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Sharp Bounds for Lebesgue Constants of Barycentric Rational Interpolation at Equidistant Points

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ABSTRACT

A rough analysis of the growth of the Lebesgue constant in the case of barycentric rational interpolation at equidistant interpolation points was made in [Bos et al. 11] and [Bos et al. 12], leading to the conclusion that it only grows logarithmically. Here we give a fine analysis, obtaining the precise growth formula

$$\frac{2}{\pi}\left(\ln(n+1)+\ln 2+\gamma\right)+o(1)$$

for the Lebesgue constant under consideration, with γ being the Euler constant. The similarity between barycentric rational interpolation at equidistant points and polynomial interpolation at Chebyshev nodes (or the like) is remarkable. After revisiting the polynomial interpolation case in Section 1 and introducing the barycentric rational interpolation case in Section 2, tight lower and upper bound estimates are given in Section 3. These fine results could only be formulated after performing very high-order numerical experiments in exact arithmetic. In Section 4, we indicate that the result can be extended to the rational interpolants introduced by Floater and Hormann in [Floater and Hormann 07]. Finally, the proof of the new tight bounds is detailed in Section 5.

KEYWORDS
barycentric rational interpolation; linear interpolation; equidistant nodes; preassigned poles

1. Sharp bounds for Lebesgue constants in polynomial interpolation

Let the function f belong to $C([-1,1])$. When approximating f by an element from a finite-dimensional $V_n = \text{span}\{\phi_0, \ldots, \phi_n\}$ with $\phi_i \in C([-1,1])$ for $0 \le i \le n$, we know that there exists at least one element $p_n^* \in V_n$ that is closest to f. When using the $\|\ \|_\infty$ norm, this element is the unique closest one if the $\phi_0, \ldots, \phi_n$ are a Chebyshev system. Since the computation of this element is more complicated than that of the interpolant

$$\sum_{i=0}^{n} \alpha_i \phi_i(x_j) = f(x_j), \qquad j = 0, \ldots, n, \qquad -1 \le x_j \le 1,$$

there is an interest in interpolation points x_j that make the interpolation error

$$\left\| f(x) - \sum_{i=0}^{n} \alpha_i \phi_i(x) \right\|_\infty = \max_{x\in[-1,1]} \left| f(x) - \sum_{i=0}^{n} \alpha_i \phi_i(x) \right|$$

as small as possible. In other words, there is an interest in using interpolating polynomials that are near-best approximants.

When $\phi_i(x) = x^i$ and f is sufficiently differentiable, then for the interpolant

$$p_n(x) = \sum_{i=0}^{n} \alpha_i x^i,$$

satisfying $p_n(x_j) = f(x_j), 0 \le j \le n$, the error $\|f - p_n\|_\infty$ is bounded by [Young and Gregory 72, p. 267]

$$\|f - p_n\|_\infty \le \max_{x\in[-1,1]} \left(\frac{|f^{(n+1)}(x)|}{(n+1)!} \right) \times \max_{x\in[-1,1]} \prod_{j=0}^{n} |x - x_j|. \qquad (1\text{–}1)$$

It is well-known that $\|(x-x_0)\cdots(x-x_n)\|_\infty$ is minimal on $[-1,1]$ if the x_j are the zeroes of the $(n+1)$-th degree Chebyshev polynomial $T_{n+1}(x) = \cos((n+1)\arccos x)$.

The operator that associates with f its interpolant p_n is linear and given by

$$P_n : C([-1,1]) \to V_n : f(x) \to p_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x)$$

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where the basic Lagrange polynomials $\ell_i(x)$,

$$\ell_i(x) = \prod_{j=0,\, i\neq j}^{n} \frac{x - x_j}{x_i - x_j},$$

satisfy $\ell_i(x_j) = \delta_{ij}$. So another bound for the interpolation error is given by

$$\|f - p_n\|_\infty \leq (1 + \|P_n\|)\, \|f - p_n^*\|_\infty,$$

$$\|P_n\| = \max_{x\in[-1,1]} \sum_{i=0}^{n} |\ell_i(x)|\,.$$

Here $\Lambda_n := \Lambda_n(x_0, \ldots, x_n) = \|P_n\|$ is called the Lebesgue constant, and $L_n(x) := L_n(x_0, \ldots, x_n; x) = |\ell_0(x)| + \ldots + |\ell_n(x)|$ is called the Lebesgue function. Both Λ_n and $L_n(x)$ clearly depend on the location of the interpolation points x_j. An explicit formula for the x_j that minimize the Lebesgue constant is not known, and if no further constraints are imposed on the interpolation points, then the solution is not even unique. But it is proved in [Vértesi 86, Szabados and Vértesi 90, pp. 110–121] that the minimal growth of the Lebesgue constant, in terms of the number of interpolation points $n + 1$, is given by

$$\frac{2}{\pi}\left(\ln(n+1) + \gamma + \ln\left(\frac{4}{\pi}\right)\right) \sim \frac{2}{\pi}\ln(n+1) + 0.52125\ldots$$

with γ the Euler constant.

