Symbolic-Numeric QD-algorithms with applications in Function theory and Linear algebra

Annie Cuyt

1. Introduction

The univariate qd-algorithm is very useful for the determination of poles of meromorphic functions and eigenvalues of certain tridiagonal matrices. Both applications are linked to the theory of orthogonal polynomials, in particular the formally orthogonal Hadamard polynomials.

When looking for the pole curves of multivariate functions, or for the eigenvalue curves of some parameterized tridiagonal matrices, the qd-algorithm has to be generalized in order to deal with multivariate data. Indeed, the univariate algorithm only involves number manipulations and these multivariate problems require the manipulation of expressions.

For the computation of the poles of a multivariate meromorphic function a symbolic generalization of the qd-algorithm implemented in floating-point polynomial arithmetic seems to be possible. For the parameterized eigenvalue problem a homogeneous version implemented in exact polynomial arithmetic can be used.

In Section 2 we summarize the univariate prerequisites for the material. The multivariate pole detection problem and the floating-point polynomial qd-algorithm are treated in Section 3 while the parameterized eigenvalue problem is solved in Section 4 using the homogeneous version of the qd-algorithm.

2. The univariate floating-point qd-algorithm.

Let us define a linear functional c from the vector space $\mathbb{C}[z]$ to $\mathbb{C}$ by

$$c(z^j) = c_j \qquad j = 0, 1, \ldots$$

Hence c is completely determined by the sequence $\{c_j\}_{j \in \mathbb{N}}$ which is called the sequence of moments of the functional c (in the sequel $c_j = 0$ when $j < 0$). With the c_j we also associate the Hankel determinants

$$H_m^{(n)} = \begin{vmatrix} c_n & \cdots & c_{n+m-1} \\ \vdots & \cdot^{\cdot^{\cdot}} & c_{n+m} \\ & & \vdots \\ c_{n+m-1} & \cdots & c_{n+2m-2} \end{vmatrix} \qquad H_0^{(n)} = 1$$

The functional c is called s-normal if

$$H_m^{(n)} \neq 0 \qquad n \geq 0 \qquad m = 0, \ldots, s$$

From now on we shall assume that this is the case.

G. Alefeld et al. (eds.), *Symbolic Algebraic Methods and Verification Methods*

With the sequence $\{c_j\}_{j\in\mathbb{N}}$ we can also set up the qd-scheme where subscripts denote columns and superscripts downward sloping diagonals [Henrici 1974]. Its start columns are given by

$$e_0^{(n)} = 0 \qquad n = 1, 2, \ldots$$
$$q_1^{(n)} = \frac{c_{n+1}}{c_n} \qquad n = 0, 1, \ldots$$

and the rhombus rules for continuation of the scheme by

$$e_m^{(n)} = q_m^{(n+1)} - q_m^{(n)} + e_{m-1}^{(n+1)} \qquad m = 1, 2\ldots \quad n = 0, 1\ldots$$
$$q_{m+1}^{(n)} = \frac{e_m^{(n+1)}}{e_m^{(n)}} q_m^{(n+1)} \qquad m = 1, 2\ldots \quad n = 0, 1, \ldots$$

In what follows we need the next lemma.

LEMMA 1:
If the functional c is s-normal, then the values $q_m^{(n)}$ and $e_m^{(n)}$ exist for $m = 1, \ldots, s$ and $n \geq 0$ and they are given by

$$q_m^{(n)} = \frac{H_m^{(n+1)} H_{m-1}^{(n)}}{H_m^{(n)} H_{m-1}^{(n+1)}}$$

$$e_m^{(n)} = \frac{H_{m+1}^{(n)} H_{m-1}^{(n+1)}}{H_m^{(n)} H_m^{(n+1)}}$$

The following properties of the qd-algorithm can be found in [Henrici 1974].

2.1. Meromorphic functions and root finding.

THEOREM 1 [Henrici 1974, pp. 612–613]:
Let

$$\sum_{j=0}^{\infty} c_j z^j$$

be the Taylor series at $z = 0$ of a function $f(z)$ meromorphic in the disk $B(0, R) = \{z : |z| < R\}$ and let the poles z_i of f in $B(0, R)$ be numbered such that

$$z_0 = 0 < |z_1| \leq |z_2| \leq \ldots < R$$

each pole occuring as many times in the sequence $\{z_i\}_{i\in\mathbb{N}}$ as indicated by its order. If f is s-normal for some integer $s > 0$, then the qd-scheme associated with f has the following properties (put $z_{s+1} = \infty$ if f has only s poles):
(a) for each m with $0 < m \leq s$ and $|z_{m-1}| < |z_m| < |z_{m+1}|$,

$$\lim_{n\to\infty} q_m^{(n)} = z_m^{-1}$$

(b) *for each m with $0 < m \leq s$ and $|z_m| < |z_{m+1}|$,*

$$\lim_{n\to\infty} e_m^{(n)} = 0$$

Any index m such that the strict inequality

$$|z_m| < |z_{m+1}|$$

holds, is called a critical index. It is clear that the critical indices of a function do not depend on the order in which the poles of equal modulus are numbered. The theorem above states that if m is a critical index and f is m-normal, then

$$\lim_{n\to\infty} e_m^{(n)} = 0$$

Thus the qd-table of a meromorphic function is divided into subtables by those e-columns tending to zero. This property motivated Rutishauser [Henrici 1974, p. 614] to apply the rhombus rules satisfied by the q- and e-values, namely

$$\begin{aligned} q_m^{(n+1)} e_m^{(n+1)} &= e_m^{(n)} q_{m+1}^{(n)} \\ e_{m-1}^{(n+1)} + q_m^{(n+1)} &= q_m^{(n)} + e_m^{(n)} \end{aligned}$$

in their progressive form [Henrici 1974]: when computing the q-values from the top down rather than from left to right, one avoids divisions by possibly small e-values that can inflate rounding errors. Other reformulations can be found in [Von Matt 1997] and [Fernando et al. 1994]. Any q-column corresponding to a simple pole of isolated modulus is flanked by such e-columns and converges to the reciprocal of the corresponding pole. If a subtable contains $j > 1$ columns of q-values, the presence of j poles of equal modulus is indicated. In [Henrici 1974] it is also explained how to determine these poles if $j > 1$.

THEOREM 2 [[Henrici 1974, p. 642]:
Let m and $m + j$ with $j > 1$ be two consecutive critical indices and let f be $(m + j)$-normal. Let the polynomials $\rho_k^{(n)}$ be defined by

$$\begin{aligned} \rho_0^{(n)}(z) &= 1 \\ \rho_{k+1}^{(n)}(z) &= z\rho_k^{(n+1)}(z) - q_{m+k+1}^{(n)} \rho_k^{(n)}(z) \qquad n \geq 0 \qquad k = 0, 1, \ldots, j-1 \end{aligned}$$

Then there exists a subsequence $\{n(\ell)\}_{\ell \in \mathbb{N}}$ such that

$$\lim_{\ell\to\infty} \rho_j^{(n(\ell))}(z) = (z - z_{m+1}^{-1}) \ldots (z - z_{m+j}^{-1})$$

The polynomials $\rho_k^{(n)}$ are closely related to the formally orthogonal Hadamard polynomials which will be discussed in the next section. From the above theorems the qd-scheme seems to be an ingenious tool for determining, under certain conditions, the poles of a meromorphic function f .

