $||\cdot||_p$ denotes one of the usual Minkowski norms. While ξ contains the directional information of X, the variable z contains the (possibly signed) distance information. With the multi-index $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{N}^n$ the notations $X^\kappa, \kappa!$ and $|\kappa|$ respectively denote

$$X^\kappa = x_1^{\kappa_1} \ldots x_n^{\kappa_n}$$
$$\kappa! = \kappa_1! \ldots \kappa_n!$$
$$|\kappa| = \kappa_1 + \ldots + \kappa_n$$

Since z can be positive as well as negative and hence two directional vectors can generate X, we also introduce a signed distance function

$$\mathrm{sd}(X) = \mathrm{sgn}(x_1)||X||_p$$

For the sequel of the discussion we need some more notation. We denote by $\mathbb{R}[z]$ the linear space of polynomials in the variable z with real coefficients, by $\mathbb{R}[\xi] = \mathbb{R}[\xi_1, \ldots, \xi_n]$ the linear space of n-variate polynomials in ξ_k with real coefficients, by $\mathbb{R}(\xi) = \mathbb{R}(\xi_1, \ldots, \xi_n)$ the commutative field of rational functions in ξ_k and with real coefficients, by $\mathbb{R}(\xi)[z]$ the linear space of polynomials in the variable z with coefficients from $\mathbb{R}(\xi)$ and by $\mathbb{R}[\xi][z]$ the linear space of polynomials in the variable z with coefficients from $\mathbb{R}[\xi]$.

Let us introduce the linear functional Γ acting on the variable z, as

$$\Gamma(z^i) = c_i(\xi)$$

where $c_i(\xi)$ is a homogeneous expression of degree i in the ξ_k:

$$c_i(\xi) = \sum_{|\kappa|=i} c_\kappa \xi^\kappa \tag{1}$$

For our purpose

$$c_\kappa = \frac{|\kappa|!}{\kappa!} \int \ldots \int_{||X||_p \leq 1} w\left(||X||_p\right) X^\kappa \, dX \tag{2}$$

where $dX = dx_1 \ldots dx_n$ and hence

$$\Gamma(z^i) = \int \ldots \int_{||X||_p \leq 1} w\left(||X||_p\right) \left(\sum_{k=1}^{n} x_k \xi_k \right)^i dX$$

The n-variate polynomials under investigation are of the form

$$V_m(X) = \mathcal{V}_m(z) = \sum_{i=0}^{m} B_{m^2-i}(\xi) z^i \tag{3a}$$

$$B_{m^2-i}(\xi) = \sum_{|\kappa|=m^2-i} b_\kappa \xi^\kappa \tag{3b}$$

The function $V_m(X)$ is a polynomial of degree m in z with polynomial coefficients from $\mathbb{R}[\xi]$. The coefficients $B_{m(m-1)}(\xi), \ldots, B_{m^2}(\xi)$ are homogeneous polynomials in the parameters ξ_k. The function $V_m(X)$ does itself not belong to $\mathbb{R}[X]$ but since $V_m(X) = \mathcal{V}_m(z)$, it belongs to $\mathbb{R}[\xi][z]$. Therefore the function $V_m(X)$ is given the name spherical polynomial: with every $\xi \in S_n$ the function $V_m(X) = \mathcal{V}_m(z)$ is associated which is a polynomial of degree m in the variable $z = \mathrm{sd}(X)$.

Imposing the orthogonality conditions

$$\Gamma(z^i \mathcal{V}_m(z)) = 0 \qquad i = 0, \ldots, m-1 \tag{4}$$

signifies that $\mathcal{V}_m(z)$ satisfies

$$
\begin{aligned}
\Gamma(z^i\mathcal{V}_m(z)) &= \sum_{j=0}^{m} B_{m^2-j}(\xi)\Gamma(z^{i+j}) \qquad i=0,\dots,m-1 \\
&= \int\cdots\int_{||X||_p\le 1} \sum_{j=0}^{m} B_{m^2-j}(\xi) w(||X||_p) \left(\sum_{k=1}^{n} x_k\xi_k\right)^{i+j} dX \\
&= \int\cdots\int_{||X||_p\le 1} w(||X||_p) \left(\sum_{k=1}^{n} x_k\xi_k\right)^{i} \mathcal{V}_m\left(\sum_{k=1}^{n} x_k\xi_k\right) dX = 0
\end{aligned}
$$