Several node sets $\{x_0, \ldots, x_n\}$ come close to realizing this minimal growth, among which the Chebyshev zeroes [Rivlin 74, Ehlich and Zeller 66, Günttner 80] and the Fekete points [Sündermann 83]. A simple node set known in closed form that approximates the optimal node set very well is the so-called extended Chebyshev node set given by

$$x_j = -\frac{\cos\left(\frac{(2j+1)\pi}{2(n+1)}\right)}{\cos\left(\frac{\pi}{2(n+1)}\right)}, \qquad j = 0, \ldots, n. \qquad (1\text{–}2)$$

The division by $\cos(\pi/(2n+2))$ guarantees that $x_0 = -1$ and $x_n = 1$. The growth of the Lebesgue constant for the extended Chebyshev nodes is bounded by [Günttner 80, Hesthaven 98]

$$\Lambda_n(x_0, \ldots, x_n) < \frac{2}{\pi}\log(n+1) + 0.5829\ldots, \qquad n \geq 4,$$

which is only slightly larger than the minimal growth. At the same time, it is known that the Lebesgue constant Λ_n for equidistant interpolation points grows exponentially [Schönhage 61, Turetskii 40].

2. Lebesgue constants for rational interpolation with preassigned poles

When moving to rational functions instead of polynomials, the approximation and interpolation problems become nonlinear unless one considers the case of an a priori fixed denominator or a priori fixed poles, as we do in this article.

So let $q_m(x) = \prod_{k=0}^{m-1}(1 - x/\xi_k)$ with $\xi_k \notin [-1, 1]$ and interpolate

$$p_n(x_j) = f(x_j)q_m(x_j), \qquad j = 0, \ldots, n \qquad (2\text{–}3)$$

with $p_n(x) \in \text{span}\{1, \ldots, x^n\}$. The rational interpolant p_n/q_m now belongs to $V_n = \text{span}\{1/q_m(x), \ldots, x^n/q_m(x)\}$. In the sequel, we restrict ourselves to polynomials $q_m(x)$ having real coefficients, in other words having poles that are real or appear in complex conjugate pairs.

With $x_j \in [-1, 1]$ and $\xi_k \notin [-1, 1]$ and since p_n now interpolates fq_m, the rational interpolation error is bounded from above by

$$\left\|f - \frac{p_n}{q_m}\right\|_\infty \leq \max_{x\in[-1,1]}\left(\frac{|(fq_m)^{(n+1)}(x)|}{(n+1)!}\right) \times \max_{x\in[-1,1]}\prod_{j=0}^{n}\frac{|x - x_j|}{|q_m(x)|}. \qquad (2\text{–}4)$$

The factor $(x - x_0)\cdots(x - x_n)/q_m(x)$ has minimal uniform norm if the x_j are the Chebyshev–Markov nodes [Lukashov 04]. These are also the zeroes of the orthogonal rational function $\mathcal{T}_{n+1}(x)$ with numerator of degree $n + 1$, denominator equal to $q_m(x)$ and satisfying [Van Deun 10]

$$\int_{-1}^{1} \mathcal{T}_{n+1}(x)\frac{p_k(x)}{q_m(x)}\frac{dx}{\sqrt{1-x^2}} = 0, \qquad k = 0, \ldots, n.$$

If the poles ξ_k are real or appear in complex conjugate pairs, then the zeroes of $\mathcal{T}_{n+1}(x)$ are indeed real, simple, and belong to the open interval $(-1, 1)$ [Van Deun 10]. In the sequel, we assume that $m = n$.

The operator that associates with f its rational interpolant p_n/q_n satisfying (2–3) is still linear and given by

$$R_n : C([-1,1]) \to V_n : f(x) \to \frac{p_n(x)}{q_n(x)} = \sum_{i=0}^{n} f(x_i)\frac{q_n(x_i)\ell_i(x)}{q_n(x)}.$$

In the same way as in Section 1, we obtain that the error in rational interpolation with preassigned poles is bounded from above by

$$\left\|f - \frac{p_n}{q_n}\right\|_\infty \leq (1 + \|R_n\|)\left\|f - \frac{p_n^*}{q_n}\right\|_\infty,$$

$$\|R_n\| = \max_{x\in[-1,1]}\sum_{i=0}^{n}\frac{|q_n(x_i)\ell_i(x)|}{|q_n(x)|},$$

where p_n^* is the best polynomial approximant of degree n to fq_n. Here $M_n := M_n(x_0, \ldots, x_n; \xi_1, \ldots, \xi_n) = \|R_n\|$ is the Lebesgue constant of rational interpolation in the points $x_0, \ldots, x_n$ with preassigned poles at $\xi_1, \ldots, \xi_n$. The function

$$M_n(x) := M_n(x_0, \ldots, x_n; \xi_1, \ldots, \xi_n; x) = \sum_{i=0}^{n} \frac{|q_n(x_i)\ell_i(x)|}{|q_n(x)|}$$

is called the Lebesgue function of rational interpolation with predetermined poles.

