Numer Algor (2011) 58:1–21
DOI 10.1007/s11075-010-9445-2

ORIGINAL PAPER

Comonotone and coconvex rational interpolation and approximation

Hoa Thang Nguyen · Annie Cuyt · Oliver Salazar Celis

Received: 7 July 2010/Accepted: 28 December 2010 /
Published online: 20 January 2011
© Springer Science+Business Media, LLC 2011

Abstract Comonotonicity and coconvexity are well-understood in uniform polynomial approximation and in piecewise interpolation. The covariance of a global (Hermite) rational interpolant under certain transformations, such as taking the reciprocal, is well-known, but its comonotonicity and its coconvexity are much less studied. In this paper we show how the barycentric weights in global rational (interval) interpolation can be chosen so as to guarantee the absence of unwanted poles and at the same time deliver comonotone and/or coconvex interpolants. In addition the rational (interval) interpolant is well-suited to reflect asymptotic behaviour or the like.

Keywords Rational interpolation · Rational approximation · Comonotonicity · Coconvexity · Copositivity · Shape preserving · Barycentric

H. T. Nguyen · A. Cuyt (✉) · O. Salazar Celis
Department of Mathematics and Computer Science,
University of Antwerp, Campus Middelheim,
Building G, Middelheimlaan 1,
2020, Antwerp, Belgium
e-mail: annie.cuyt@ua.ac.be

H. T. Nguyen
e-mail: hoa.nguyenthang@ua.ac.be

O. Salazar Celis
e-mail: oliver.salazarcelis@ua.ac.be

1 Introduction

Global rational interpolation has its advantages and disadvantages compared to, on the one hand, global linear techniques and, on the other hand, piecewise (rational) techniques. It is superior compared to linear techniques in the presence of singularities or steep changes in the function values, and allows the easy incorporation of asymptotic behaviour (vertical, horizontal, slant). It also outperforms piecewise techniques when it comes to convergence: exponential convergence can be achieved [2]. But splines are famous for their flexibility when it comes to shape control.

Of course rational interpolation may suffer from the presence of unwanted poles in the domain of interest, and this should be avoided. The most stable formula for the rational interpolant for use on a finite interval is the barycentric form [5, 17]. Besides guaranteeing the absence of unwanted poles, as in [4], we show that the barycentric form of the rational interpolant can also guarantee comonotonicity and coconvexity.

In case of uniform polynomial approximation [12, 13] or piecewise approximation [14], these properties are extensively studied. Their computation is a 2-tier process. First a comonotone or coconvex continuous piecewise polynomial is constructed that approximates the given function well. Next this continuous piecewise polynomial is approximated by a comonotone or coconvex global polynomial.

In the case of rational (interval) interpolation, the construction is somewhat simpler and boils down to the computation of the barycentric weights in the representation of the interpolant. Our focus in the next sections is on the existence and construction of these interpolants. Asymptotic behaviour, which is typical for rational functions, is included in the discussion. We do not state convergence properties. This issue is dealt with succesfully in [10]. In the concluding section we make the connection with Bézier curves.

1.1 Barycentric rational interpolation

Given $n+1$ points $x_0, \ldots, x_n$ and function values $f_0, \ldots, f_n$, the rational functions

$$r_n(x) = \frac{\displaystyle\sum_{i=0}^{n} f_i \frac{w_i}{x - x_i}}{\displaystyle\sum_{i=0}^{n} \frac{w_i}{x - x_i}} \tag{1}$$

interpolate the values f_i at the points x_i for any nonzero weights w_i, in other words $r_n(x_i) = f_i$. Hence, with respect to interpolation of the given data, the function $r_n(x)$, when represented as in (1), is immune to rounding errors in the computation of the coefficients. If we denote

$$\ell(x) = (x - x_0) \cdots (x - x_n)$$
$$\ell_i(x) = \ell(x)/(x - x_i),$$

then $r_n(x)$ can be written as $r_n(x) = p_n(x)/q_n(x)$ with

$$p_n(x) = \sum_{i=0}^{n} f_i w_i \ell_i(x)$$
$$q_n(x) = \sum_{i=0}^{n} w_i \ell_i(x).$$

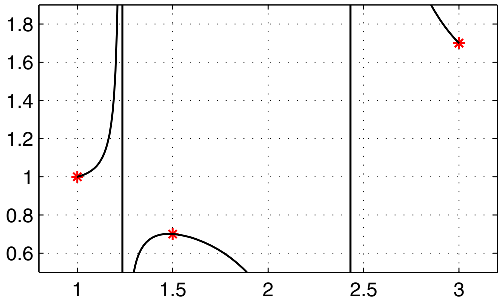
Hence it is easy to see that the degree in numerator and denominator of $r_n(x)$ is at most n. Also, the rational function $r_n(x)$, and consequently the barycentric weights, are only determined up to a multiplicative constant in numerator and denominator.

A rational function of the form (1) clearly does not deliver the minimal degree solution of the rational interpolation problem. We have $n + 1$ additional degrees of freedom at our disposal in the barycentric weights $w_0, \ldots, w_n$ that we can make use of. In [6, 15, 17, 18] degree conditions are imposed in order to obtain the minimal degree solution. In [16] the minimal degree solution, in case the point data f_i are replaced by realistic interval data $[f_{i,<}, f_{i,>}]$, is obtained from a quadratic programming problem.

In the next sections we propose to add conditions to control the shape of the rational function $r_n(x)$ while keeping it polefree in the region of interest. For global approximants (as opposed to piecewise approximants) this is much less studied [8]. This shape control can be combined with the classical rational interpolation [7] as well as with the rational interval interpolation developed in [16].

To illustrate the possible variation in rational curves from changing the barycentric weights, we take the example where $n = 2$ and $x_0 = 1, x_1 =$

Fig. 1 $r_2(x)$ with pole



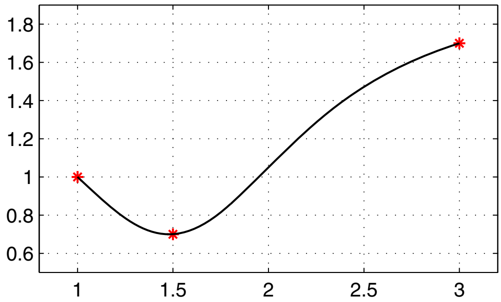


Fig. 2 $r_2(x)$ polefree

1.5, $x_2 = 3$, $f_0 = 1$, $f_1 = 0.7$, $f_2 = 1.7$. With $w_0 = w_1 = w_2 = 1$ the function $r_2(x)$ has a pole amidst the data points (see Fig. 1). It is known from [4] that with $w_0 = -w_1 = w_2 = 1$ the function $r_2(x)$ does not have real poles (see Fig. 2). Changing the weights to $w_0 = 1$, $w_1 = -1.5$, $w_2 = 0.9$ makes $r_2(x)$ in addition convex in $[x_0, x_2]$ (see Fig. 3).