As in the univariate case the orthogonality conditions (4) only determine $\mathcal{V}_m(z)$ up to a kind of normalization: $m+1$ polynomial coefficients $B_{m^2-i}(\xi)$ must be determined from the m parameterized conditions (4). How this is done, is shown now. For more information on this issue we refer to [3, 5].

With the $c_i(\xi)$ we define the polynomial Hankel determinants

$$
H_m(\xi) = \begin{vmatrix} c_0(\xi) & \cdots & c_{m-1}(\xi) \\ \vdots & ⋰ & c_m(\xi) \\ & & \vdots \\ c_{m-1}(\xi) & \cdots & c_{2m-2}(\xi) \end{vmatrix} \qquad H_0(\xi) = 1
$$

We call the functional Γ definite if

$$
H_m(\xi) \not\equiv 0 \qquad m \ge 0
$$

In the sequel of the text we assume that $\mathcal{V}_m(z)$ satisfies (4) and that Γ is a definite functional. Also we shall assume that $\mathcal{V}_m(z)$ as given by (3) is primitive, meaning that its polynomial coefficients $B_{m^2-i}(\xi)$ are relatively prime. This last condition can always be satisfied, because for a definite functional Γ a solution of (4) is given by [3]

$$
\mathcal{V}_m(z) = \frac{1}{p_m(\xi)} \begin{vmatrix} c_0(\xi) & \cdots & c_{m-1}(\xi) & c_m(\xi) \\ \vdots & ⋰ & & \vdots \\ c_{m-1}(\xi) & & \cdots & c_{2m-1}(\xi) \\ 1 & z & \cdots & z^m \end{vmatrix} \qquad \mathcal{V}_0(z) = 1 \tag{5}
$$

where the polynomial $p_m(\xi)$ is a polynomial greatest common divisor of the polynomial coefficients of the powers of z in this determinant expression.

In the sequel we use both the notation $V_m(X)$ and $\mathcal{V}_m(z)$ interchangeably to refer to (3). In [3] the following 3-term recurrence relation was proved for the spherical orthogonal polynomials $V_m(X) = \mathcal{V}_m(z)$.

Theorem 1. *Let the functional Γ be definite and let the polynomials $\mathcal{V}_m(z)$ and $p_m(\xi)$ be defined as in (4) and (5). Then the polynomial sequence $\{\mathcal{V}_m(z)\}_m$*

obeys the recurrence relation

$$V_{m+1}(X) = \alpha_{m+1}(\xi)\left((z - \beta_{m+1}(\xi))V_m(X) - \gamma_{m+1}(\xi)V_{m-1}(X)\right)$$
$$V_{-1}(X) = 0 \qquad V_0(X) = 1$$

with

$$\alpha_{m+1}(\xi) = \frac{p_m(\xi)}{p_{m+1}(\xi)}\frac{H_{m+1}(\xi)}{H_m(\xi)} \qquad\qquad \beta_{m+1}(\xi) = \frac{\Gamma\left(z\,[V_m(X)]^2\right)}{\Gamma\left([V_m(X)]^2\right)}$$
$$\gamma_{m+1}(\xi) = \frac{p_{m-1}(\xi)}{p_m(\xi)}\frac{H_{m+1}(\xi)}{H_m(\xi)} \qquad\qquad \gamma_1(\xi) = c_0(\xi)$$