In [Cuyt et al. 11], the behavior of M_n is investigated in case the x_j are the extended Chebyshev–Markov nodes for some predetermined $q_n(x)$. The notion *extended* is again to be understood in the way as in (1–2). It is important to note that $\mathcal{T}_{n+1}(x)$ is the rational function with monic numerator of degree $n+1$ and denominator $q_m(x)$ having minimal $||\cdot||_\infty$ on $[-1, 1]$. So $\mathcal{T}_{n+1}(x)$ minimizes the bound (2–4) in the same way as $T_{n+1}(x)$ minimizes (1–1).

In [Berrut and Mittelmann 97], the poles ξ_k are determined in order to minimize M_n in the case of equidistant interpolation points x_j. So there the location of the poles is adapted to the given equidistant interpolation points, while in [Cuyt et al. 11] the location of the interpolation points is optimized for given poles. It depends on the numerical application of course, whether it is more important to have equidistant data available than to make use of predetermined poles that dictate the shape and the behavior of the interpolant.

Here, we want to give sharp bounds on the growth of the Lebesgue constant M_n in the case of $n+1$ equidistant interpolation points x_j and n poles fixed either by [Berrut 88]

$$q_n(x) = \sum_{i=0}^{n}(-1)^i \prod_{j=0,\, j\neq i}^{n} (x - x_j) \tag{2–5}$$

as in Section 3 or by [Floater and Hormann 07]

$$s_n^{(d)}(x) = \sum_{i=0}^{n}(-1)^i\sigma_i \prod_{j=0,\, j\neq i}^{n} (x - x_j),$$
$$\sigma_i = \sum_{j=\max(i-d,0)}^{\min(i,n-d)} \binom{d}{i-j},$$
$$n \geq 2d, \quad d = 1, 2, \ldots \tag{2–6}$$

as in Section 4. It is well-known that neither the polynomial $q_n(x)$ [Berrut 88] nor the polynomial $s_n^{(d)}(x)$ [Floater and Hormann 07] have zeroes on the real line. Hence, in both cases $\xi_k \notin [-1, 1]$.

A first analysis of M_n for equidistant interpolation points and poles preassigned by (2–5) or (2–6) is given in [Bos et al. 11] and [Bos et al. 12], respectively. We denote the former Lebesgue constant by

$$M_n^{(0)} := M_n(x_0, \ldots, x_n; q_n(\xi_k) = 0)$$

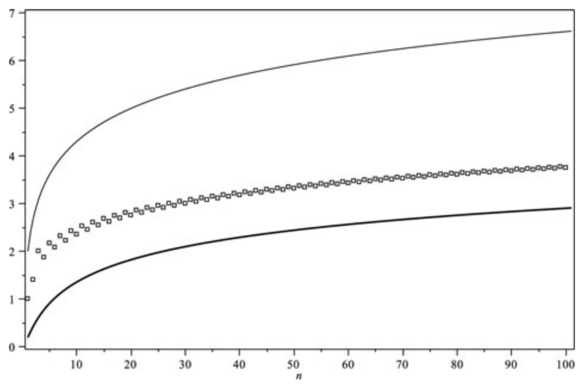

Figure 1. Bounds for $M_n^{(0)}$ as in [Bos et al. 11] with $M_n^{(0)}$ denoted by □.

and the latter by

$$M_n^{(d)} := M_n(x_0, \ldots, x_n; s_n^{(d)}(\xi_k) = 0), \qquad d \geq 1.$$

In both cases, we denote the Lebesgue function by $M_n(x)$, as it is clear from the context in which case we are.

3. Precise growth formula for Berrut's rational interpolant

For $q_n(x)$ in $||R_n||$ given by (2–5), the expression for the Lebesgue function $M_n(x)$ can be simplified to

$$M_n(x) = \frac{\sum_{i=0}^{n} 1/|x - x_i|}{\left|\sum_{i=0}^{n}(-1)^i/(x - x_i)\right|}. \tag{3–7}$$

In [Bos et al. 11], crude lower and upper bounds are given for $M_n^{(0)}$:

$$\frac{2}{\pi + \frac{4}{n}} \ln(n+1) \leq M_n^{(0)} \leq 2 + \ln(n).$$

We illustrate these in Figure 1, where $M_n^{(0)}$, for subsequent values of n, is indicated with the symbol □.