1.2 Limiting behaviour and derivatives of the rational interpolant

The numerator and denominator polynomials of $r_n(x)$ can also be written as

$$p_n(x) = \sum_{i=0}^{n} a_i x^i,$$

$$q_n(x) = \sum_{i=0}^{n} b_i x^i.$$

When the degree of numerator and denominator of the rational interpolant is exactly n, it has a horizontal asymptote given by

$$\lim_{x \to \infty} r_n(x) = \frac{a_n}{b_n}.$$

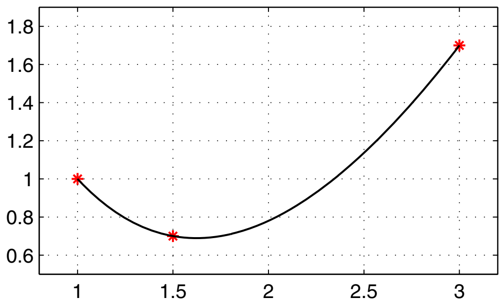


Fig. 3 $r_2(x)$ polefree and convex

When the degree of the numerator is n and that of the denominator is $n-1$, which can be obtained by imposing the appropriate degree condition, the rational interpolant has a slant asymptote

$$\frac{a_n}{b_{n-1}}x + \frac{a_{n-1}}{b_{n-1}}.$$

To control monotonicity or convexity/concavity, we need information about the derivatives of $r_n(x)$. Let x not be a pole of $r_n(x)$ and let $w_0 \cdots w_n \neq 0$. It is well-known from [17] that the k-th derivative of $r_n(x)$ at x is given by

$$\frac{r_n^k(x)}{k!} = \begin{cases} \dfrac{\displaystyle\sum_{i=0}^{n} r_n[x, \ldots, x, x_i] \frac{w_i}{x - x_i}}{\displaystyle\sum_{i=0}^{n} \frac{w_i}{x - x_i}}, & x \neq x_j, \quad j = 0, \ldots, n \\ \dfrac{-\left(\displaystyle\sum_{i=0, i\neq j}^{n} w_i r_n[x, \ldots, x, x_i]\right)}{w_j}, & x = x_j, \end{cases}$$

where $x, \ldots, x$ stands for k instances of x. So in the first derivative the divided differences

$$r_n[x, x_i] = \frac{r_n(x) - r_n(x_i)}{x - x_i}$$

appear and in the second derivative the

$$r_n[x, x, x_i] = \frac{r_n'(x) - r_n[x, x_i]}{x - x_i}.$$

Our aim is to compute barycentric weights $w_0, \ldots, w_n$ such that some derivative is guaranteed to be positive or negative in a specific interval.

A local maximum/minimum at $x = c$, characterized by $r_n'(c) = 0$, can be enforced by demanding that $r_n(x)$ be increasing/decreasing till c and decreasing/increasing thereafter.

2 Guaranteeing polefree interpolants

A necessary condition for the barycentric weights to satisfy when $r_n(x)$ is irreducible and polefree in $[x_0, x_n]$ is [17]

$$w_i w_{i+1} < 0, \qquad i = 0, \ldots, n-1.$$

In practice, it is the rule rather than the exception that $r_n(x)$ is irreducible, due to the accumulation of rounding errors. So in the sequel we sometimes replace

$w_i = (-1)^i \omega_i$ and look for appropriate values ω_i instead of w_i. It is well-known that the weights $w_i = (-1)^i$ guarantee rational interpolants which are polefree on the real line [4].

2.1 Descartes' rule of signs

Let us write the denominator in the form

$$q_n(x) = \sum_{i=0}^{n} b_i x^i \tag{2}$$

and let us denote $b = (b_0, \ldots, b_n)^t$, $w = (w_0, \ldots, w_n)^t$ and by $\phi_i^j(x_0, \ldots, x_n)$ the cumulative sum of all products of j factors x_k where $k \neq i$. For instance

$$\phi_0^2(x_0, \ldots, x_n) = x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n.$$

We put $\phi_i^0(x_0, \ldots, x_n) = 1$ for all i and fill the $(n+1) \times (n+1)$ matrix Φ with the entries

$$\Phi_{ij} = (-1)^{n-i} \phi_j^{n-i}(x_0, \ldots, x_n).$$

Coefficient b_i in (2) is given by

$$b_i = (-1)^{n-i} \sum_{j=0}^{n} w_j \phi_j^{n-i}(x_0, \ldots, x_n), \qquad i = 0, \ldots, n$$

or in short

$$b = \Phi w.$$

Another expression for the b_i can be found from the evaluations

$$q_n(x_j) = w_j \ell_j(x_j) = \sum_{i=0}^{n} b_i x_j^i$$

or

$$\begin{pmatrix} 1 & x_0 & \ldots & x_0^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \ldots & x_n^n \end{pmatrix} b = \begin{pmatrix} \ell_0(x_0) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \ell_n(x_n) \end{pmatrix} w. \tag{3}$$

Denoting the Vandermonde matrix in (3) by V and the diagonal matrix by L we also find

$$\Phi = V^{-1} L.$$

Since Descartes' rule of signs says that a polynomial in monomial form has at most as many positive real roots as sign changes in the coefficients, we now shift the data points x_i such that the region of interest is the positive real axis. Imposing that $b_i > 0, i = 0, \ldots, n$ then guarantees that $q_n(x) > 0$ since $q_n(0) = b_0 > 0$. Barycentric weights satisfying this condition can be found by solving the system of linear inequalities

$$\Phi w > 0 \tag{4}$$

where the vector inequality is taken componentwise. It is clear that (4) always has a nontrivial solution since each $q_n(x)$ with preassigned zeroes on the negative real axis leads to a vector of weights w_i satisfying (4).