2.2. Hadamard polynomials and eigenvalue problems.

With the sequence $\{c_j\}_{j\in\mathbb{N}}$ we can also associate the Hadamard polynomials

$$p_m^{(n)}(z) = \frac{H_m^{(n)}(z)}{H_m^{(n)}} \qquad m \geq 0, n \geq 0$$

where

$$H_m^{(n)}(z) = \begin{vmatrix} c_n & \cdots & c_{n+m-1} & c_{n+m} \\ \vdots & \iddots & & \\ & & \vdots & \vdots \\ c_{n+m-1} & \cdots & & c_{n+2m-1} \\ 1 & \cdots & z^{m-1} & z^m \end{vmatrix} \qquad H_0^{(n)}(z) = 1$$

These monic polynomials are formally orthogonal with respect to the linear functional c because they satisfy [Brezinski 1980, pp. 40–41]

$$c\left(z^i p_m^{(n)}(z)\right) = 0 \qquad i = 0, \ldots, m-1$$

In [Henrici 1974, pp. 634–636] it was shown that

$$p_m^{(n)}(z) = \det(zI - A_m^{(n)})$$

where $A_m^{(n)}$ denotes the matrix

$$A_m^{(n)} = \begin{pmatrix} q_1^{(n)} + e_0^{(n)} & q_1^{(n)} e_1^{(n)} & & & 0 \\ 1 & q_2^{(n)} + e_1^{(n)} & q_2^{(n)} e_2^{(n)} & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & q_{m-1}^{(n)} + e_{m-2}^{(n)} & q_{m-1}^{(n)} e_{m-1}^{(n)} \\ 0 & & & 1 & q_m^{(n)} + e_{m-1}^{(n)} \end{pmatrix}$$

Hence the zeros of the Hadamard polynomials are the eigenvalues of the matrix $A_m^{(n)}$, or equivalently of the matrix $B_m^{(n)}$ where

$$B_m^{(n)} = \begin{pmatrix} q_1^{(n)} + e_0^{(n)} & -q_1^{(n)} & & & 0 \\ -e_1^{(n)} & q_2^{(n)} + e_1^{(n)} & -q_2^{(n)} & & \\ & \ddots & \ddots & \ddots & \\ & & -e_{m-2}^{(n)} & q_{m-1}^{(n)} + e_{m-2}^{(n)} & -q_{m-1}^{(n)} \\ 0 & & & -e_{m-1}^{(n)} & q_m^{(n)} + e_{m-1}^{(n)} \end{pmatrix}$$

The next theorem tells us that the qd-algorithm can be an ingenious way to compute the eigenvalues of such tridiagonal matrices.

THEOREM 3 [Henrici 1974, pp. 634–636]:
Let the functional c be m-normal and let the eigenvalues z_i of $B_m^{(n)}$ be numbered such that

$$|z_1| \geq |z_2| \geq \ldots \geq |z_m| \geq 0 = |z_{m+1}|$$

each eigenvalue occuring as many times in this sequence as indicated by its multiplicity. Then the qd-scheme associated with the sequence $\{c_j\}_{j \in \mathbb{N}}$ has the following properties:

(a) for each k with $0 < k \leq m$ and $|z_k| > |z_{k+1}|$, it holds that

$$\lim_{n \to \infty} e_k^{(n)} = 0$$

(b) for each k with $0 < k \leq m$ and $|z_{k-1}| > |z_k| > |z_{k+1}|$, it holds that

$$\lim_{n \to \infty} q_k^{(n)} = z_k$$

(c) for each k and $j > 1$ such that $0 < k < k+j \leq m$ and $|z_{k-1}| > |z_k| = \ldots = |z_{k+j-1}| > |z_{k+j}|$, it holds that for the polynomials $\pi_i^{(n)}$ defined by

$$\pi_0^{(n)}(z) = 1$$
$$\pi_{i+1}^{(n)}(z) = z\pi_i^{(n+1)}(z) - q_{k+i+1}^{(n)}\pi_i^{(n)}(z) \qquad n \geq 0 \qquad i = 0, 1, \ldots, j-1$$

there exists a subsequence $\{\pi_j^{(n_\ell)}\}_{\ell \in \mathbb{N}}$ such that

$$\lim_{\ell \to \infty} \pi_j^{(n_\ell)}(z) = (z - z_{k+1}) \ldots (z - z_{k+j})$$

3. Multivariate pole detection.

Let a multidimensional table $\{c_{i_1,\ldots,i_k}\}_{i_j \in \mathbb{N}}$ be given and let us introduce the notations

$$\vec{i} = (i_1, \ldots, i_k) \qquad |\vec{i}| = i_1 + \ldots + i_k$$
$$\vec{x} = (x_1, \ldots x_k) \qquad \vec{x}^{\,\vec{i}} = x_1^{i_1} \ldots x_k^{i_k}$$
$$c_{\vec{i}} = c_{i_1 \ldots i_k}$$

For a formal Taylor series expansion

$$f(\vec{x}) = \sum_{\vec{i} \in \mathbb{N}^k} c_{\vec{i}}\, \vec{x}^{\,\vec{i}}$$

we must specify in which order we shall deal with the index tuples $\vec{i}$ in $\mathbb{N}^k$. Let us enumerate them as

$$f(\vec{x}) = \sum_{\ell=0}^{\infty} c_{\vec{i}(\ell)}\, \vec{x}^{\,\vec{i}(\ell)}$$

with the only requirement for the enumeration that it satisfies the inclusion property. By this we mean that when a point is added, every point in the polyrectangular subset emanating from the origin with the added point as its furthermost corner, must already be enumerated previously. As a consequence, always $\vec{i}(0) = (0, \ldots, 0)$. Let us also introduce the monomials

$$C_{m,n}(\vec{x}) = c_{\vec{i}(n)-\vec{i}(m)} \, \vec{x}^{\,\vec{i}(n)-\vec{i}(m)} \qquad i_j(n) \geq i_j(m) \quad j = 1, \ldots, k$$

which are comparable to the univariate terms $c_{n-m} z^{n-m}$. In what follows we consider functions $f(\vec{x})$ which are meromorphic in a polydisc $B(0, R_{\vec{i}}) = \{\vec{x} : |x_{i_j}| < R_{i_j}\}$, meaning that there exists a polynomial

$$D(\vec{x}) = \sum_{\vec{i} \in M \subseteq \mathbb{N}^2} r_{\vec{i}} \, \vec{x}^{\,\vec{i}} = \sum_{\ell=0}^{m} r_{\vec{i}(\ell)} \, \vec{x}^{\,\vec{i}(\ell)} \qquad r_{\vec{i}(0)} r_{\vec{i}(m)} \neq 0 \tag{1}$$

indexed by the finite set M such that $(fD)(\vec{x})$ is analytic in the polydisc above. The index tuples belonging to M are determined by the powers of the variables x_{i_j} occurring in the polynomial $D(\vec{x})$. As a consequence of the inclusion property satisfied by the enumeration, one must allow for some of the coefficients $r_{\vec{i}}$ to be zero.