As proved in Section 5, the growth rate of $M_n^{(0)}$ is given more precisely by

$$\frac{2\left(\ln(n+1) + \ln 2 + \gamma\right)}{\pi + \frac{4}{n+3}} \leq M_n^{(0)} \simeq \frac{2\left(\ln(n+1) + \ln 2 + \gamma + \frac{1}{24n}\right)}{\pi - \frac{4}{n+2}}. \tag{3–8}$$

This is the exact asymptotic growth of the Lebesgue constant $M_n^{(0)}$. The new bounds are illustrated in Figure 2.

The tight formulation (3–8) was only possible after carrying out numerical experiments in exact arithmetic up to $n = O(10^{1000})$! The proof follows in Section 5. The advantage of exact arithmetic here (besides the absence

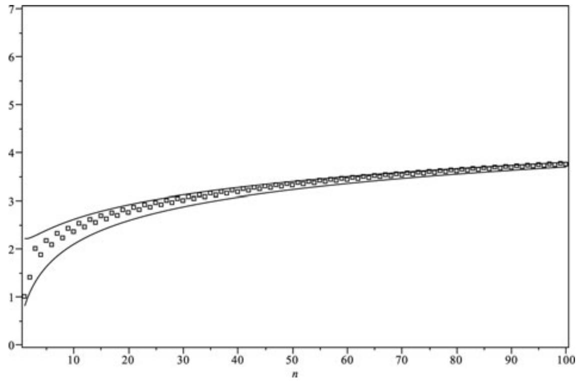

Figure 2. Sharpened bounds for $M_n^{(0)}$ with $M_n^{(0)}$ denoted by □.

of rounding errors) is that, in computer algebra software, there is a nice compact expression for the evaluation of $M_n(x)$ halfway between two neighboring interpolation points in terms of the digamma function $\Psi(x)$. This expression allows us to evaluate it for very high values of n. When making such a detailed analysis, the true problem to obtain an accurate bound becomes clear. The maximum value of the Lebesgue function $M_n(x)$ does not occur near a fixed location independent of n, like the midpoint or the endpoints of the interval: for n even, the positive argument of the maximum is a function of $\log_{10} n$. More precisely, it changes with n mod 4 and moves up with $\log_{10}\lfloor n/4\rfloor$ when n is even! To illustrate this, we display the value of $M_n(x)$ near the many local maxima (also see Figure 5).

For $n = 4 \times 10^{100}$ (situation as in Figure 5 top left) and $x = (20i + 9)/n$, $i = 0, \ldots, 9$, the values can be found in Table 1: a global maximum occurs (not even at, but) near $x = 149/n$ (we also show the values of $M_n(x)$ at $x = 145/n$ and $x = 153/n$ for comparison). For $n = 4 \times 10^{1000} + 2$ (situation as in Figure 5 bottom left) and $x = (200i + 67)/n$, $i = 0, \ldots, 7$, the values can be found in Table 2: a global maximum occurs very near $x = 1467/n$ (compare with the value of $M_n(x)$ at $x = 1463/n$ and $x = 1471/n$).

Table 1. Values of $M_n(x)$, $n = 4 \times 10^{100}$, 209 digits.

$M_n(189/n)$	148.2784002 (189 digits) 1347141070
$M_n(169/n)$	148.2784002 (189 digits) 1371588542
$M_n(153/n)$	148.2784002 (189 digits) 1379687364
$M_n(149/n)$	148.2784002 (189 digits) 1380120520
$M_n(145/n)$	148.2784002 (189 digits) 1379917056
$M_n(129/n)$	148.2784002 (189 digits) 1372737004
$M_n(109/n)$	148.2784002 (189 digits) 1349437993
$M_n(89/n)$	148.2784002 (189 digits) 1310223488
$M_n(69/n)$	148.2784002 (189 digits) 1255093489
$M_n(49/n)$	148.2784002 (189 digits) 1184047995
$M_n(29/n)$	148.2784002 (189 digits) 1097087007
$M_n(9/n)$	148.2784002 (189 digits) 0994210525

Table 2. Values of $M_n(x)$, $n = 4 \times 10^{1000} + 2$, 2012 digits.