2.2 The Lorentz representation

In [3] it is pointed out that a representation for the degree n polynomial $q_n(x)$ of the form

$$q_n(x) = \sum_{i=0}^{m} c_i (x-a)^i (b-x)^{m-i}, \qquad m \geq n, \quad c_i > 0 \tag{5}$$

satisfies $q_n(x) > 0$ in the open interval (a, b). Note that such a representation need not be unique because with $a = 0$ and $b = 1$,

$$1 = \sum_{i=0}^{k} \binom{k}{i} x^i (1-x)^{k-i}.$$

Choosing $m = n$ makes the representation unique but also imposes a restriction on the location of the complex zeroes of $q_n(x)$ [9]. The smallest number m for which the positive polynomial $q_n(x)$ can be written in the form (5) is called the Lorentz degree of $q_n(x)$. We denote it by $\partial_{[a,b]} q_n$ to distinguish it from the standard degree ∂q_n.

Let us denote $c = (c_0, \ldots, c_m)^t$. Then we find from the evaluations

$$q_n(x_j) = w_j \ell_j(x_j) = \sum_{i=0}^{m} c_i (x_j - a)^i (b - x_j)^{m-i}$$

that

$$Cc = Lw$$

with the $(n+1) \times (m+1)$ matrix C given by

$$C = \begin{pmatrix} (b-x_0)^m & (x_0-a)(b-x_0)^{m-1} & \ldots & (x_0-a)^m \\ \vdots & \vdots & & \vdots \\ (b-x_n)^m & (x_n-a)(b-x_n)^{m-1} & \ldots & (x_n-a)^m \end{pmatrix}$$

and L as before. We denote the pseudo-inverse of C, computed from its singular value decomposition, by $C^{-\uparrow}$ and so

$$c = C^{-\uparrow} L w.$$

Barycentric weights guaranteeing that $c_i > 0, i = 0, \ldots, m$ can be found by solving the system of linear inequalities

$$C^{-\uparrow} L w > 0 \tag{6}$$

where again the vector inequality is taken componentwise.

2.3 Balancing the $w_i/(x - x_i)$

We now assume that the points x_i are ordered such that $x_0 < x_1 < \ldots < x_n$. With the interpolation points $x_0, \ldots, x_n$ we can also associate open intervals $I_i = (x_{i-1}, x_i), i = 1, \ldots, n$ if we put $x_{-1} = -\infty$ and $x_{n+1} = +\infty$. Then for $x \in I_i$, we can write

$$q_n(x) = \ell(x)\,(s(x) + t(x))$$

where

$$s_j(x) = \begin{cases} 0, & x < x_j \\ w_j/(x - x_j), & x > x_j \end{cases}$$

$$t_j(x) = \begin{cases} w_j/(x - x_j), & x < x_j \\ 0, & x > x_j \end{cases}$$

$$s(x) = \sum_{j=0}^{i-1} s_j(x)$$

$$t(x) = \sum_{j=i}^{n} t_j(x).$$

If the weights w_i are such that for $x \in I_i, i = 1, \ldots, n$ we have

$$\begin{aligned} |s_j(x)| &> |s_{j-1}(x)|, \qquad j = 1, \ldots, i-1 \\ |t_j(x)| &> |t_{j+1}(x)|, \qquad j = i, \ldots, n-1, \end{aligned} \tag{7}$$

then the signs of both $s(x)$ and $t(x)$ on I_i are that of $(-1)^{i-1}$. This is easily seen by summing $s(x)$ from $j = 0$ to $j = i - 1$ and summing $t(x)$ from $j = n$ to

$j = i$. So $s(x) + t(x)$ changes sign at each interpolation point, as does $\ell(x)$, and consequently $q_n(x)$ does not change sign. It is zerofree in (a, b). Making use of the fact that the $s_j(x)$ and $t_j(x)$ are hyperbola and that their vertical asymptotes are ordered since $x_0 < x_1 < \ldots < x_n$, the conditions (7) are satisfied if we have with $w_i = (-1)^i \omega_i$ and $a < x_0$, $b > x_n$ that

$$\begin{aligned} \frac{\omega_{j-1}}{b - x_{j-1}} &< \frac{\omega_j}{b - x_j}, \qquad j = 1, \ldots, n \\ \frac{\omega_j}{x_j - a} &> \frac{\omega_{j+1}}{x_{j+1} - a}, \qquad j = 0, \ldots, n-1. \end{aligned} \tag{8}$$

This is because from (8) we obtain

$$\left(\omega_j - \omega_{j-1}\right)\left(b - x_{j-1}\right) > -\omega_{j-1}\left(x_j - x_{j-1}\right), \qquad j = 1, \ldots, i-1$$

$$\left(\omega_{j+1} - \omega_j\right)\left(x_j - a\right) < \omega_j\left(x_{j+1} - x_j\right), \qquad j = i, \ldots, n-1$$

where, for $x \in I_i$, we can replace in the above $x_j - a$ by $x_j - x$ (for $j = i, \ldots, n-1$) and $b - x_{j-1}$ by $x - x_{j-1}$ (for $j = 0, \ldots, i-1$). Consequently we obtain for $x \in I_i$ that

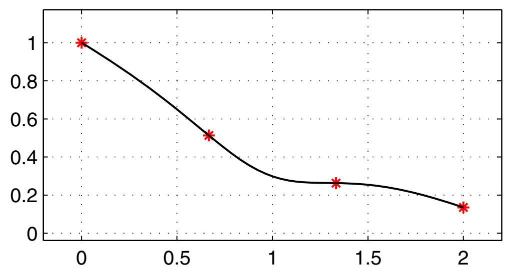
$$\frac{\omega_{j-1}}{x - x_{j-1}} < \frac{\omega_j}{x - x_j}, \qquad j = 1, \ldots, i-1$$

$$\frac{\omega_j}{x_j - x} > \frac{\omega_{j+1}}{x_{j+1} - x}, \qquad j = i, \ldots, n-1$$

which ensures (7). With $a = x_0$ and $b = x_n$ the conditions (8) need to be satisfied for $j = 1, \ldots, n-1$.