3.1. The general order qd-algorithm.

Let us introduce help entries $g_{0,m}^{(n)}$ by:

$$g_{0,m}^{(n)} = \sum_{k=0}^{n} C_{m,k}(\vec{x}) - \sum_{k=0}^{n} C_{m-1,k}(\vec{x})$$

$$g_{m,r}^{(n)} = \frac{g_{m-1,r}^{(n)} g_{m-1,m}^{(n+1)} - g_{m-1,r}^{(n+1)} g_{m-1,m}^{(n)}}{g_{m-1,m}^{(n+1)} - g_{m-1,m}^{(n)}} \qquad r = m+1, m+2, \ldots$$

keeping in mind that $c_{\vec{i}} = 0$ if some $i_j < 0$. The general order multivariate qd-algorithm is then defined by [Cuyt 1988]:

$$Q_1^{(n)}(\vec{x}) = \frac{C_{0,n+1}(\vec{x})}{C_{0,n}(\vec{x})} \, \frac{g_{0,1}^{(n+1)}}{g_{0,1}^{(n+1)} - g_{0,1}^{(n+2)}}$$

$$Q_m^{(n+1)}(\vec{x}) = \frac{E_{m-1}^{(n+2)}(\vec{x}) Q_{m-1}^{(n+2)}(\vec{x})}{E_{m-1}^{(n+1)}(\vec{x})} \, \frac{g_{m-2,m-1}^{(n+m-1)} - g_{m-2,m-1}^{(n+m)}}{g_{m-2,m-1}^{(n+m-1)}} \, \frac{g_{m-1,m}^{(n+m)}}{g_{m-1,m}^{(n+m)} - g_{m-1,m}^{(n+m+1)}} \tag{2}$$

$$m \geq 2$$

$$E_m^{(n+1)}(\vec{x}) + 1 = \frac{g_{m-1,m}^{(n+m)} - g_{m-1,m}^{(n+m+1)}}{g_{m-1,m}^{(n+m)}} \left(Q_m^{(n+2)}(\vec{x}) + 1 \right) \qquad m \geq 1 \tag{3}$$

If we arrange the values $Q_m^{(n)}(\vec{x})$ and $E_m^{(n)}(\vec{x})$ as in the univariate case, where subscripts indicate columns and superscripts indicate downward sloping diagonals, then (2) links the elements in the rhombus

$$\begin{array}{ccc} & E_{m-1}^{(n+1)}(\vec{x}) & \\ Q_{m-1}^{(n+2)}(\vec{x}) & & Q_m^{(n+1)}(\vec{x}) \\ & E_{m-1}^{(n+2)}(\vec{x}) & \end{array}$$

and (3) links two elements on an upward sloping diagonal

$$\begin{array}{cc} & E_m^{(n+1)}(\vec{x}) \\ Q_m^{(n+2)}(\vec{x}) & \end{array}$$

In analogy with the univariate case it is also possible to give explicit determinant formulas for the multivariate Q- and E-functions. Let us define the determinants

$$H_{0,m}^{(n)}(\vec{x}) = \begin{vmatrix} C_{0,n}(\vec{x}) & \dots & C_{m-1,n}(\vec{x}) \\ \vdots & & \vdots \\ C_{0,n+m-1}(\vec{x}) & \dots & C_{m-1,n+m-1}(\vec{x}) \end{vmatrix} \qquad H_{0,0}^{(n)} = 0$$

$$H_{1,m}^{(n)}(\vec{x}) = \begin{vmatrix} 1 & \dots & 1 \\ C_{0,n}(\vec{x}) & \dots & C_{m,n}(\vec{x}) \\ \vdots & & \vdots \\ C_{0,n+m-1}(\vec{x}) & \dots & C_{m,n+m-1}(\vec{x}) \end{vmatrix} \qquad H_{1,-1}^{(n)} = 0$$

$$H_{2,m}^{(n)}(\vec{x}) = \begin{vmatrix} \sum_{k=0}^{n-1} C_{0,k}(\vec{x}) & \dots & \sum_{k=0}^{n-1} C_{m,k}(\vec{x}) \\ C_{0,n}(\vec{x}) & \dots & C_{m,n}(\vec{x}) \\ \vdots & & \vdots \\ C_{0,n+m-1}(\vec{x}) & \dots & C_{m,n+m-1}(\vec{x}) \end{vmatrix} \qquad H_{2,-1}^{(n)} = 0$$

$$H_{3,m}^{(n)}(\vec{x}) = \begin{vmatrix} 1 & \dots & 1 \\ \sum_{k=0}^{n-1} C_{0,k}(\vec{x}) & \dots & \sum_{k=0}^{n-1} C_{m,k}(\vec{x}) \\ C_{0,n}(\vec{x}) & \dots & C_{m,n}(\vec{x}) \\ \vdots & & \vdots \\ C_{0,n+m-2}(\vec{x}) & \dots & C_{m,n+m-2}(\vec{x}) \end{vmatrix} \qquad T_{3,-1}^{(n)} = 0$$

and the polynomial

$$\hat{H}_{1,m}^{(n)}(\vec{x}) = \frac{H_{1,m}^{(n)}(\vec{x})}{\vec{x}^{\sigma}} \qquad \sigma = \sum_{j=1}^{m} \left(\vec{\imath}(n+j) - \vec{\imath}(j)\right)$$

By factoring out $\vec{x}^{\sigma}$, we obtain a polynomial that satisfies

$$\hat{H}_{1,m}^{(n)}(\vec{0}) \neq 0$$

By means of recurrence relations for these determinants we can prove the following lemma [Cuyt 1988].

LEMMA 2:
For well-defined $Q_m^{(n)}(\vec{x})$ and $E_m^{(n)}(\vec{x})$ the following determinant formulas hold:

$$-Q_m^{(n)}(\vec{x}) = \frac{H_{0,m}^{(n+m)} H_{1,m-1}^{(n+m-1)} H_{3,m}^{(n+m)}}{H_{0,m}^{(n+m-1)} H_{1,m}^{(n+m)} H_{3,m-1}^{(n+m)}}(\vec{x})$$

$$-E_m^{(n)}(\vec{x}) = \frac{H_{0,m+1}^{(n+m)} H_{1,m-1}^{(n+m)} H_{3,m}^{(n+m+1)}}{H_{0,m}^{(n+m)} H_{1,m}^{(n+m+1)} H_{3,m}^{(n+m)}}(\vec{x})$$

3.2. Application to pole detection.

For the chosen enumeration that satisfies the inclusion property, we also define the k functions

$$\nu_j(n) = \max\{i_j(\ell) \mid 0 \leq \ell \leq n\} \qquad j = 1, \ldots, k$$

Let m zeros $\vec{x}(h)$ of $D(\vec{x})$, specified by (1), be given in $B(0, R_{\vec{i}})$:

$$D(\vec{x}(h)) = 0 \qquad h = 1, \ldots, m \tag{4}$$

The next theorems are formulated for the so-called "simple pole" case where no information on derivatives at pole points $\vec{x}(h)$ is used. It is of course true that the following results can also be written down for the so-called "multipole" case introduced in [Cuyt 1992].

THEOREM 4:
Let $f(\vec{x})$ be a function which is meromorphic in the polydisc $B(0, R_{\vec{i}}) = \{\vec{x} : |x_j| < R_{i_j}, j = 1, \ldots, k\}$. Let (1) be satisfied and let m zeros $\vec{x}(h)$ of $D(\vec{x})$ in $B(0, R_{\vec{i}})$ be given, satisfying

$$(fD)(\vec{x}(h)) \neq 0 \qquad h = 1, \ldots, m \tag{5a}$$

and

$$\begin{vmatrix} \vec{x}(1)^{\vec{i}(1)} & \ldots & \vec{x}(1)^{\vec{i}(m)} \\ \vdots & & \vdots \\ \vec{x}(m)^{\vec{i}(1)} & \ldots & \vec{x}(m)^{\vec{i}(m)} \end{vmatrix} \neq 0 \tag{5b}$$

Then if $\lim_{n\to\infty} \nu_j(n) = \infty$ for $j = 1, \ldots, k$, one finds

$$\lim_{n\to\infty} \hat{H}_{1,m}^{(n+m)}(\vec{x}) = D(\vec{x})$$

From Lemma 2 and Theorem 4 we see that if f is a meromorphic function, then in some column the expressions $Q_m^{(n+1)}(\vec{x})$ contain information on the poles of f, because in that case the factor $\hat{H}_{1,m}^{(n+m)}(\vec{x})$ in the denominator of $Q_m^{(n+1)}(\vec{x})$ converges to the poles of the meromorphic f. This particular factor $\hat{H}_{1,m}^{(n+m)}$ is easily isolated because it is the only one in the denominator of $Q_m^{(n+1)}(\vec{x})$ (except for a constant) that evaluates different from zero in the origin. Let us also define the functions

$$\hat{E}_m^{(n+1)}(\vec{x}) = \frac{H_{0,m+1}^{(n+m)} H_{3,m}^{(n+m+1)}}{H_{0,m}^{(n+m)} H_{3,m}^{(n+m)}}(\vec{x})$$

which contain only some of the factors in $E_m^{(n+1)}(\vec{x})$, namely those factors that do not contain direct pole curve information. Again these specific factors are easily isolated in $E_m^{(n+1)}(\vec{x})$ because they all evaluate to zero in the origin and the remaining factors don't.