2 Spherical Legendre and Tchebyshev polynomials

Let us first consider $w(||X||_p) = 1$ and construct so-called spherical Legendre polynomials $\mathcal{L}_m(X)$. For the purpose of some graphical illustrations we switch to the bivariate case. For the ℓ_2-norm the expressions $c_i(\xi)$ equal zero for odd i and are given by the following expressions for even i:

$$c_0(\xi) = \pi \qquad c_2(\xi) = \frac{\pi}{4}(\xi_1^2 + \xi_2^2) \qquad c_4(\xi) = \frac{\pi}{8}(\xi_1^2 + \xi_2^2)^2$$
$$c_6(\xi) = \frac{5\pi}{64}(\xi_1^2 + \xi_2^2)^3 \qquad c_8(\xi) = \frac{7\pi}{128}(\xi_1^2 + \xi_2^2)^4 \qquad \ldots$$

Using the signed distance function $\mathrm{sd}(x, y)$, the first few orthogonal polynomials satisfying (4), can be written as :

$$\begin{aligned}
\mathcal{L}_0(z) &= 1\\
\mathcal{L}_1(z) &= z\\
&= \mathrm{sd}(x, y)\\
\\
\mathcal{L}_2(z) &= z^2 - \tfrac{1}{4}\ \ \xi_1^2 + \xi_2^2\\
&= \ \ \mathrm{sd}(x, y) - \tfrac{1}{2}\ \ \ \mathrm{sd}(x, y) + \tfrac{1}{2}\\
\\
\mathcal{L}_3(z) &= z^3 - \tfrac{1}{2}\ \ \xi_1^2 + \xi_2^2\ \ z\\
&= \mathrm{sd}(x, y)\left(\mathrm{sd}(x, y) - \tfrac{1}{\sqrt{2}}\right)\left(\mathrm{sd}(x, y) + \tfrac{1}{\sqrt{2}}\right)\\
\\
\mathcal{L}_4(z) &= z^4 - \tfrac{3}{4}\ \ \xi_1^2 + \xi_2^2\ \ z^2 + \tfrac{1}{16}\ \ \xi_1^2 + \xi_2^2\ \ ^2\\
&= \left(\mathrm{sd}(x, y) - \tfrac{\sqrt{3-\sqrt{5}}}{2\sqrt{2}}\right)\left(\mathrm{sd}(x, y) + \tfrac{\sqrt{3-\sqrt{5}}}{2\sqrt{2}}\right)\\
&\quad\left(\mathrm{sd}(x, y) - \tfrac{\sqrt{3+\sqrt{5}}}{2\sqrt{2}}\right)\left(\mathrm{sd}(x, y) + \tfrac{\sqrt{3+\sqrt{5}}}{2\sqrt{2}}\right)
\end{aligned} \tag{6}$$

$$
\begin{aligned}
\mathcal{L}_5(z) &= z^5 - \ \xi_1^2 + \xi_2^2 \ \ z^3 + \tfrac{3}{16} \ \ \xi_1^2 + \xi_2^2 \ ^2 z \\
&= \mathrm{sd}(x,y) \ \ \mathrm{sd}(x,y) - \tfrac{1}{2} \ \ \ \mathrm{sd}(x,y) + \tfrac{1}{2} \ \ \left(\mathrm{sd}(x,y) - \tfrac{\sqrt{3}}{2}\right)\left(\mathrm{sd}(x,y) + \tfrac{\sqrt{3}}{2}\right)
\end{aligned}
$$