Although the simple choice $w_i = (-1)^i$, suggested in [4], satisfies (8) it is useful to have a method of obtaining more and other barycentric weights that offer the same guarantee. Let us for instance take a look at different polefree rational interpolants $r_3(x)$ of the exponential function through the points $(x_i, \exp(-x_i))$ where $x_i = 2i/3$, $i = 0, \ldots, 3$. With $w_i = (-1)^i$ (see Fig. 4) the rational interpolant $r_3(x)$ does not match the look and feel of $\exp(-x)$

Fig. 4 $r_3(x)$ with $w_i = (-1)^i$



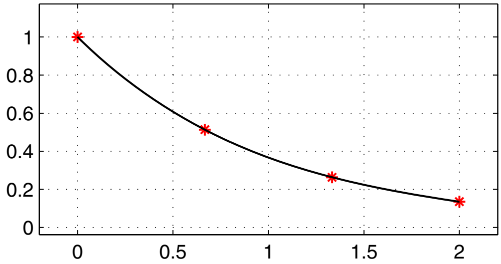


Fig. 5 $r_3(x)$ with w_i satisfying (8)

while with $w_0 = 9/32$, $w_1 = -1$, $w_2 = 3/2$, $w_3 = -7/8$ (see Fig. 5) it has the desired behaviour. In the next sections we concentrate on this shape control in combination with the guarantee that $r_n(x)$ is polefree.

3 Controlling monotonicity

The formula for the first derivative of $r_n(x)$, with x not being a pole or interpolation point of $r_n(x)$, can be rewritten as

$$r'_n(x) = \frac{\sum_{i=0}^{n} r_n[x, x_i] w_i \ell_i(x)}{q_n(x)}.$$

If we denote by

$$\ell_{ij}(x) = \ell(x) / \big((x - x_i)(x - x_j)\big)$$

then we can write for each $i = 0, \ldots, n$

$$r_n[x, x_i] = \frac{\sum_{j=0,\, j \neq i}^{n} (f_j - f_i) w_j \ell_{ij}(x)}{q_n(x)}$$

and consequently

$$r'_n(x) = \frac{\sum_{i,j=0, i<j}^{n} w_i w_j (f_i - f_j)(x_j - x_i) \ell_{ij}^2(x)}{q_n^2(x)}.$$

In order to guarantee that the numerator of $r'_n(x)$ does not change sign in a specified open interval (a, b), being positive for an increasing function and negative for a decreasing one, we now represent it in Lorentz form, taking the Lorentz degree equal to the actual degree,

$$r'_n(x) q_n^2(x) = \sum_{i=0}^{2(n-1)} c_i (x-a)^i (b-x)^{2(n-1)-i}.$$

Let the $2n-1$ distinct points $y_0, \ldots, y_{2(n-1)} \in (a, b)$ not be poles or interpolation points of $r_n(x)$. Then

$$
\begin{aligned}
r_n'(y_k) q_n^2(y_k) &= \sum_{i=0}^{2(n-1)} c_i (y_k - a)^i (b - y_k)^{2(n-1)-i} \\
&= \sum_{i,j=0,i<j}^{n} w_i w_j (f_i - f_j)(x_j - x_i) \ell_{ij}^2(y_k).
\end{aligned}
$$

We introduce the new vector of $n(n+1)/2$ unknowns

$$
v = (w_0 w_1, \ldots, w_0 w_n, w_1 w_2, \ldots, w_1 w_n, \ldots, w_{n-1} w_n)
$$

and denote $m = 2(n-1)$, $c = (c_0, \ldots, c_m)^t$,

$$
C = \begin{pmatrix} (b-y_0)^m & (y_0 - a)(b - y_0)^{m-1} & \ldots & (y_0 - a)^m \\ \vdots & \vdots & & \vdots \\ (b - y_m)^m & (y_m - a)(b - y_m)^{m-1} & \ldots & (y_m - a)^m \end{pmatrix}
$$

$$
D = \begin{pmatrix} (f_0 - f_1)(x_1 - x_0)\ell_{01}^2(y_0) & \ldots & (f_{n-1} - f_n)(x_n - x_{n-1})\ell_{n-1n}^2(y_0) \\ \vdots & & \vdots \\ (f_0 - f_1)(x_1 - x_0)\ell_{01}^2(y_m) & \ldots & (f_{n-1} - f_n)(x_n - x_{n-1})\ell_{n-1n}^2(y_m) \end{pmatrix}.
$$

Then

$$
c = C^{-1} D v.
$$

From ranges for each of the components in the vector v, deduced from the conditions that the c_i do not change sign, instances for the weights w_i can easily be obtained as follows. Since the barycentric weights are only determined up to a multiplicative factor, we can fix $w_0 = 1$. Consequently solutions for the unknowns $w_0 w_1, \ldots, w_0 w_n$ reduce to solutions for $w_1, \ldots, w_n$. Combining the range obtained for w_1 with those for $w_1 w_2, \ldots, w_1 w_n$ updates the solution set for the weights $w_2, \ldots, w_n$. Continuing in this way finally delivers valid ranges for all the weights $w_1, \ldots, w_n$ with $w_0 = 1$.

The componentwise inequalities

$$
C^{-1} D v \geq 0
$$

can be solved using for instance Mathematica's `Reduce` over the domain $\{w_i w_{i+1} < 0, w_0 > 0\} \cap [-1, 1]^{n+1}$. To find at least one particular solution in each connected component of the solution set, Mathematica's `Semialgebraic ComponentInstances` can be used. In Fig. 6 we show some rational interpolants for different choices of the w_i that all guarantee the