THEOREM 5:
Let $f(\vec{x})$ be a function which is meromorphic in the polydisc $B(0, R_{\vec{i}})$, meaning that there exists a polynomial $D(\vec{x})$ such that $(fD)(\vec{x})$ is analytic in the polydisc above. Let the polynomial $D(\vec{x})$ be given by (1) and let the conditions (5) be satisfied. If the first m columns of the general order multivariate qd-scheme are defined, then with the enumeration of $\mathbb{N}^k$ satisfying the same conditions as in Theorem 4, one finds:

(a)

$$\lim_{n\to\infty} \hat{E}_m^{(n+1)}(\vec{x}) = 0$$

in measure in a neighbourhood of the origin

(b)

$$\lim_{n\to\infty} \hat{H}_{1,m}^{(n+m)}(\vec{x}) = D(\vec{x})$$

How the general order multivariate qd-scheme is used as a tool to detect successive factors of the polar singularities of a multivariate function is further detailed in Section 5. We shall discuss how to deal with the situation where several algebraic curves define polar singularities and we shall also indicate which columns have to be inspected. How the general order qd-scheme can be implemented in floating-point polynomial arithmetic instead of symbolically is indicated in [Cuyt et al. 1999].

4. Parameterized eigenvalue problems.

For the multidimensional table $\{c_{i_1,\ldots,i_k}\}_{i_j\in\mathbb{N}}$ we introduce the additional notation

$$\vec{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k \qquad ||\vec{\lambda}||_p = 1$$

where $||.||_p$ is one of the Minkowski-norms on $\mathbb{C}^k$. In the sequel we shall often switch from a cartesian coordinate system to a spherical one (and back) and hence we also introduce

$$\vec{x} = (\lambda_1 z, \ldots, \lambda_k z) \qquad x_1, \ldots, x_k, z \in \mathbb{C} \qquad ||\vec{\lambda}||_p = 1$$

$$c_j(\vec{\lambda}) = \sum_{|\vec{i}|=j} c_{\vec{i}}\, \vec{\lambda}^{\vec{i}}$$

$$C_j(\vec{x}) = \sum_{|\vec{i}|=j} c_{\vec{i}}\, \vec{x}^{\vec{i}} \tag{6}$$

$$= \left(\sum_{|\vec{i}|=j} c_{\vec{i}}\, \vec{\lambda}^{\vec{i}} \right) z^j$$

$$= c_j(\vec{\lambda}) z^j \tag{7}$$

The expressions (6) and (7) will be used interchangeably for the homogeneous polynomial $C_j(\vec{x})$. We denote by $\mathbb{C}[z]$ the linear space of polynomials in the variable z with complex coefficients, by $\mathbb{C}[\lambda_1, \ldots, \lambda_k]$ the linear space of multivariate polynomials in $\vec{\lambda}$ with complex coefficients, by $\mathbb{C}(\lambda_1, \ldots, \lambda_k)$ the commutative field of rational functions in $\vec{\lambda}$ with complex coefficients, by $\mathbb{C}(\lambda_1, \ldots, \lambda_k)[z]$ the linear space of polynomials in the variable z with coefficients from $\mathbb{C}(\lambda_1, \ldots, \lambda_k)$ and finally by $\mathbb{C}[\lambda_1, \ldots, \lambda_k][z]$ the linear space of polynomials in the variable z with coefficients from $\mathbb{C}[\lambda_1, \ldots, \lambda_k]$.

With the $c_j(\vec{\lambda})$ we can now define a parameterized linear functional Γ acting on the space $\mathbb{C}[z]$ by

$$\Gamma(z^j) = c_j(\vec{\lambda})$$

and parameterized Hankel determinants in $\mathbb{C}[\lambda_1, \ldots, \lambda_k]$,

$$\mathcal{H}_m^{(n)} = \begin{vmatrix} c_n(\vec{\lambda}) & \cdots & c_{n+m-1}(\vec{\lambda}) \\ \vdots & \iddots & c_{n+m}(\vec{\lambda}) \\ & & \vdots \\ c_{n+m-1}(\vec{\lambda}) & \cdots & c_{n+2m-2}(\vec{\lambda}) \end{vmatrix} \qquad \mathcal{H}_0^{(n)} = 1$$

as well as parameterized Hadamard polynomials in $\mathbb{C}[\lambda_1, \ldots, \lambda_k][z]$,

$$\mathcal{P}_m^{(n)}(z) = \frac{\mathcal{H}_m^{(n)}(z)}{h_m^{(n)}(\vec{\lambda})} \qquad m \geq 0 \qquad n \geq 0 \tag{8}$$

where

$$\mathcal{H}_m^{(n)}(z) = \begin{vmatrix} c_n(\vec{\lambda}) & \cdots & c_{n+m-1}(\vec{\lambda}) & c_{n+m}(\vec{\lambda}) \\ \vdots & \iddots & & c_{n+m+1}(\vec{\lambda}) \\ c_{n+m-1}(\vec{\lambda}) & & \cdots & c_{n+2m-1}(\vec{\lambda}) \\ 1 & z & \cdots & z^m \end{vmatrix} \qquad \mathcal{H}_0^{(n)}(z) = 1$$

and where the polynomial $h_m^{(n)}(\vec{\lambda})$ is a polynomial greatest common divisor of the polynomial coefficients of the powers of z in $\mathcal{P}_m^{(n)}(z)$. In this way its polynomial coefficients are relatively prime and hence $\mathcal{P}_m^{(n)}(z)$ is primitive. Note that $\mathcal{P}_m^{(n)}(z)$ belongs to $\mathbb{C}[\lambda_1,\ldots,\lambda_k][z]$ but does not belong to $\mathbb{C}[x_1,\ldots,x_k]$ because the powers in $\vec{\lambda}$ and z do not match. These parameterized Hadamard polynomials $\mathcal{P}_m^{(n)}(z)$ were introduced in [Benouahmane et al. 1999] and satisfy the formal orthogonality conditions

$$\Gamma\left(z^i\mathcal{P}_m^{(n)}(z)\right) = 0 \qquad i = 0,\ldots,m-1$$

We will call the functional Γ s-normal if

$$\mathcal{H}_m^{(n)}(\vec{\lambda}) \not\equiv 0 \qquad n \geq 0 \qquad m = 0,\ldots,s$$