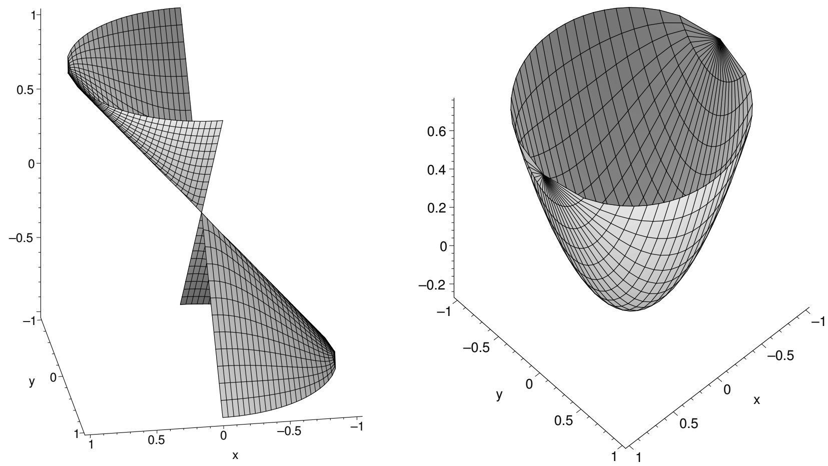

Fig. 1. $\mathcal{L}_1(z)$ for $x^2 + y^2 \leq 1$ $\qquad$ $\mathcal{L}_2(z)$ for $x^2 + y^2 \leq 1$

With $w(||X||_p) = 1/\sqrt{1 - ||X||_p^2}$, we obtain spherical Tchebyshev polynomials $\mathcal{T}_m(X)$. Again, for $p = 2$, the odd-numbered $c_i(\xi)$ equal zero. The $c_i(\xi)$ for even i and the $\mathcal{T}_m(z)$ are given by:

$$
c_0(\xi) = 2\pi \qquad c_2(\xi) = \frac{2\pi}{3}(\xi_1^2 + \xi_2^2) \qquad c_4(\xi) = \frac{2\pi}{5}(\xi_1^2 + \xi_2^2)^2
$$
$$
c_6(\xi) = \frac{2\pi}{7}(\xi_1^2 + \xi_2^2)^3 \qquad c_8(\xi) = \frac{2\pi}{9}(\xi_1^2 + \xi_2^2)^4 \qquad \ldots
$$

and

$$
\begin{aligned}
\mathcal{T}_0(z) &= 1 \\
\mathcal{T}_1(z) &= z \\
&= \mathrm{sd}(x,y)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_2(z) &= z^2 - \tfrac{1}{3} \ \ \xi_1^2 + \xi_2^2 \\
&= \left(\mathrm{sd}(x,y) - \tfrac{1}{\sqrt{3}}\right)\left(\mathrm{sd}(x,y) + \tfrac{1}{\sqrt{3}}\right)
\end{aligned}
$$

$$\begin{aligned}\mathcal{T}_3(z) &= z^3 - \tfrac{3}{5}\;\; \xi_1^2 + \xi_2^2 \;\; z \\ &= \mathrm{sd}(x,y)\left(\mathrm{sd}(x,y) - \tfrac{\sqrt{3}}{\sqrt{5}}\right)\left(\mathrm{sd}(x,y) + \tfrac{\sqrt{3}}{\sqrt{5}}\right)\end{aligned}$$

$$\begin{aligned}\mathcal{T}_4(z) &= z^4 - \tfrac{6}{7}\;\; \xi_1^2 + \xi_2^2 \;\; z^2 + \tfrac{3}{35}\;\; \xi_1^2 + \xi_2^2 \;\;{}^2 \\ &= \left(\mathrm{sd}(x,y) - \tfrac{\sqrt{525+70\sqrt{30}}}{35}\right)\left(\mathrm{sd}(x,y) + \tfrac{\sqrt{525+70\sqrt{30}}}{35}\right) \\ &\quad \left(\mathrm{sd}(x,y) - \tfrac{\sqrt{525-70\sqrt{30}}}{35}\right)\left(\mathrm{sd}(x,y) + \tfrac{\sqrt{525-70\sqrt{30}}}{35}\right)\end{aligned}$$

$$\begin{aligned}\mathcal{T}_5(z) &= z^5 - \tfrac{10}{9}\;\; \xi_1^2 + \xi_2^2 \;\; z^3 + \tfrac{5}{21}\;\; \xi_1^2 + \xi_2^2 \;\;{}^2\, z \\ &= \mathrm{sd}(x,y)\left(\mathrm{sd}(x,y) - \tfrac{\sqrt{245+14\sqrt{70}}}{21}\right)\left(\mathrm{sd}(x,y) + \tfrac{\sqrt{245+14\sqrt{70}}}{21}\right) \\ &\quad \left(\mathrm{sd}(x,y) - \tfrac{\sqrt{245-14\sqrt{70}}}{21}\right)\left(\mathrm{sd}(x,y) + \tfrac{\sqrt{245-14\sqrt{70}}}{21}\right)\end{aligned}$$