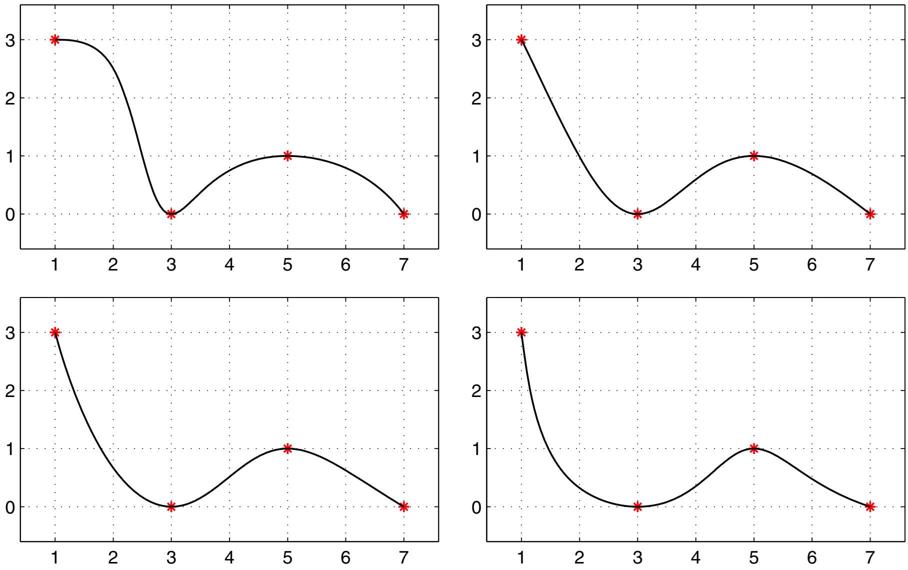


Fig. 6 A selection of comonotone interpolants $r_2(x)$

desired monotonicity behaviour: decreasing in (x_0, x_1), increasing in (x_1, x_2) and decreasing again in (x_2, x_3). The data are $x_i = 1 + 2i, i = 0, \ldots, 3$ and $f_0 = 3,\ f_1 = 0,\ f_2 = 1,\ f_3 = 0$. For the functions $r_3(x)$ shown, we kept w_0 and w_2 fixed and varied w_1 and w_3 in the solution set

$$\begin{aligned} w_0 &> 0, \\ w_1 &\leq -w_0/2, \\ w_2 &= 3w_0, \\ w_3 &= -w_0 + w_1. \end{aligned}$$

The respective weight vectors (w_0, w_1, w_2, w_3) from left to right and from top to bottom for the 4 graphs are $(1/3, -1/6, 1, -1/2)$, $(1/3, -49/96, 1, -27/32)$, $(1/3, -149/192, 1, -71/64)$ and $(1/3, -5/3, 1, -2)$.

It is clear that a computer algebra system is not capable of handing large problems. Memory overload problems quickly occur. Just consider adding one point, namely $x_i = 1 + 2i, i = 0, ...4$ and changing the function values to $f_0 = 3,\ f_1 = 0.5,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1.5$ and we need to switch to floating-point arithmetic. The inequalities can be treated as polynomial constraints in an optimization problem and solved using Matlab's add-on `GloptiPoly` [11]. To the latter an objective function to optimize the numerical stability of the evaluation of $r_n(x)$ can be added [6]. `GloptiPoly` solves global polynomial optimization problems subject to polynomial inequality or equality constrains. When only linear and quadratic expressions are involved, simple Matlab scripts [11] can be used to enter them. In Fig. 7 we show $r_4(x)$ satisfying

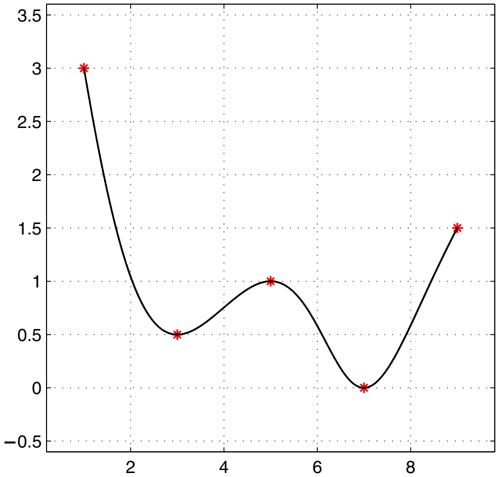


Fig. 7 $r_4(x)$ with w_i satisfying monotonicity constraints

the above interpolation conditions with an additional increasing constraint on (x_3, x_4) and obtained from `GloptiPoly`: the computed weights equal $w_0 = 0.3674$, $w_1 = -1$, $w_2 = 1$, $w_3 = -0.6812$, $w_4 = 0.7450$.

4 Imposing convexity or concavity

The second derivative $r_n''(x)$ in a point x that is not a pole or an interpolation point, equals

$$r_n''(x) = \frac{2\sum_{i=0}^{n} r_n[x, x, x_i] w_i \ell_i(x)}{q_n(x)}.$$

For each $i = 0, \ldots, n$ we can write

$$r_n[x, x, x_i] w_i \ell_i(x) = \frac{\sum_{i,j=0, i\neq j}^{n} \left(r_n[x, x_j] - r_n[x, x_i]\right) w_i w_j \ell_i(x) \ell_{ij}(x)}{q_n(x)}.$$

After some rearrangements of the terms we find for the numerator of $r_n''(x)$ the following polynomial of degree $3(n-1)$ in x and homogeneous and cubic in the weights w_i:

$$\begin{aligned} &2 \sum_{i,j=0, i<j}^{n} w_i w_j (w_i + w_j)(f_i - f_j)(x_i - x_j) \ell_{ij}^3(x) \\ &\quad + 2 \sum_{i,j,k=0, i<j<k} w_i w_j w_k \ell_{ijk}(x) \left(F_{ijk}(x) + F_{ikj}(x) + F_{jki}(x)\right) \end{aligned} \qquad (9)$$

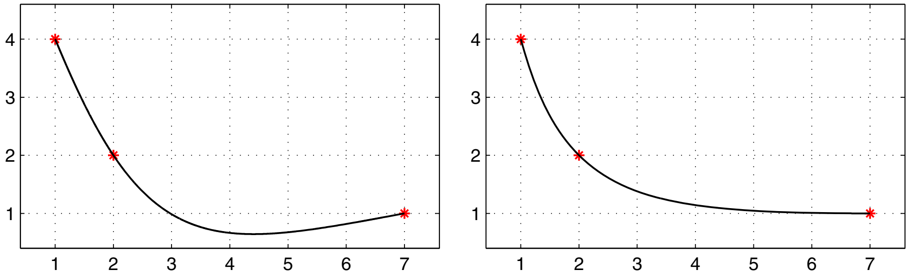


Fig. 8 A selection of convex interpolants $r_3(x)$

with

$$F_{ijk}(x) = (x_i - x_j)\ell_{ij}^2(x)\left[x_i(f_j - f_k) + x_j(f_k - f_i) + x(f_i - f_j)\right].$$

The denominator of $r_n''(x)$ is $q_n^3(x)$. Convexity in a certain interval is obtained by imposing that (9) has the same sign in that interval as $q_n(x)$. For concavity the signs need to be opposite. As in the previous section these conditions lead to polynomial inequalities for the weights w_i, in this case cubic inequalities.