$M_n(1471/n)$	1467.562478 (1987 digits) 700047035547169
$M_n(1467/n)$	1467.562478 (1987 digits) 700047058426038
$M_n(1463/n)$	1467.562478 (1987 digits) 700047017642930
$M_n(1267/n)$	1467.562478 (1987 digits) 699967033348511
$M_n(1067/n)$	1467.562478 (1987 digits) 699727853327892
$M_n(867/n)$	1467.562478 (1987 digits) 699329518364181
$M_n(667/n)$	1467.562478 (1987 digits) 698772028457378
$M_n(467/n)$	1467.562478 (1987 digits) 698055383607483
$M_n(267/n)$	1467.562478 (1987 digits) 697179583814496
$M_n(67/n)$	1467.562478 (1987 digits) 696144629078418

4. Growth formulas for Floater and Hormann's rational interpolant

For $q_n(x)$ in $||R_n||$ given by (2–6), the expression for the Lebesgue function $M_n(x)$ simplifies to

$$M_n(x) = \frac{\sum_{i=0}^{n} \sigma_i/|x - x_i|}{\left|\sum_{i=0}^{n}(-1)^i \sigma_i/(x - x_i)\right|}, \tag{4–9}$$

with σ_i, $i = 0, \ldots, n$ given by

$$\sigma_i = \begin{cases} \sum_{j=0}^{i} \binom{d}{j}, & i \le d, \\ 2^d, & d \le i \le n - d, \\ \sigma_{n-i}, & i \ge n - d. \end{cases}$$

For n odd and $d = 1$, the maximum of the Lebesgue function occurs near the origin, not precisely at the origin. We illustrate this in Figure 3: with $n = 11$, $d = 1$ the Lebesgue function $M_n(x)$ achieves its maximum at about $\pm 2/11$. A more precise statement is that for n mod $4 = 1$ the maximum is at $x^* = 0$ and for n mod $4 = 3$ the maximum occurs near $\pm 2/n$. With n even the maximum occurs near $1/n$.

The same sharp lower and upper bound estimates from (3–8) apply to $M_n^{(1)}$ with $d = 1$ in (2–6). An illustration is given in Figure 4, where $M_n^{(1)}$ is indicated by ⋄.

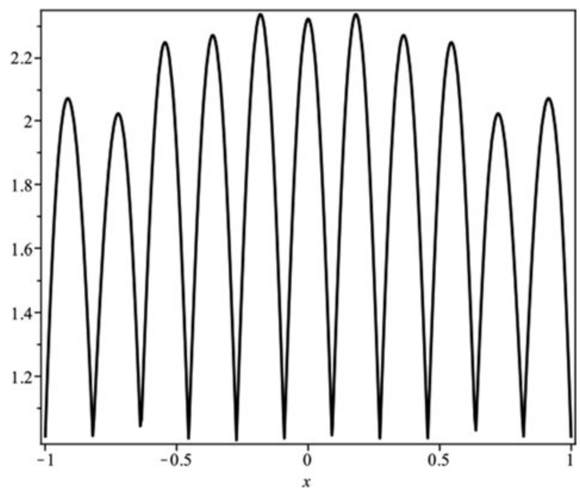

Figure 3. $M_{11}(x)$.

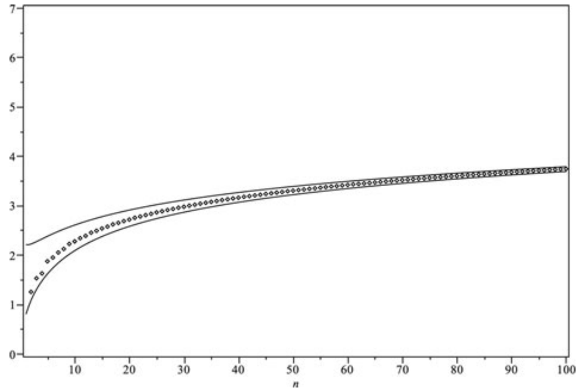

Figure 4. Sharpened bounds for $M_n^{(1)}$ with $M_n^{(1)}$ denoted by $\diamond$.

When $d > 1$, then an improved (but not yet fine) upper bound is given by:

$$M_n^{(d)} \le 2^{d-1} \frac{2}{\pi - \frac{4}{n+2}} \left(\ln(n+1) + \ln 2 + \gamma + \frac{1}{24n} \right), \quad d > 1. \tag{4–10}$$

The results mentioned in this section for $d \ge 1$ can be proved in a similar way as we prove (3–8) in Section 5.