4.1. The homogeneous qd-algorithm.

The homogeneous multivariate qd-algorithm is defined by the starting values

$$\begin{aligned} E_0^{(n)}(\vec{x}) &= 0 \\ Q_1^{(n)}(\vec{x}) &= \frac{C_{n+1}(\vec{x})}{C_n(\vec{x})} \qquad n = 1,2,\ldots \end{aligned}$$

and the continuation rules

$$\begin{aligned} E_m^{(n)}(\vec{x}) &= Q_m^{(n+1)}(\vec{x}) - Q_m^{(n)}(\vec{x}) + E_{m-1}^{(n+1)}(\vec{x}) \qquad m = 1,2\ldots \quad n = 0,1\ldots \\ Q_{m+1}^{(n)}(\vec{x}) &= \frac{E_m^{(n+1)}(\vec{x})Q_m^{(n+1)}(\vec{x})}{E_m^{(n)}(\vec{x})} \qquad m = 1,2,\ldots \quad n = 0,1,\ldots \end{aligned}$$

which can be executed symbolically or for one particular $\vec{x}$ numerically. It was shown in [Chaffy 1984, pp. 22–28] that the homogeneous qd-algorithm satisfies

$$\begin{aligned} Q_m^{(n)}(\vec{x}) &= Q_m^{(n)}(\lambda_1 z,\ldots,\lambda_k z) \\ &= q_m^{(n)}(\vec{\lambda})z \qquad (9) \\ E_m^{(n)}(\vec{x}) &= E_m^{(n)}(\lambda_1 z,\ldots,\lambda_k z) \\ &= e_m^{(n)}(\vec{\lambda})z \qquad (10) \end{aligned}$$

where

$$\begin{aligned} e_0^{(n)}(\vec{\lambda}) &= 0 \qquad n = 1,2,\ldots \\ q_1^{(n)}(\vec{\lambda}) &= \frac{c_{n+1}(\vec{\lambda})}{c_n(\vec{\lambda})} \qquad n = 0,1,\ldots \\ e_m^{(n)}(\vec{\lambda}) &= q_m^{(n+1)}(\vec{\lambda}) - q_m^{(n)}(\vec{\lambda}) + e_{m-1}^{(n+1)}(\vec{\lambda}) \qquad m = 1,2\ldots \quad n = 0,1\ldots \\ q_{m+1}^{(n)}(\vec{\lambda}) &= \frac{e_m^{(n+1)}(\vec{\lambda})q_m^{(n+1)}(\vec{\lambda})}{e_m^{(n)}(\vec{\lambda})} \qquad m = 1,2,\ldots \quad n = 0,1,\ldots \end{aligned}$$

In other words the homogeneous qd-algorithm can be regarded as a parameterized univariate qd-algorithm. It is then easy to write down the following parameterized version of a result proved in [Cuyt 1994].

LEMMA 2:
If the functional Γ is s-normal, then the values $Q_m^{(n)}(\vec{x})$ and $E_m^{(n)}(\vec{x})$ exist for $m = 1, \ldots, s$ and $n \geq 0$ and they are given by

$$Q_m^{(n)}(\vec{x}) = \frac{\mathcal{H}_m^{(n+1)}\mathcal{H}_{m-1}^{(n)}}{\mathcal{H}_m^{(n)}\mathcal{H}_{m-1}^{(n+1)}} z$$

$$E_m^{(n)}(\vec{x}) = \frac{\mathcal{H}_{m+1}^{(n)}\mathcal{H}_{m-1}^{(n+1)}}{\mathcal{H}_m^{(n)}\mathcal{H}_m^{(n+1)}} z$$

From this lemma it is easy to see that $q_m^{(n)}(\vec{\lambda})$ and $e_m^{(n)}(\vec{\lambda})$ introduced in (9) and (10) are rational expressions in $\lambda_1, \ldots, \lambda_k$ and hence that they belong to $\mathbb{C}(\lambda_1, \ldots, \lambda_k)$.

4.2. Application to parameterized eigenvalue problems.

Theorem 6 will generalize Theorem 3 for the computation of the eigenvalues of $B_m^{(n)}(\vec{x})$ given by (to save space we have respectively denoted $Q_m^{(n)}(\vec{x})$ and $E_m^{(n)}(\vec{x})$ in the matrix by $Q_m^{(n)}$ and $E_m^{(n)}$)

$$\begin{pmatrix} (Q_1^{(n)} + E_0^{(n)}) & -Q_1^{(n)} & & & 0 \\ -E_1^{(n)} & (Q_2^{(n)} + E_1^{(n)}) & -Q_2^{(n)} & & \\ & \ddots & \ddots & \ddots & \\ & & -E_{m-2}^{(n)} & (Q_{m-1}^{(n)} + E_{m-2}^{(n)}) & -Q_{m-1}^{(n)} \\ 0 & & & -E_{m-1}^{(n)} & (Q_m^{(n)} + E_{m-1}^{(n)}) \end{pmatrix} \tag{11}$$

Let us consider the following rational function, belonging to $\mathbb{C}(\lambda_1, \ldots, \lambda_k)[z]$,

$$\begin{aligned} r^{(n)}(\vec{\lambda}z) &= \frac{1|}{|z} + \frac{-q_1^{(n)}(\vec{\lambda})z|}{|z} + \frac{-e_1^{(n)}(\vec{\lambda})z|}{|z} + \frac{-q_2^{(n)}(\vec{\lambda})z|}{|z} + \ldots + \frac{-q_m^{(n)}(\vec{\lambda})z|}{|z} \\ &= \frac{1|}{|z} + \frac{-Q_1^{(n)}(\vec{x})|}{|z} + \frac{-E_1^{(n)}(\vec{x})|}{|z} + \frac{-Q_2^{(n)}(\vec{x})|}{|z} + \ldots + \frac{-Q_m^{(n)}(\vec{x})|}{|z} \end{aligned} \tag{12}$$

The denominator $r_D(\vec{\lambda}z)$ of $r^{(n)}(\vec{\lambda}z)$ is monic of degree m in z and is of the form

$$r_D(\vec{\lambda}z) = \sum_{i=0}^{m} b_i(\vec{\lambda})z^i \qquad b_i(\vec{\lambda}) \in \mathbb{C}(\lambda_1, \ldots, \lambda_k) \qquad b_m(\vec{\lambda}) = 1$$

A special collection of vectors $\vec{\lambda}$ is the following one. If we carefully examine Theorem 3, then we notice that the difference between the cases (b) and (c) stems from the fact that (c) deals with poles of equal modulus whereas (b) doesn't. Since $\mathbb{C}(\lambda_1, \ldots, \lambda_k)[z]$ is a unique factorization domain, $r_D(\vec{\lambda} z)$ can be factored as

$$r_D(\vec{\lambda} z) = \sum_{i=0}^{m} b_i(\vec{\lambda}) z^i = \prod_{i=1}^{\ell} \beta_i(\vec{\lambda} z) \qquad \beta_i(\vec{\lambda} z) \in \mathbb{C}(\lambda_1, \ldots, \lambda_k)[z]$$

For some $\vec{\lambda}$ it may happen that a zero $z^*(\vec{\lambda})$ of $\beta_i(\vec{\lambda} z)$ is at the same time a zero of $\beta_j(\vec{\lambda} z)$ with $i \neq j$ and this $\vec{\lambda}$ then gives rise to a pole of higher order than the multiplicity of $z^*(\vec{\lambda})$ as a zero of $\beta_i(\vec{\lambda} z)$ alone. For this $\vec{\lambda}$ the pole $z^*(\vec{\lambda})$ is then at least a multipole and its inverse a degenerate eigenvalue.

If $z^*(\vec{\lambda})$ is a multipole of only $\beta_i(\vec{\lambda} z)$ and not of another factor, then the parameterized interpretation of the $q_m^{(n)}(\vec{\lambda})$- and $e_m^{(n)}(\vec{\lambda})$-values is not disturbed, although $1/z^*(\vec{\lambda})$ is still a degenerate eigenvalue. If it nullifies at least two different factors, then for this $\vec{\lambda}$ the qd-table has to be interpreted differently. Let us collect the vectors $\vec{\lambda}$ that give rise to such an occasional pole of higher order in the set

$$M = \{\vec{\lambda} \in \mathbb{C}^k : ||\vec{\lambda}|| = 1,\ r^{(n)}(\vec{\lambda} z) \text{ has a pole that cancels more than one factor of } r_D(\vec{\lambda} z)\}$$

It is important to see that the set M is a finite set [??].