In the example shown in Fig. 8 we take $x_0 = 1, x_1 = 2, x_2 = 7$ and $f_0 = 4$, $f_1 = 2$, $f_2 = 1$ and add the convexity conditions. The current version of `GloptiPoly` is aimed at small and medium-scale problems. To handle the cubic inequalities from the convexity and concavity requirements, we need another Matlab add-on for sparse polynomial optimization, namely `SparsePOP`. The number of weights that can be dealt with is 30, and since our polynomial inequalities have a sizable number of zero coefficients, `SparsePOP` is more efficient than the previous solvers we mentioned. The computed solution is $w_0 = 0.3816$, $w_1 = -0.5659$, $w_2 = 0.4148$ (at the left). Another solution is given by $w_0 = 2/5$, $w_1 = -1$, $w_2 = 1$ (at the right).

5 Asymptotes and extrema

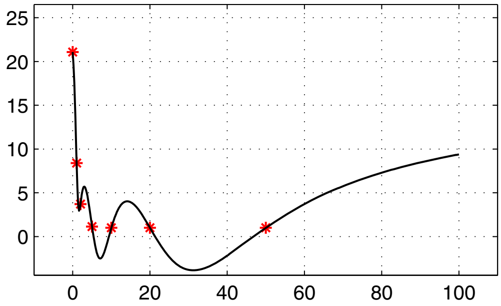
We recall that the highest degree coefficients in numerator and denominator of $r_n(x)$ are given by

$$a_n = \sum_{i=0}^{n} w_i f_i, \qquad b_n = \sum_{i=0}^{n} w_i.$$

When $b_n \neq 0$, a horizontal asymptote at $y = \theta$ can be imposed by choosing w_i such that

$$\sum_{i=0}^{n} (f_i - \theta) w_i = 0. \tag{10}$$

Fig. 9 $r_6(x)$ with $w_i = (-1)^i$



For a slant asymptote we need $b_n = 0$ and $b_{n-1} \neq 0$ with

$$b_{n-1} = -\sum_{i=0}^{n} \left(w_i \sum_{j \neq i} x_j \right).$$

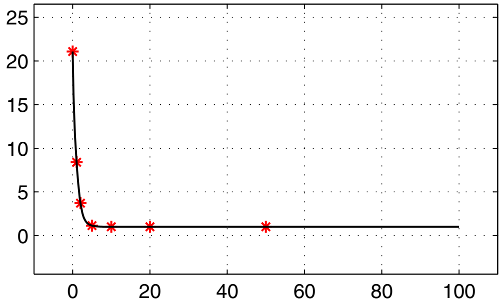
Hence a slant asymptote with slope ξ is obtained with

$$\sum_{i=0}^{n} w_i = 0,$$

$$\sum_{i=0}^{n} \left(f_i + \xi \sum_{j \neq i} x_j \right) w_i = 0.$$

We illustrate by means of an example how the above can be combined with the requirement to have a zero-free denominator. Take $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 5$, $x_4 = 10$, $x_5 = 20$, $x_6 = 50$ and $f_i = 1 + \exp(-x_i + 3)$. In Fig. 9 we give the rational interpolant with $w_i = (-1)^i$ which guarantees a polefree $r_6(x)$, and in Fig. 10 we compute w_i for $r_6(x)$ from (8) and (10). From the solution set we take $w_0 = 0.0005$, $w_1 = -0.0095$, $w_2 = 0.0324$, $w_3 = -0.2175$, $w_4 = 0.8880$, $w_5 = -1.5962$, $w_6 = 1.2657$. The results clearly do not need commenting.

Fig. 10 $r_6(x)$ with w_i as above



6 Barycentric rational interval interpolation

Now let us take a look at the interpolation of interval data. Then at the points x_i we are given intervals $F_i = [f_{i<}, f_{i>}]$ instead of point values f_i. The advantage of considering interval data is in the context of measured or simulated data: intervals give us a way to take the inherent data error into consideration whilst guaranteeing an upper bound on the tolerated range of uncertainty. The latter is the main difference with a least squares technique which does as well as it can, but without respecting an imposed threshold on the approximation error. Problem statement (1) then changes to the following. Given $N+1$ points x_i and intervals $[f_{i<}, f_{i>}]$, find $n+1$ points x_{i_j} among them with $n < N$ and $n+1$ values $g_{i_j} \in F_{i_j}$ and nonzero weights w_j, $j = 0, \ldots, n$ such that

$$R_n(x) = \frac{\displaystyle\sum_{j=0}^{n} g_{i_j} \frac{w_j}{x - x_{i_j}}}{\displaystyle\sum_{j=0}^{n} \frac{w_j}{x - x_{i_j}}} \tag{11}$$

satisfies

$$R_n(x_i) \in F_i, \qquad i = 0, \ldots, N. \tag{12}$$

We denote

$$Q_n(x) = \sum_{j=0}^{n} w_j \ell_{i_j}(x),$$
$$\ell_{i_j}(x) = \prod_{k=0, i_k \neq i_j}^{n} \left(x - x_{i_k}\right).$$

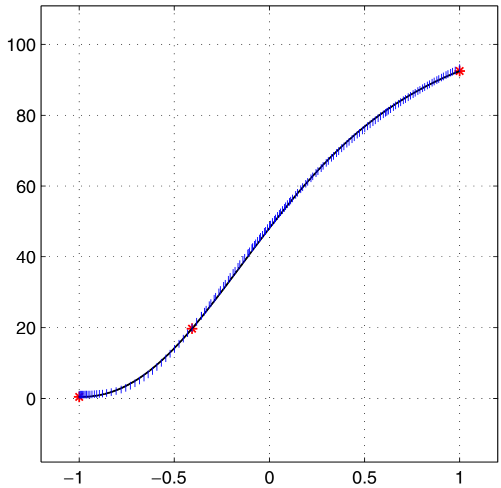


Fig. 11 Rational interval interpolant $R_2(x)$ (polefree, positive, increasing) with $w = [0.0086, -0.0109, 0.0079]$ and $g = [0.4734, 19.7216, 92.4635]$

Table 1 Data from waveform distortion study

x	0	1	2	3	4	5	6	7	8	9	10
f	10	10	10	10	10	10	10.5	15	50	60	85

If for a certain n a solution $R_n(x)$ exists, then it clearly exists for any selection of the x_{i_j}. The crucial thing is to find the function values g_{i_j} and the weights w_j such that the representation (11) can be given and the interpolation conditions (12) are satisfied. With $w_j = (-1)^j \omega_j$, and assuming that

$$Q_n(x_i) = \sum_{j=0}^{n} w_j \ell_{i_j}(x_i) > 0, \qquad i = 0, \dots, N$$

a polefree rational interval interpolant is obtained by imposing (8) on the weights, together with

$$f_{i<} Q_n(x_i) \le \sum_{j=0}^{n} g_{i_j} w_j \ell_{i_j}(x_i) \le f_{i>} Q_n(x_i), \qquad i = 0, \dots, N$$

on the g_{i_j} which are also restricted by $f_{i_j<} \le g_{i_j} \le f_{i_j>}$.