5. Proof of sharp growth estimates for $M_n^{(0)}$

Let the interpolation points x_j be equidistant, $x_j = -1 + 2j/n$, $j = 0, \ldots, n$ and let the poles $\xi_1, \ldots, \xi_n$ lie outside $[-1, 1]$. The Lebesgue function $M_n(x)$ given by (3–7) takes the (minimum) value 1 at the interpolation points x_j, $j = 0, \ldots, n$ and has n local maxima in between each pair of consecutive interpolation points. It is clear that the Lebesgue function $M_n(x)$ is symmetric with respect to the origin: $M_n(-x) = M_n(x)$. The graph of $M_n(x)$ essentially takes four different shapes, depending on the value of n, and the proof of the growth rate distinguishes these four different cases. In Figure 5, we show $M_n(x)$ for $n = 4k, 4k+1, 4k+2, 4k+3$ with $k = 1$. Because of the symmetry of $M_n(x)$, whenever a maximum is attained at x^*, so it is at $-x^*$. We focus on the positive argument of the maximum.

As we prove further down and as is illustrated in Figure 5, the position of $x^* = \arg\max_{0 \le x \le 1} M_n(x)$ changes with n mod 4 and is (except for $n = 4k+3$) located near (not precisely at!) a midpoint of two interpolation points (note that $2/n$ is the distance between two consecutive interpolation points).

For small $n \ge 3$ (for $n = 1$, $x^* = 0$ and for $n = 2$, $x^* \approx {}^1/_2$), the statement can be made rather precise:

$$\begin{aligned} n \bmod 4 = 0 &: x^* \approx \tfrac{1}{n}, \quad n \le 24, \\ n \bmod 4 = 1 &: x^* \approx \tfrac{2}{n}, \\ n \bmod 4 = 2 &: x^* \approx \tfrac{3}{n}, \quad n \le 718, \\ n \bmod 4 = 3 &: x^* = 0. \end{aligned} \tag{5–11}$$

And more generally, for $k = \lfloor n/4 \rfloor > 1$:

$$\begin{aligned} n \bmod 4 = 0 &: \; x^* \in]x_{2k}, x_{2k+\lceil \log_{10} k \rceil + 2}[=]\,0, \frac{2(\lceil \log_{10} k \rceil + 2)}{n}[, \\ n \bmod 4 = 1 &: \; x^* \approx \frac{2}{n}, \\ n \bmod 4 = 2 &: \; x^* \in]x_{2k+1}, x_{2k+\lceil \log_{10} k \rceil + 1}[=]\,0, \frac{2\lceil \log_{10} k \rceil}{n}[, \\ n \bmod 4 = 3 &: \; x^* = 0. \end{aligned} \tag{5–12}$$

To determine the location of x^* where a maximum is attained, we further make use of some simple rules.

Rules

$$\frac{N}{D} < \frac{A}{B} \Rightarrow \frac{N}{D} < \frac{N+A}{D+B}, \quad N, D, A, B > 0, \tag{5–13a}$$

$$D \le N \Rightarrow \frac{N+C}{D+C} \le \frac{N}{D}, \quad N, D, C > 0, \tag{5–13b}$$

$$D \le N, B < D \Rightarrow \frac{N+A}{D+A} \le \frac{N+B}{D-B}, \quad N, D, A, B > 0, \tag{5–13c}$$

$$D \le N, B < A \Rightarrow \frac{N+A}{D+A} \le \frac{N+B}{D+B}, \quad N, D, A, B > 0. \tag{5–13d}$$

To prove (3–8), once the location of x^* is known, we also need a lemma [Günttner 88] and bounds on the partial sums of the Leibniz series.

Lemma

$$\sum_{k=0}^{n} \frac{1}{2k+1} < \frac{1}{2} \ln(n+1) + \ln 2 + \frac{\gamma}{2} + \frac{1}{48(n+1)^2}. \tag{5–14}$$

Series

$$\frac{\pi}{4} - \frac{1}{2n+3} < \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} < \frac{\pi}{4} + \frac{1}{2n+3}.$$

Now let's start the proof of (3–8). In order to simplify the computations, we make a change of variable, from $x \in [-1, 1]$ to $y \in [0, 1]$ by $y := (x+1)/2$. This way we are dealing only with positive values in the subsequent sums. The interpolation points x_i are then mapped to equidistant points y_i at a distance $1/n$ of each other. Because there is no risk of ambiguity, when consistently using y-values with evaluations expressed in the transformed variable and x-values with evaluations expressed in the original variable, the same notation M_n is used for the Lebesgue function in the variable x and the function after the transformation of x to y.

