Tutorial

What do Sparse Interpolation, Padé Approximation, Gaussian Quadrature and Tensor Decomposition Have in Common?

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We present the problem statement of sparse interpolation of data f_i collected uniformly at points $x_i = i\Delta$, as

$$\sum_{j=1}^{n} \alpha_j \exp(\phi_j x_i) = f_i, \quad \alpha_j, \phi_j \in \mathbb{C}, \quad |\Im(\phi_j)\Delta| < \pi, \qquad (1)$$

and its basic mathematical and computational methods to solve it.

The original solution is presented already in 1795 by de Prony [6]. Much later it is expressed in terms of the generalized eigenvalue problem [8]

$$H_{n,n}^{(1)} v_j = \exp(\phi_j \Delta) H_{n,n}^{(0)} v_j, \quad j = 1, \dots, n,$$

$$H_{n_1,n_2}^{(r)} = \begin{pmatrix} f_r & \cdots & f_{r+n_2-1} \\ \vdots & \iddots & \vdots \\ f_{r+n_1-1} & \cdots & f_{r+n_1+n_2-2} \end{pmatrix}$$

for the ϕ_j and the subsequent solution of the structured linear system (1) for the α_j.

When considering the limited number of regularly collected samples f_i as Taylor series coefficients,

$$\sum_{i=0}^{\infty} f_i z^i = \sum_{j=1}^{n} \frac{\alpha_j}{1 - \exp(\phi_j \Delta) z},$$

the problem statement easily connects to Padé approximation [4, 9].

The Padé approximant denominators are in turn closely related to the formally orthogonal Hadamard polynomials and Gaussian quadrature [1, 7] by

$$\prod_{j=1}^{n} (z - \exp(\phi_j \Delta)) = \begin{vmatrix} f_0 & \cdots & f_n \\ \vdots & & \vdots \\ f_{n-1} & \cdots & f_{2n-1} \\ 1 & \cdots & z^n \end{vmatrix} / |H_{n,n}^{(0)}|.$$

Results on their zeroes and certain convergence properties shed new light [2] on some computational problems in sparse interpolation.

The problem statement can also be viewed as an m-order tensor decomposition problem where $3 \le m \le 2n-1$ [3, 5], namely

$$\sum_{j=1}^{n} \alpha_j \begin{pmatrix} 1 \\ \exp(\phi_j \Delta) \\ \vdots \\ \exp(\phi_j \Delta)^{n_1-1} \end{pmatrix} \circ \cdots \circ \begin{pmatrix} 1 \\ \exp(\phi_j \Delta) \\ \vdots \\ \exp(\phi_j \Delta)^{n_m-1} \end{pmatrix}$$
$$= (f_{k_1+\ldots+k_m-m})_{1 \le k_\ell \le n_\ell}, \quad 1 \le \ell \le m, 2 \le n_\ell \le n,$$

with the connection

$$H_{n_1,n_2}^{(k_3+\ldots+k_m-m+2)} = (f_{i+j+k_3+\ldots+k_m-m})_{1 \le i \le n_1, 1 \le j \le n_2}.$$

Here $\circ$ denotes the outer product and the decomposition problem is solved using techniques from multilinear algebra. Through the latter reformulation the toolkit of algorithms for sparse interpolation is further enlarged.

REFERENCES

[1] C. Brezinski. 1980. *Padé type approximation and general orthogonal polynomials.* ISNM 50, Birkhäuser Verlag, Basel.
[2] M. Briani, A. Cuyt, and W.-s. Lee. 2017. *VEXPA: Validated EXPonential Analysis through regular subsampling.* ArXiv e-print 1709.04281 [math.NA]. Universiteit Antwerpen.
[3] A. Cuyt, F. Knaepkens, and W.-s. Lee. 2018. From exponential analysis to Padé approximation and Tensor decomposition, in one and more dimensions. In *LNCS11077*, V.P. Gerdt et al. (Eds.). 116–130. Proceedings CASC 2018, Lille (France).
[4] A. Cuyt and W.-s. Lee. 2016. Sparse interpolation and Rational approximation *(Contemporary Mathematics)*, D. Hardin, D. Lubinsky, and B. Simanek (Eds.), Vol. 661. American Mathematical Society, Providence, RI, 229–242. https://doi.org/10.1090/conm/661/13284
[5] A. Cuyt, W.-s. Lee, and X. Yang. 2016. On tensor decomposition, sparse interpolation and Padé approximation. *Jaén J. Approx.* 8, 1 (2016), 33–58.
[6] R. de Prony. 1795. Essai expérimental et analytique sur les lois de la dilatabilité des fluides élastiques et sur celles de la force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures. *J. Ec. Poly.* 1 (1795), 24–76.
[7] P. Henrici. 1974. *Applied and computational complex analysis I.* John Wiley & Sons, New York.
[8] Y. Hua and T. K. Sarkar. 1990. Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise. *IEEE Trans. Acoust., Speech, Signal Process.* 38 (1990), 814–824. https://doi.org/10.1109/29.56027
[9] L. Weiss and R. N. McDonough. 1963. Prony's method, Z-transforms, and Padé approximation. *SIAM Rev.* 5 (1963), 145–149.

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