These conditions can be complemented with additional constraints guaranteeing positivity, monotonicity, convexity, concavity or some kind of asymptotic behaviour, as in the preceding paragraphs. We illustrate the above with a larger dataset taken from the webpage www.itl.nist.gov/div898/strd/nls/data/kirby2.shtml. Given are $N + 1 = 151$ datapoints with positive function values displaying a monotonely increasing behaviour. It is indicated that the data can be modeled by a rational function of degree 2 in numerator and denominator. On the website such a rational function is computed using a discrete rational least squares technique, starting from an appropriate initial

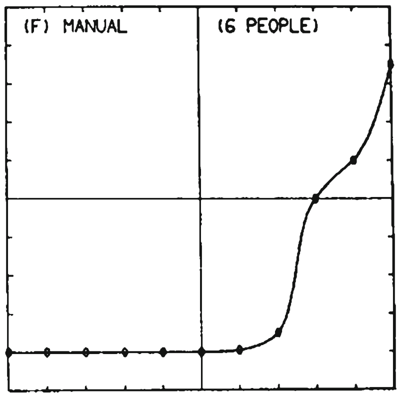


Fig. 12 Manual curve

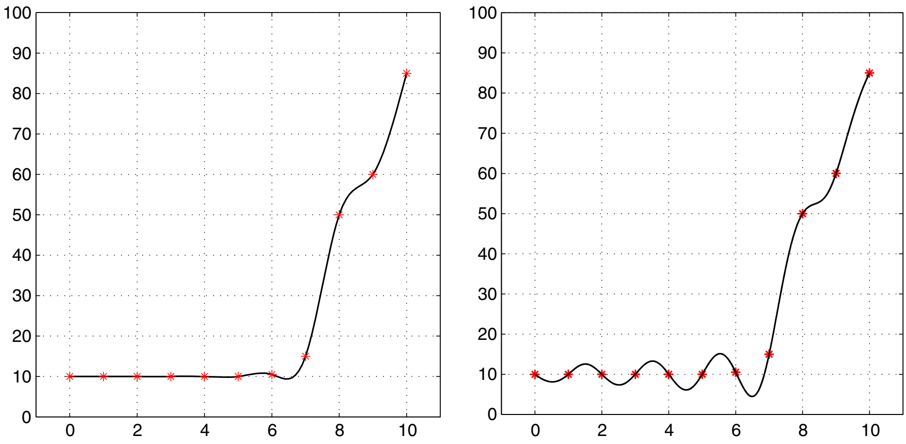


Fig. 13 Natural cubic spline and rational interpolation with $w_i = (-1)^i$

guess for the numerator and denominator coefficients. Control of the error is only a posteriori.

We allow a priori error control by replacing the point data by intervals. We take care of the positivity of $R_2(x)$ by having all left interval endpoints $f_{i<} \geq 0$ and imposing monotonicity. The interpolant $R_2(x)$ is also polefree in $[x_0, x_N]$. The result is displayed in Fig. 11. The chosen datapoints x_{i_j}, $j = 0, 1, 2$ are indicated with an asterisk and the computed weights w_j and function values g_{i_j} are given beneath the figure.

7 Benchmark problem

A study of waveform distortion in electronic circuits conducted by Akima [1] has delivered a benchmark problem for several modelling techniques. The data, which we display in Table 1, were also interpolated by a curve manually drawn by a number of scientists. We show the average curve of this procedure in Fig. 12. In Fig. 13 we display the curves obtained from natural cubic spline interpolation (at the left) and from barycentric rational interpolation with the weights $w_i = (-1)^i$, $i = 0, ..., 10$ (at the right).

We apply the new technique of rational interval interpolation ($N = 10$) with the added requirements of a zerofree denominator (linear constraints as in Section 2.3) and a monotonely increasing function (quadratic constraints as in Section 3) in the interval $[x_0, x_{10}]$. For the points x_{i_j}, $j = 0, \ldots, n$ we

Table 2 Data subset

x	0	7	8	9	10
f	10	15	50	60	85

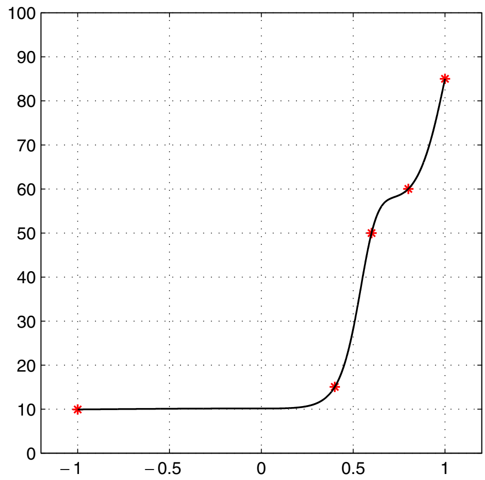


Fig. 14 Rational interval interpolant $R_4(x)$

choose $n = 4$ and the values given in Table 2. The interpolation intervals are taken as $[f_i - 1/6, f_i + 1/6]$, $i = 0, \ldots, 10$. In Fig. 14 we show the rational interpolant with the weights $w_0 = 0.8511$, $w_1 = -0.2688$, $w_2 = 0.1922$, $w_3 = -0.2081$, $w_4 = 0.0447$ obtained from `SparsePOP` and the function values $g_{i_0} = 9.9254$, $g_{i_1} = 15.0612$, $g_{i_2} = 49.9845$, $g_{i_3} = 60.0069$, $g_{i_4} = 85.0003$ in the interpolation intervals. With these values our barycentric rational interpolant is given by (11). The 11 intervals $[f_i - 1/6, f_i + 1/6]$ are too small to be visible on the graph. The results clearly do not need commenting.