Let us now fix $\xi = \xi^*$ and take a look at the projected spherical polynomials

$$\mathcal{V}_{m,\xi^*}(z) = V_m(\xi_1^* z, \ldots, \xi_n^* z)$$

From the definition of $\mathcal{V}_m(X)$ in general, and the formulas and graphs for $\mathcal{L}_m(X)$ and $\mathcal{T}_m(X)$ in particular, it is clear that for each $\xi = \xi^*$ the functions $\mathcal{V}_{m,\xi^*}(z)$ are polynomials of degree m in z. Are these projected polynomials themselves orthogonal? If so, what is their relationship to the univariate Legendre and Tchebyshev polynomials? The answer to the first question is given in Theorem 2 while the answer to the second question, which follows partly from Theorem 2, is further elaborated in the next section.

Let us introduce the (univariate) linear functional c^* acting on the variable z, by

$$c^*(z^i) = c_i(\xi^*) = \Gamma(z^i)\mid_{\xi=\xi^*} \tag{7}$$

In what follows we use the notation $V_m(z)$ to denote the univariate polynomials of degree m orthogonal with respect to the linear functional c^*. The reader should not confuse these polynomials with the $\mathcal{V}_m(z)$ or the $V_m(X)$. Note that the $V_m(z)$ are computed from orthogonality conditions with respect to c^*, which is a particular projection of Γ, while the $\mathcal{V}_{m,\xi^*}(z)$ are a particular instance of the spherical polynomials orthogonal with respect to Γ.

Theorem 2. *Let the monic univariate polynomials $V_m(z)$ satisfy the orthogonality conditions*

$$c^*(z^i V_m(z)) = 0 \qquad i = 0, \ldots, m-1$$

with c^* given by (7)*, and let the multivariate functions $V_m(X) = \mathcal{V}_m(z)$ satisfy the orthogonality conditions* (4)*. Then*

$$\begin{aligned}H_m(\xi^*)V_m(z) &= p_m(\xi^*)\mathcal{V}_{m,\xi^*}(z) \\ &= p_m(\xi^*)V_m(X^*) \qquad X^* = (\xi_1^* z, \ldots, \xi_n^* z)\end{aligned}$$

In words, Theorem 2 says that the $V_m(z)$ and $\mathcal{V}_{m,\xi^*}(z)$ coincide up to a normalizing factor $p_m(\xi^*)/H_m(\xi^*)$. Or reformulated in yet another way, it says that the orthogonality conditions and the projection operator commute.

With respect to the projection property it is important to point out that $c^*(z^i)$ does not coincide with the one-dimensional version of c_κ given by (2), meaning (2) for $n = 1$ and $\kappa = i$. While in the one-dimensional situation, the linear functional

$$c(z^i) = c_i = \int_{-1}^{1} w(|x|)x^i \, dx \tag{8}$$

gives rise to the classical orthogonal polynomials, we do not immediately retrieve these classical polynomials from the projection, because the projected functional c^* given by (7) does not coincide with the functional c given by (8). Then what is the connection between the spherical orthogonal polynomials $\mathcal{L}_m(z)$ or $\mathcal{T}_m(z)$ and their univariate counterparts, the Legendre polynomials $L_m(z)$ or the Tchebyshev polynomials $T_m(z)$? This is explained in the next section.