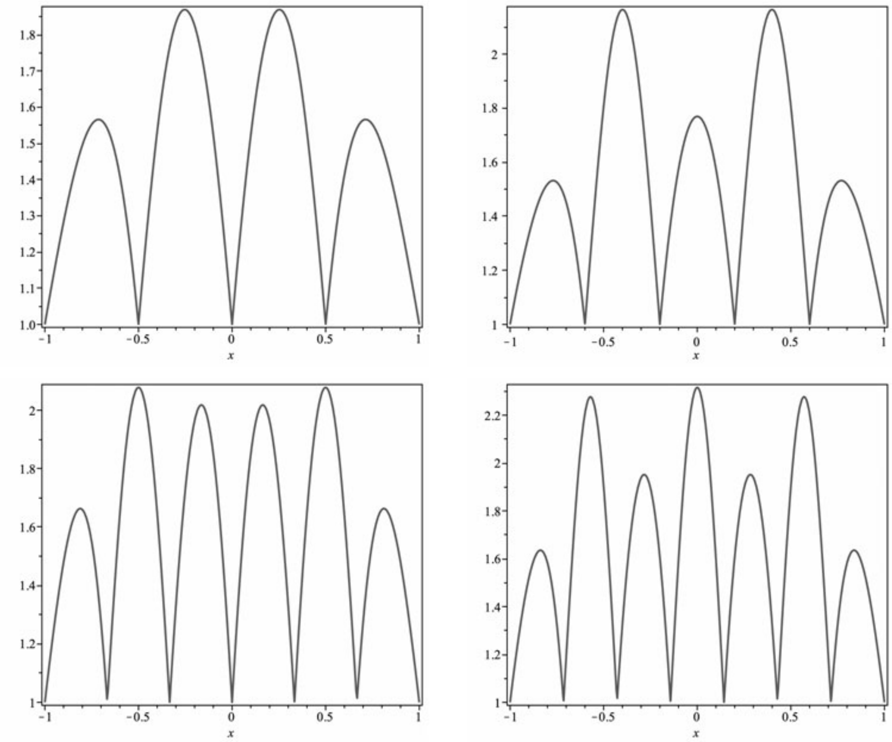

Figure 5. Graphs of $M_4(x)$, $M_5(x)$, $M_6(x)$, $M_7(x)$ from left to right and top to bottom.

We now investigate the value of the Lebesgue function $M_n(y)$ at the midpoints $\hat{y}_i = (y_{i-1} + y_i)/2$, $i = 1, \ldots, n$ which are very close to the arguments of the local maxima of the Lebesgue function (the values displayed in Table 1 are for instance $M_n(\hat{y}_{2k+\ell})$ for $\ell = 10i + 5$, $i = 0, \ldots, 9$). It is easy to verify that

$$M_n(\hat{y}_i) = \frac{N_n(\hat{y}_i)}{D_n(\hat{y}_i)}, \qquad i = 1, \ldots, n,$$

where

$$N_n(\hat{y}_i) = \sum_{j=0}^{i-1} \frac{1}{2j+1} + \sum_{j=0}^{n-i} \frac{1}{2j+1}, \qquad (5\text{–}15a)$$

$$D_n(\hat{y}_i) = \sum_{j=0}^{i-1} \frac{(-1)^j}{2j+1} + \sum_{j=0}^{n-i} \frac{(-1)^j}{2j+1}. \qquad (5\text{–}15b)$$

We write $n = 4k + (n \bmod 4)$. For $n \bmod 4 = 1$ and $n \bmod 4 = 2$, we have

$$N_n(\hat{y}_1) < N_n(\hat{y}_2) < \ldots < N_n(\hat{y}_{2k+1})$$

and

$$D_n(\hat{y}_{2i+1}) > D_n(\hat{y}_{2i+2}), \qquad i = 0, \ldots, k-1,$$
$$D_n(\hat{y}_{2i+1}) > D_n(\hat{y}_{2i+3}), \qquad i = 0, \ldots, k-1.$$

For $n \bmod 4 = 0$, the statements about N_n and D_n at $\hat{y}_{2k+1}$ are dropped, and for $n \bmod 4 = 3$, similar statements at $\hat{y}_{2k+2}$ are added. From the above, we can deduce that for $n \bmod 4 = 1$ and $n \bmod 4 = 2$,

$$M_n(\hat{y}_{2i+1}) < M_n(\hat{y}_{2i+2}), \qquad i = 0, \ldots, k-1,$$
$$M_n(\hat{y}_{2i+1}) < M_n(\hat{y}_{2i+3}), \qquad i = 0, \ldots, k-1,$$

with a similar adjustment for $n \bmod 4 = 0$ and $n \bmod 4 = 3$ as before. Now we treat the cases n odd and n even separately.

When n is odd, we find that for $n \bmod 4 = 1$,

$$M_n(\hat{y}_{2i}) \le M_n(\hat{y}_{2k}), \qquad i = 1, \ldots, k-1 \quad (5\text{–}16)$$

by combining (5–13a) for $N = N_n(\hat{y}_{2i}), D = D_n(\hat{y}_{2i})$ with

$$\frac{N_n(\hat{y}_{2k}) - N_n(\hat{y}_{2i})}{D_n(\hat{y}_{2k}) - D_n(\hat{y}_{2i})} \le \frac{N_n(\hat{y}_{2k}) - N_n(\hat{y}_{2i+2})}{D_n(\hat{y}_{2k}) - D_n(\hat{y}_{2i+2})}, \quad i = 1, \ldots, k-2.$$

Analogously, for $n \bmod 4 = 3$, the statement (5–16) holds for $i = 1, \ldots, k$ with $\hat{y}_{2k}$ in the right-hand side replaced by $\hat{y}_{2k+2}$.