8 Conclusion

We have seen how the barycentric weights w_i can be chosen to influence the shape of the rational function:

- positive denominator,
- increasing,
- decreasing,
- convex,
- concave,
- local extrema,
- horizontal or slant asymptote.

In addition, we have illustrated (see for instance Fig. 6) that the rational interpolant $r_n(x)$, and more specific its shape, is a continuous function of the weights $w_0, \ldots, w_n$. For instance, keeping the weights $w_0 = 1/3$, $w_1 = -1/6$ fixed in Fig. 6 and varying w_2 (left to right and top to bottom in Fig. 15) from $w_2 = 1/2$ over $w_2 = 1$ and $w_2 = 1.5$ to $w_2 = \frac{9}{4}$ slowly transforms the overall concave look into an overall convex look.

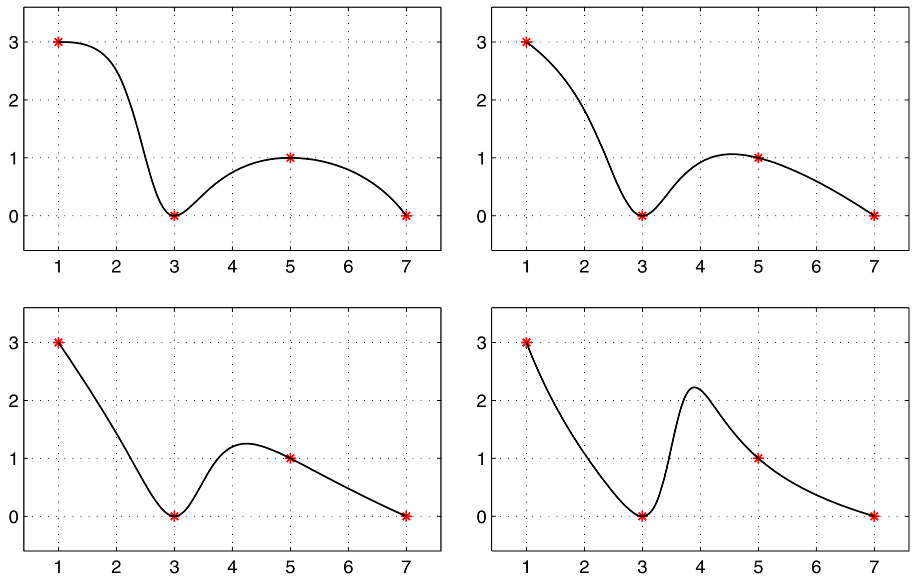


Fig. 15 From overall concave to overall convex

This property connects the barycentric rational interpolant to rational Bézier curves. In the latter the Lagrange basis in x is replaced by a parameterized Bernstein basis and a weighted curve connects the points $(x_i,\ f_i)$:

$$s_n(t) = \frac{\displaystyle\sum_{i=0}^{n} w_i \binom{n}{i} t^{n-i}(1-t)^i (x_i,\ f_i)}{\displaystyle\sum_{i=0}^{n} w_i \binom{n}{i} t^{n-i}(1-t)^i}, \qquad 0 \le t \le 1.$$

This form is not used for (interval) interpolation though, which is one of the strengths of our technique.

References

1. Akima, H.: A new method of interpolation and smooth curve fitting based on local procedures. J. ACM **17**(4), 589–602 (1970)
2. Baltensperger, R., Berrut, J.P., Noël, B.: Exponential convergence of a linear rational interpolant between transformed Chebyshev points. Math. Comput. **68**(227), 1109–1120 (1999)
3. Bernstein, S.N.: Sur la représentation des polynômes positifs. Commun. Soc. Math. Kharkow **2**(14), 227–228 (1915)
4. Berrut, J.P.: Rational functions for guaranteed and experimentally well-conditioned global interpolation. Comput. Math. Appl. **15**, 1–16 (1988)
5. Berrut, J.P., Baltensperger, R., Mittelmann, H.: Recent developments in barycentric rational interpolation. In: de Bruin, M., Mache, D., Szabados, J. (eds.) Trends and Applications in Constructive Approximation. International Series of Numerical Mathematics, pp. 27–52. Birkhäuser, Basel (2005)

6. Berrut, J.P., Mittelmann, H.: Matrices for the direct determination of the barycentric weights of rational interpolation. J. Comput. Appl. Math. **78**(2), 355–370 (1997)
7. Cuyt, A., Wuytack, L.: Nonlinear Methods in Numerical Analysis. North-Holland, Amsterdam (1987)
8. Edelman, A., Micchelli, C.: Admissible slopes for monotone and convex interpolation. Numer. Math. **51**, 441–458 (1987)
9. Erdélyi, T., Szabados, J.: On polynomials with positive coefficients. J. Approx. Theory **54**(1), 107–122 (1988)
10. Floater, M.S., Hormann, K.: Barycentric rational interpolation with no poles and high rates of approximation. Numer. Math. **107**(2), 315–331 (2007)
11. Henrion, D., Lasserre, J.B.: GloptiPoly: Global optimization over polynomials with Matlab and SeDuMi. ACM Trans. Math. Softw. **29**(2), 165–194 (2003)
12. Kopotun, K., Leviatan, D.: Comonotone polynomial approximation in $L_p[-1, 1], 0 < p \leq \infty$. Acta Math. Hung. **77**(4), 301–310 (1997)
13. Leviatan, D., Shevchuk, I.A.: Coconvex polynomial approximation. J. Approx. Theory **121**(1), 100–118 (2003)
14. Manni, C.: On shape preserving C^2 Hermite interpolation. BIT **41**(1), 127–148 (2001)
15. Polezzi, M., Sri Ranga, A.: On the denominator values and barycentric weights of rational interpolants. J. Comput. Appl. Math. **200**, 576–590 (2007)
16. Salazar Celis, O., Cuyt, A., Verdonk, B.: Rational approximation of vertical segments. Numer. Algorithms **45**, 375–388 (2007)
17. Schneider, C., Werner, W.: Some new aspects of rational interpolation. Math. Comput. **47**(175), 285–299 (1986)
18. Zhu, X., Zhu, G.: A method for directly finding the denominator values of rational interpolants. J. Comput. Appl. Math. **148**(2), 341–348 (2002)