3 Radial orthogonal polynomials

For another choice of c_κ it is possible to retrieve the classical families of orthogonal polynomials. At the same time the spherical orthogonal polynomials, for this particular c_κ, coincide with some particular radial basis functions. Let for simplicity $n = 2$ in $X = (x_1, \ldots, x_n)$ and $p = 2$ in $||X||_p$. With

$$c_{j,i-j} = \begin{cases} 0 \text{ for } j \text{ odd or } i-j \text{ odd} \\ \begin{pmatrix} \frac{i}{2} \\ \frac{j}{2} \end{pmatrix} \int_{-1}^{1} u^i \, du \text{ elsewhere} \end{cases} \tag{9}$$

we obtain for the first few even-numbered $c_i(\xi)$:

$$c_0(\xi) = 2 \qquad c_2(\xi) = \frac{2}{3}(\xi_1^2 + \xi_2^2) \qquad c_4(\xi) = \frac{2}{5}(\xi_1^2 + \xi_2^2)^2 \qquad \ldots$$

while the odd-numbered $c_i(\xi)$ are zero. With the functional Γ still defined as before, $\Gamma(z^i)$ takes for $c_{j,i-j}$ as given by (9) and $w(||X||_p) = 1$, the form

$$\Gamma(z^i) = \left(\int_{-1}^{1} u^i \, du \right) (\xi_1^2 + \xi_2^2)^{i/2}$$

and we obtain from (4) and (5) the orthogonal bivariate Legendre functions

$$\begin{aligned}
R_0(x,y) = \mathcal{R}_0(z) =& 1 \\
R_1(x,y) = \mathcal{R}_1(z) =& z = \mathrm{sd}(x,y) \\
R_2(x,y) = \mathcal{R}_2(z) =& z^2 - \frac{1}{3} = \mathrm{sd}^2(x,y) - \frac{1}{3} \\
R_3(x,y) = \mathcal{R}_3(z) =& z^3 - \frac{3}{5}z = \mathrm{sd}(x,y)\left(\mathrm{sd}^2(x,y) - \frac{3}{5}\right)
\end{aligned}$$

The projection property as formulated in Theorem 2 is still valid, now with the functional c^* equal to the functional c given in (8). Hence these $\mathcal{R}_m(z)$ coincide on every one-dimensional subspace of $\mathbb{R}^2$ with the well-known univariate Legendre polynomials $L_m(z)$. The main difference between the $\mathcal{R}_m(X)$ and $\mathcal{L}_m(X)$ is that they satisfy different orthogonality conditions. While the $\mathcal{R}_m(X)$ satisfy

$$\int_{-1}^{1} z^i \mathcal{R}_m(z)\, dz = 0 \qquad z = \mathrm{sd}(X) \qquad i = 0, \ldots, m-1$$

which is a radial version of the classical orthogonality condition

$$\int_{-1}^{1} z^i L_m(z)\, dz = 0 \qquad i = 0, \ldots, m-1$$

for the Legendre polynomials $L_m(z)$, the spherical Legendre polynomials $\mathcal{L}_m(z) = L_m(x, y)$ satisfy (4) which is a truly multivariate orthogonality. The $\mathcal{R}_m(X)$ are also related to the radial version of the classical Legendre polynomials $L_m(||X||_p)$ and appear as such in [6].