THEOREM 6:

Let the functional Γ be m-normal and fix $\vec{\lambda}$ on the unit sphere in $\mathbb{C}^k$ excluding M. Let the eigenvalues of $B_m^{(n)}(\vec{x})$ in $\{\vec{\lambda} z : ||\vec{\lambda}||_p = 1, \vec{\lambda} \notin M, z \in \mathbb{C}\}$ be numbered such that

$$|z_1(\vec{\lambda})| \geq |z_2(\vec{\lambda})| \geq \ldots \geq |z_m(\vec{\lambda})| \geq 0 = |z_{m+1}(\vec{\lambda})|$$

each eigenvalue occuring as many times in this sequence as indicated by its multiplicity. Then the homogeneous qd-scheme associated with the sequence $\{c_j(\vec{\lambda})\}_{j \in \mathbb{N}}$ has the following properties:

(a) for each k with $0 < k \leq m$ and $|z_k(\vec{\lambda})| > |z_{k+1}(\vec{\lambda})|$, it holds that

$$\lim_{n \to \infty} e_k^{(n)}(\vec{\lambda}) = 0$$

(b) for each k with $0 < k \leq m$ and $|z_{k-1}(\vec{\lambda})| > |z_k(\vec{\lambda})| > |z_{k+1}(\vec{\lambda})|$, it holds that

$$\lim_{n \to \infty} q_k^{(n)}(\vec{\lambda}) = z_k(\vec{\lambda})$$

(c) for each k and $j > 1$ such that $0 < k < k+j \leq m$ and $|z_{k-1}(\vec{\lambda})| > |z_k(\vec{\lambda})| = \ldots = |z_{k+j-1}(\vec{\lambda})| > |z_{k+j}(\vec{\lambda})|$, it holds that for the polynomials $\pi_i^{(n)}(\vec{\lambda}, z)$ defined by

$$\pi_0^{(n)}(\vec{\lambda}, z) = 1$$
$$\pi_{i+1}^{(n)}(\vec{\lambda}, z) = z\pi_i^{(n+1)}(\vec{\lambda}, z) - q_{k+i+1}^{(n)}(\vec{\lambda})\pi_i^{(n)}(\vec{\lambda}, z) \qquad n \geq 0, 0 \leq i \leq j-1$$

there exists a subsequence $\{\pi_j^{(n_\ell)}\}_{\ell \in \mathbb{N}}$ *such that*

$$\lim_{\ell \to \infty} \pi_j^{(n_\ell)}(\vec{\lambda}, z) = (z - z_{k+1}(\vec{\lambda})) \ldots (z - z_{k+j}(\vec{\lambda}))$$

For more details we refer to [Benouahmane et al. 2000].

5. Numerical illustration.

Each of the above techniques will be illustrated by means of a numerical example. In order to let the reader fully understand the difference between the two generalizations of the qd-algorithm presented above, we have chosen the examples such that the solution of both the pole detection problem and the parameterized eigenvalue problem is identical, meaning that our example is chosen such that for both problems the solution curves are identical. Nevertheless, the symbolic output of both generalizations of the qd-algorithm will not be identical, because they treat the multivariate data in a different way, as explained in respectively Section 3 and 4.

5.1. Pole detection mechanism.

Since we want to illustrate the techniques with some graphs, we shall now restrict ourselves to the case of two real variables. Consider a meromorphic function with polar singularities, given by

$$f(x_1, x_2) = \frac{\exp(x_1 + 2x_2)}{-5x_1^3 - x_1^2 x_2 - 5x_1 x_2^2 - x_2^3 - 5x_1^2 + 8x_1 x_2 + 7x_2^2 + 40x_1 - 4x_2 - 30}$$

The set M indexing the denominator polynomial $D(x_1, x_2)$ indicates the order in which the coefficients in the Taylor series representation of f are dealt with:

$$M = \{(0,0), (1,0), (0,1), \ldots, (3,0), \ldots, (0,3)\} = \{\vec{\imath}(0), \ldots, \vec{\imath}(9)\}$$

Furthermore, the denominator polynomial can be factored as the product of a "straight line" $D_1(x_1, x_2)$ and a "circle" $D_2(x_1, x_2)$:

$$D(x_1, x_2) = D_1(x_1, x_2) D_2(x_1, x_2) = (5 - 5x_1 - x_2)(x_1^2 + 2x_1 + x_2^2 - 2x_2 - 6)$$

For the factors D_1 and D_2 the sets M_1 and M_2 as defined in (1) for D, are respectively given by:

$$\begin{aligned} M_1 &= \{(0,0), (1,0), (0,1)\} \\ M_2 &= \{(0,0), (1,0), (0,1), (2,0), (0,2)\} \end{aligned}$$

From Lemma 3 and Theorem 5 we know that the information on the poles of $f(x_1, x_2)$ is to be found in the denominator of some Q-expressions. We need only look at the factor $H_{1,m}^{(n+m)}$ and more particularly at $\hat{H}_{1,m}^{(n+m)}$. From now on, when we say that a Q-column delivers pole information, we mean that the

$\hat{H}$-function in question has been extracted from its denominator. One cannot know from the factored form of $D(\vec{x})$ in what order one is to expect the factors $D_\ell(\vec{x})$. However Theorem 5 states that a Q-column delivering in its denominator as a limiting value the product of some pole factors is immediately followed by an $\hat{E}$-column tending to zero. The general order multivariate qd-table is divided into subtables by these vanishing $\hat{E}$-columns. The column number of such a vanishing $\hat{E}$-column is called a critical index.

From M_1, M_2 and M we can easily obtain the candidates for critical index:

$$\begin{aligned} m &= \#M_1 - 1 = 2 \\ m &= \#M_2 - 1 = 4 \\ m &= \#M - 1 = 9 \end{aligned}$$

After inspection of the $\hat{E}$-expressions, it is clear that the true critical indices in this example are $m = 2$ and $m = 9$.

We remark that in a real-life case the degree of the factors $D_\ell(x_1, x_2)$ is not necessarily known in advance and hence that it is very important (much more important than in the univariate case) to have vanishing $\hat{E}$-columns signaling where to look for the pole information. Numerical experiments have also shown that the enumeration of $\mathbb{N}^k$ is closely linked to the order in which the pole factors are to be traced by the algorithm. Indeed the sequence of index tuples $\{\vec{i}(\ell)\}_{\ell \in \mathbb{N}}$ indicates in which order the algorithm deals with the input Taylor coefficients of $f(\vec{x})$.

We print out the approximants obtained for the factor $D_1(x_1, x_2)$ and for the denominator $D(x_1, x_2) = D_1(x_2, x_2) D_2(x_1, x_2)$ after inputting the first 351 coefficients of $f(x_1, x_2)$, namely $\{c_{i_1 i_2}\}_{0 \le i_1 + i_2 \le 25}$:

$$\begin{aligned} D_1(x_1, x_2) &\approx \hat{H}_{1,2}^{(333)}(x_1, x_2) \\ &= -1.018 + x_1 + 0.2120 x_2 \\ (D_1 D_2)(x_1, x_2) &\approx \hat{H}_{1,9}^{(340)}(x_1, x_2) \\ &= (6.000 - 8.000 x_1 + 0.8000 x_2 + 1.000 x_1^2 - 1.600 x_1 x_2 - 1.400 x_2^2 \\ &\quad + x_1^3 + 0.2000 x_1^2 x_2 + 1.000 x_1 x_2^2 + 0.2000 x_2^3) \end{aligned}$$

Some of these functions are better understood from their graphs. The zeros of $\hat{H}_{1,2}^{(333)}$ and $\hat{H}_{1,9}^{(340)}$ are plotted in $[-5, 3] \times [-2, 6] \subset \mathbb{R}^2$. The surfaces $\hat{E}_2^{(332)}$ and $\hat{E}_9^{(332)}$ are shown in $[-1, 1] \times [-1, 1] \times [-1, 1] \subset \mathbb{R}^3$.