The situation is more complicated when n is even though. But fortunately the following inequalities for the near-maxima at $\hat{y}_{2k}$, and the other local maxima near $\hat{y}_{2i}$, help us out. Using (5–13a), we find

$$M_{4k+1}(\hat{y}_{2k}) \le M_{4k+3}(\hat{y}_{2k+2}).$$

From (5–13b), we obtain

$$M_{4k+2}(\hat{y}_{2i}) \leq M_{4k+1}(\hat{y}_{2i}), \qquad i = 1, \ldots, k.$$

And finally (5–13c) gives

$$M_{4k}(\hat{y}_{2i}) \leq M_{4k+2}(\hat{y}_{2i}), \qquad i = 1, \ldots, k.$$

Remains to investigate $M_n(\hat{y}_{2k+1})$ in case $n \bmod 4 = 1$ or $n \bmod 4 = 2$. Using (5–13c), we obtain

$$M_{4k+1}(\hat{y}_{2k+1}) \leq M_{4k+1}(\hat{y}_{2k})$$

and at last from (5–13d)

$$M_{4k+2}(\hat{y}_{2k+1}) \leq M_{4k+2}(\hat{y}_{2k}).$$

Since we know that when $n \bmod 4 = 3$ a maximum occurs exactly at $M_n(\hat{y}_{2k+2})$, we can use this value to compute an upper bound estimate for the Lebesgue constant $M_n^{(0)}$. Likewise, a lower bound for $M_n^{(0)}$ can be obtained because $M_{4k}(\hat{y}_{2k}) \leq M_{4k}^{(0)} \leq M_n^{(0)}$ for general n.

To conclude:

$$\max_n \max_{x\in[-1,1]} M_n(x) \approx M_{4k+3}(0)$$

and

$$\min_n \max_{x\in[-1,1]} M_n(x) \geq M_{4k}(1/n).$$

In other words, a sharp upper bound for $M_{4k+3}(0)$ is an accurate estimate for $M_n(x^*)$, $n = 4k + i$, $0 \leq i \leq 3$, and a lower bound for $M_{4k}(1/n)$ is a lower bound for $M_n(x^*)$, $n = 4k + i$, $0 \leq i \leq 3$.

To prove the actual bounds, we make use of the transformed variable y again. For the upper bound we have:

$$M_{4k+3}(^1/_2) = \frac{N_{4k+3}(^1/_2)}{D_{4k+3}(^1/_2)},$$

$$N_{4k+3}(^1/_2) = 2\sum_{j=0}^{2k+1} \frac{1}{2j+1} \leq \ln(8k+8) + \gamma + \frac{1}{24(2k+2)^2} \leq \ln(2n+2) + \gamma + \frac{1}{24n},$$

$$D_{4k+3}(^1/_2) = 2\left|\sum_{j=0}^{2k+1} \frac{(-1)^{j+2k+1}}{2j+1}\right| \geq \frac{\pi}{2} - \frac{2}{n+2}.$$

From these inequalities, it follows that (stated in the variable x now)

$$\max_n \max_{x\in[-1,1]} M_n(x) \simeq \frac{2}{\pi - \frac{4}{n+2}} \times \left(\ln(n+1) + \ln 2 + \gamma + \frac{1}{24n}\right).$$

For the lower bound, expressed in the transformed variable y, we use the fact that the numerator of $M_{4k}(^{n+1}/_{2n})$ can be expressed using the digamma function $\Psi(y)$ where for $y > 0$ it holds that $\ln(y) - 1/y \leq \Psi(y)$:

$$M_{4k}(^{n+1}/_{2n}) = \frac{N_{4k}(^{n+1}/_{2n})}{D_{4k}(^{n+1}/_{2n})},$$

$$N_{4k}(^{n+1}/_{2n}) = \sum_{j=0}^{2k+2} \frac{1}{2j+1} + \sum_{j=0}^{2k+1} \frac{1}{2j+1} \geq \ln(8k+2) + \gamma = \ln(2n+2) + \gamma,$$

$$D_{4k}(^{n+1}/_{2n}) \leq 2\left|\sum_{j=0}^{2k+2} \frac{(-1)^j}{2j+1}\right| \leq \frac{\pi}{2} + \frac{2}{n+3}$$

from which (3–8) follows.

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