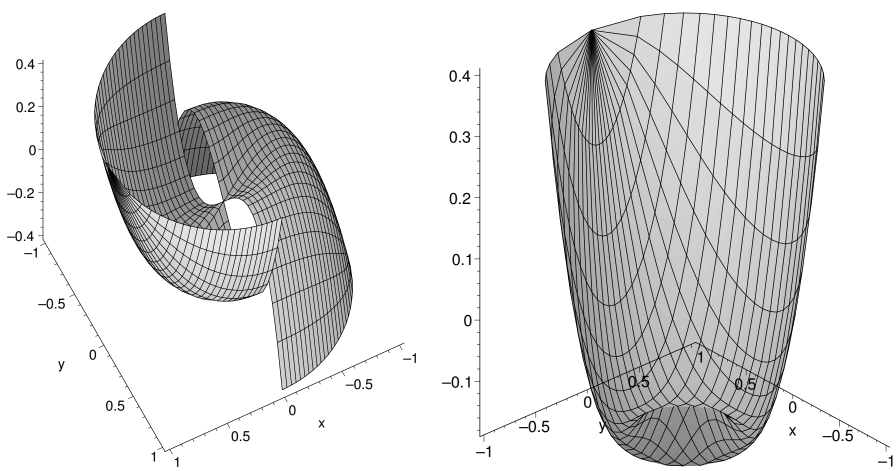

Fig. 2. $\mathcal{R}_3(z)$ for $x^2 + y^2 \leq 1$ $\qquad$ $L_3(||X||_2)$ for $x^2 + y^2 \leq 1$

In a similar way, the univariate Tchebyshev polynomials can be retrieved with

$$c_{j,i-j} = \begin{cases} 0 \text{ for } j \text{ odd or } i-j \text{ odd} \\ \dbinom{\frac{i}{2}}{\frac{j}{2}} \displaystyle\int_{-1}^{1} \frac{u^i}{\sqrt{1-u^2}}\, du \text{ elsewhere} \end{cases}$$

4 Gaussian cubature formulas

Let us again concentrate on the real-valued c_κ given by (2). If the functional Γ is positive definite, meaning that

$$\forall \xi \in \mathbb{R}^2 : H_m(\xi) > 0 \qquad m \geq 0$$

then the zeroes $z_i^{(m)}(\xi^*)$ of $\mathcal{V}_{m,\xi^*}(z)$ are real and simple because the functional c^* given by (7) is positive definite. Then according to the implicit function theorem, there exists for each $z_i^{(m)}(\xi^*)$ a unique holomorphic function $\phi_i^{(m)}(\xi^*)$ such that in a neighbourhood of $z_i^{(m)}(\xi^*)$,

$$\mathcal{V}_{m,\xi^*}(z) = 0 \Longleftrightarrow z = \phi_i^{(m)}(\xi^*) \tag{10}$$

Since this is true for each $\xi = \xi^*$ because Γ is positive definite, this implies that for each $i = 1, \ldots, m$ the zeroes $z_i^{(m)}$ can be viewed as a holomorphic function of ξ, namely $z_i^{(m)} = \phi_i^{(m)}(\xi)$. Let us denote

$$\mathcal{W}_{m-1}(u) = \Gamma\left(\frac{\mathcal{V}_m(z) - \mathcal{V}_m(u)}{z-u}\right)$$

$$A_i^{(m)}(\xi) = \frac{\mathcal{W}_{m-1,\xi}(z_i^{(m)})}{\mathcal{V}'_{m,\xi}(z_i^{(m)})} = \frac{\mathcal{W}_{m-1}(\phi_i^{(m)}(\xi))}{\mathcal{V}'_m(\phi_i^{(m)}(\xi))} \tag{11}$$

Here the functions $\mathcal{W}_{m-1}(z)$ are also spherical polynomials, now of degree $m-1$ in z. Then the following cubature formula can rightfully be called a Gaussian cubature formula. The proof of this fact can be found in [2].

Theorem 3. *Let $\mathcal{P}(z)$ be a polynomial of degree $2m-1$ belonging to $\mathbb{R}(\xi)[z]$, the set of polynomials in the variable z with coefficients from the space of multivariate rational functions in the real ξ_k with real coefficients. Let the functions $\phi_i^{(m)}(\xi)$ be given as in (10) and be such that*

$$\forall \xi \in S_2 : j \neq i \Longrightarrow \phi_j^{(m)}(\xi) \neq \phi_i^{(m)}(\xi)$$

Then

$$\int \cdots \int_{||X||_p \leq 1} w(||X||_p)\, \mathcal{P}\left(\sum_{k=1}^n \xi_k x_k\right) dX = \sum_{i=1}^m A_i^{(m)}(\xi)\mathcal{P}(\phi_i^{(m)}(\xi))$$