Figure 1: $\hat{H}_{1,2}^{(333)}$

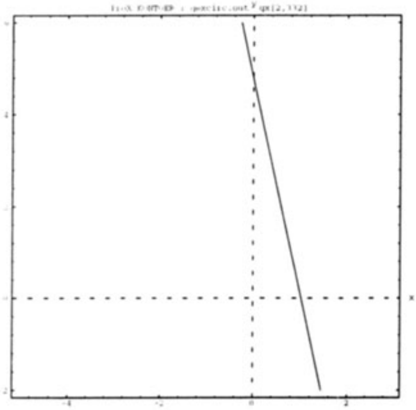


Figure 2: $\hat{E}_{2}^{(332)}$

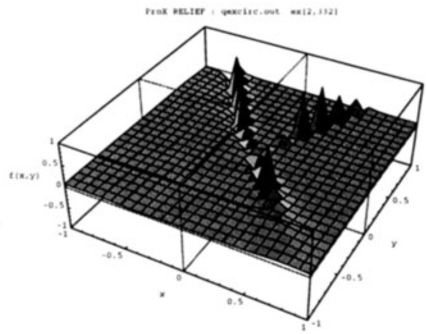


Figure 3: $\hat{H}_{1,9}^{(340)}$

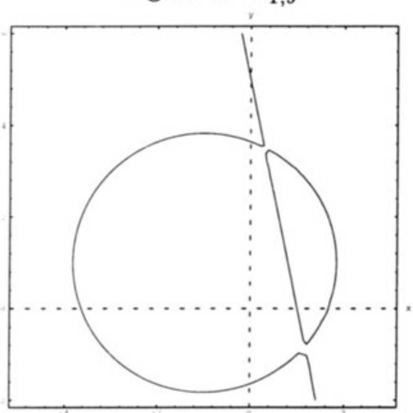
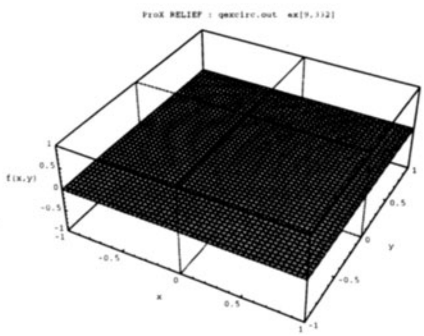


Figure 4: $\hat{E}_{9}^{(332)}$



5.2. Parameterized eigenvalue technique.

We again take the dimension of our multivariate problem $k = 2$ and consider the matrix $B_3^{(1)}(\vec{x})$ of the form (11) with

$$Q_1^{(1)}(x_1, x_2) = \frac{-225x_1^4+60x_1^3x_2-190x_1^2x_2^2-132x_1x_2^3-45x_2^4}{(5x_1+x_2)(5x_1^2-8x_1x_2-7x_2^2)(x_1^2+x_2^2)}$$

$$Q_2^{(1)}(x_1, x_2) = \frac{2(5x_1^2-8x_1x_2-7x_2^2)(11875x_1^3+75x_1^2x_2+3075x_1x_2^2-661x_2^3)}{(725x_1^2-40x_1x_2+113x_2^2)(225x_1^4-60x_1^3x_2+190x_1^2x_2^2+132x_1x_2^3+45x_2^4)}$$

$$Q_3^{(1)}(x_1, x_2) = \frac{15(725x_1^2-40x_1x_2+113x_2^2)}{11875x_1^3+75x_1^2x_2+3075x_1x_2^2-661x_2^3}$$

$$E_1^{(1)}(x_1, x_2) = \frac{2(5x_1+x_2)(x_1^2+x_2^2)(725x_1^2-40x_1x_2+113x_2^2)}{(5x_1^2-8x_1x_2-7x_2^2)(225x_1^4-60x_1^3x_2+190x_1^2x_2^2+132x_1x_2^3+45x_2^4)}$$

$$E_2^{(1)}(x_1, x_2) = \frac{3375(-225x_1^4+60x_1^3x_2-190x_1^2x_2^2-132x_1x_2^3-45x_2^4)}{(725x_1^2-40x_1x_2+113x_2^2)(11875x_1^3+75x_1^2x_2+3075x_1x_2^2-661x_2^3)}$$

The eigenvalue problem for this 3×3 matrix can easily be solved exactly by hand, or by calling a computer algebra system. But we want to illustrate the use of the symbolic homogeneous qd-algorithm which remains applicable to large-scale problems and delivers an accurate approximation. If we look at the exact eigenvalues of the matrix $B_3^{(1)}(\vec{x})$ defined above, then we find the expressions

$$\begin{aligned} z_1(x_1, x_2) &= \frac{5}{5x_1 + x_2} \\ z_2(x_1, x_2) &= \frac{-x_1 + x_2 - \sqrt{7x_1^2 - 2x_1x_2 + 7x_2^2}}{x_1^2 + x_2^2} \\ z_3(x_1, x_2) &= \frac{-x_1 + x_2 + \sqrt{7x_1^2 - 2x_1x_2 + 7x_2^2}}{x_1^2 + x_2^2} \end{aligned}$$

To give the reader an idea of what this looks like, we give a parametric plot of the eigenvalues for $\vec{x}$ varying over the unit disk in $I\!R^2$, in other words for $\vec{x} = e^{i\theta}$ or $x_1 = \cos\theta$ and $x_2 = \sin\theta$ with $\theta \in [0, 2\pi[$. This results in the pictures

Figure 5: $z_1(x_1, x_2)$

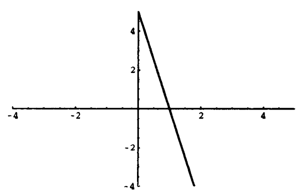


Figure 6: $z_2(x_1, x_2)$

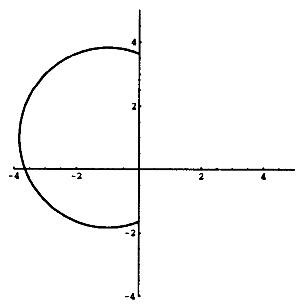
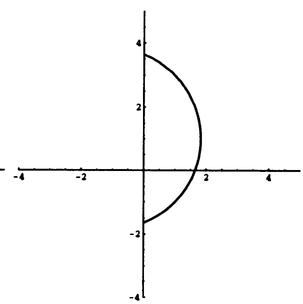


Figure 7: $z_3(x_1, x_2)$



When on the other hand the qd-technique is used, we first switch to polar coordinates. By the homogeneous qd-algorithm the eigenvalues are then being approximated, and more important, being regrouped such that, for each vector $\vec{\lambda}$, and because of the symbolic computation for all vectors $\vec{\lambda}$ at the same time (except for those in M), the expression

$$z_1(\vec{\lambda}) = \lim_{n\to\infty} q_1^{(n)}(\vec{\lambda})$$

delivers the in modulus largest eigenvalues and the expression

$$z_3(\vec{\lambda}) = \lim_{n\to\infty} q_3^{(n)}(\vec{\lambda})$$

delivers the in modulus smallest eigenvalues. When applied to the matrix $B_3^{(1)}$ this results for instance in the approximations

$$z_1(\lambda_1, \lambda_2) \approx q_1^{(9)}(\vec{\lambda})$$

$$z_2(\lambda_1, \lambda_2) \approx q_2^{(7)}(\vec{\lambda})$$

$$z_3(\lambda_1, \lambda_2) \approx q_3^{(5)}(\vec{\lambda}) = \frac{\begin{array}{c}15(177359375\lambda_1^6+68062500\lambda_1^5\lambda_2+108140625\lambda_1^4\lambda_2^2-20047000\lambda_1^3\lambda_2^3+\\ 25741125\lambda_1^2\lambda_2^4-8636220\lambda_1\lambda_2^5+2417987\lambda_2^6)\end{array}}{\begin{array}{c}2689140625\lambda_1^7+1469890625\lambda_1^6\lambda_2+1996921875\lambda_1^5\lambda_2^2-206283125\lambda_1^4\lambda_2^3+\\ 555975875\lambda_1^3\lambda_2^4-222944925\lambda_1^2\lambda_2^5+93486145\lambda_1\lambda_2^6-21496039\lambda_2^7\end{array}}$$