Let us illustrate Theorem 3 with a bivariate example to render the achieved result more understandable. Take

$$\mathcal{P}(z) = \frac{\xi_1}{\xi_2+1}z^3 + \frac{\xi_2}{\xi_1^2+1}z^2 + z + 10$$

and consider again the ℓ_2-norm. Then

$$\int\int_{||(x,y)|| \leq 1} \mathcal{P}(\xi_1 x + \xi_2 y)dx\, dy = \frac{\pi\ \xi_2^3 + \xi_2\xi_1^2 + 40\xi_1^2 + 40}{4\,(\xi_1^2+1)} \tag{12}$$

The exact integration rule given in Theorem 3 applies to (12) with $w(||X||_p) = 1$ and $m = 2$. From the orthogonal function $V_2(x, y) = \mathcal{V}_2(z)$ given in (6), we obtain the zeroes

$$\phi_1^{(2)}(\xi) = \frac{1}{2}\sqrt{\xi_1^2 + \xi_2^2} \qquad\qquad \phi_2^{(2)}(\xi) = -\frac{1}{2}\sqrt{\xi_1^2 + \xi_2^2}$$

and the weights

$$A_1^{(2)}(\xi) = A_2^{(2)}(\xi) = \frac{\pi}{2}$$

The integration rule

$$A_1^{(2)}\mathcal{P}(\phi_1^{(2)}(\xi)) + A_2^{(2)}\mathcal{P}(\phi_2^{(2)}(\xi))$$

then yields the same result as (12). In fact, the Gaussian m-point cubature formula given in Theorem 3 exactly integrates a parameterized family of polynomials, over a domain in $\mathbb{R}^2$, or more generally $\mathbb{R}^n$. The m nodes and weights are themselves functions of the parameters ξ_1 and ξ_2. To illustrate this we graph two instances of this family $\mathcal{P}(\xi_1 x + \xi_2 y)$, namely for the choices $(\xi_1, \xi_2) = (3/5, 4/5)$ and $(\xi_1, \xi_2) = (-\sqrt{2}/2, -\sqrt{2}/2)$.

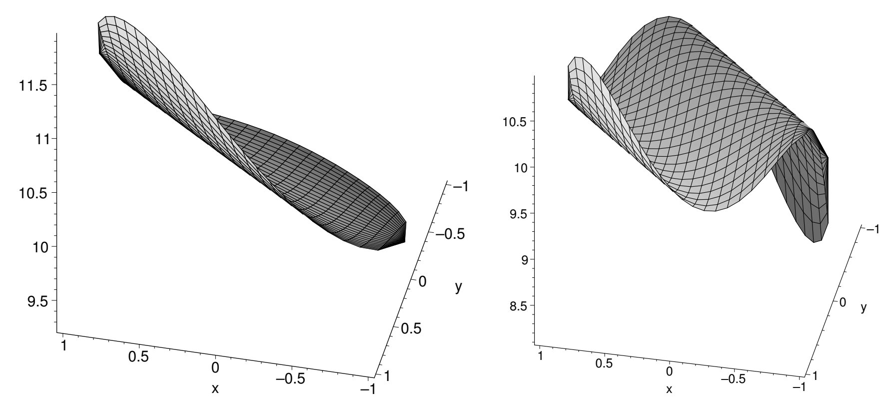

Fig. 3. $(\xi_1, \xi_2) = (3/5, 4/5)$ $\qquad$ $(\xi_1, \xi_2) = (-\sqrt{2}/2, -\sqrt{2}/2)$

For the ℓ_1- and ℓ_∞-norm similar computations can be performed: after obtaining the $c_i(\xi)$ for these norms, the orthogonal polynomial $\mathcal{V}_2(z)$ constructed from the $c_i(\xi)$ delivers all necessary ingredients for the application of the Gaussian cubature rule.

More properties of the spherical orthogonal functions $V_m(x, y)$ can be proved, such as the fact that they are the characteristic polynomials of certain parametrized tridiagonal matrices [4]. The connection between their theory and the theory of the univariate orthogonal polynomials is very close, while more multivariate in nature than their radial counterparts.

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