For $\vec{\lambda}$ varying over the unit disk in $I\!R^2$ the parameterized approximations for the eigenvalues now look like

Figure 8: $q_1^{(24)}(\lambda_1, \lambda_2)$

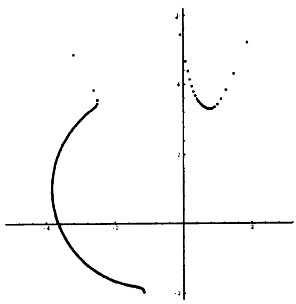


Figure 9: $q_2^{(22)}(\lambda_1, \lambda_2)$

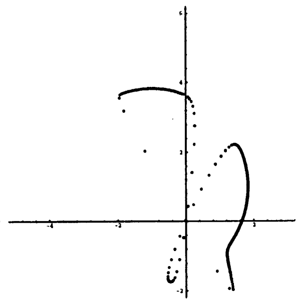
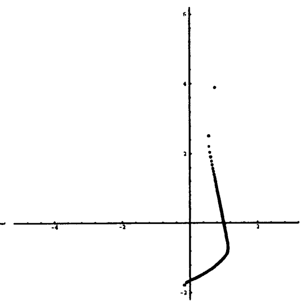


Figure 10: $q_3^{(20)}(\lambda_1, \lambda_2)$



It is clear that the superposition of the figures 5, 6 and 7 and the superposition of 8, 9 and 10 result in the same total set of eigenvalues.

Figure 11: join of 5, 6 and 7

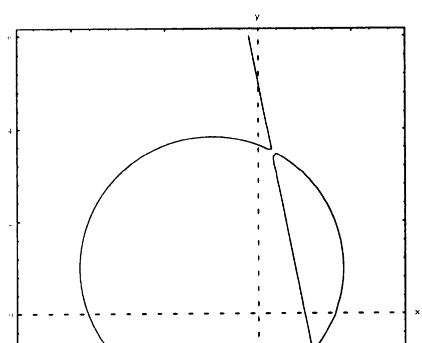
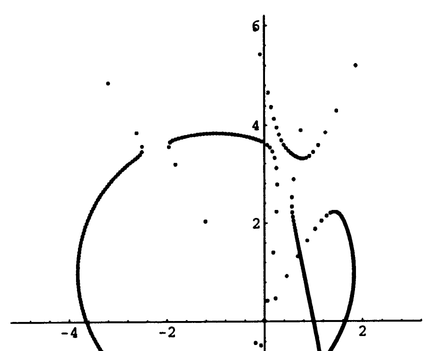


Figure 12: join of 8, 9 and 10

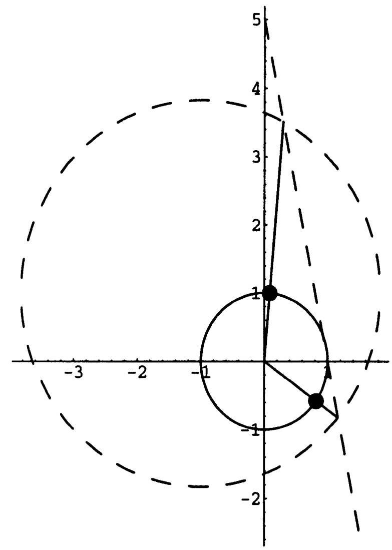


The difference between the two approaches lies in the fact that in the second approach the eigenvalues are automatically ordered in modulus, and that it is therefore an approximation method, not an explicit solution method. Remains the problem of the problematic vectors in M. For our example

$$M = \{\vec{\lambda}_1^* = (0.0843271, 0.996438), \vec{\lambda}_2^* = (0.817216, -0.576331)\}$$

as shown in Figure 13.

Figure 13: set M of problematic parameters



If we carefully examine the expression $e_1^{(23)}(\vec{\lambda})$ then we notice that for $\vec{\lambda} \neq \vec{\lambda}_2^*$, the value $e_1^{(23)}(\vec{\lambda})$ tends to zero. For instance, again restricting ourselves to the unit circle in $I\!R^2$,

$$e_1^{(23)}(1,0) \approx 1.3 \times 10^{-7}$$

On the other hand for $\vec{\lambda} = \vec{\lambda}_2^*$,

$$e_1^{(23)}(\vec{\lambda}_2^*) \approx -0.00572$$

Hence $q_1^{(24)}(\vec{\lambda})$ delivers the largest eigenvalues except for $\vec{\lambda} = \vec{\lambda}_2^*$ where a double pole occurs in $r(\vec{\lambda}z)$. Examining $e_2^{(21)}(\vec{\lambda})$ shows that this expression tends to zero for $\vec{\lambda} \neq \vec{\lambda}_1^*$. But careful: for $\vec{\lambda} = \vec{\lambda}_2^*$, the expression $q_2^{(22)}(\vec{\lambda})$ must be interpreted differently. If $\vec{\lambda} \notin M$, then $q_2^{(22)}(\vec{\lambda})$ delivers the second eigenvalue. For $\vec{\lambda} = \vec{\lambda}_2^*$ Theorem 1(c) must be applied with $k = 1$ and $j = 2$. Continuing our investigation reveals that $e_2^{(21)}(\vec{\lambda}_1^*)$ does not tend to zero,

$$e_2^{(21)}(1,0) \approx -2.3 \times 10^{-6}$$
$$e_2^{(21)}(\vec{\lambda}_1^*) \approx -0.00252$$

while $e_3^{(19)}(\vec{\lambda}_1^*)$ does. Hence the expression $q_3^{(20)}(\vec{\lambda})$ must now be interpreted differently for $\vec{\lambda} = \vec{\lambda}_1^*$ and for all other $\vec{\lambda}$. For $\vec{\lambda} \neq \vec{\lambda}_1^*$ the expression $q_3^{(20)}(\vec{\lambda})$ delivers the smallest eigenvalues. For $\vec{\lambda} = \vec{\lambda}_1^*$ Theorem 1(c) must be applied with $k = 2$ and $j = 2$.

References

Benouahmane, B. and Cuyt, A. (1999): Properties of multivariate homogeneous orthogonal polynomials. J. Approx. Th. submitted.

Benouahmane, B., Cuyt, A. and Verdonk, B. (2000): On the solution of parameterized (smallest) eigenvalue problems. SIAM J. Matrix Anal. & Applcs. submitted.

Brezinski, C. (1980): Padé type approximation and general orthogonal polynomials. Birkhauser Verlag, Basel.

Chaffy, C. (1984): Interpolation polynomiale et rationnelle d'une fonction de plusieurs variables complexes. Thèse. Inst. Polytech., Grenoble.

Cuyt, A. (1988): A multivariate qd-like algorithm. BIT 28: 98–112.

Cuyt, A. (1992): Extension of "A multivariate convergence theorem of the de Montessus de Ballore type" to multipoles. J. Comput. Appl. Math.41: 323–330.

Cuyt, A. (1994): On the convergence of the multivariate "homogeneous" qd-algorithm. BIT 34: 535–545.

Cuyt, A. and Verdonk, B. (1999): Extending the qd-algorithm to tackle multivariate problems. In: Papamichael, N., Ruscheweyh, S., and Saff, B. (eds.): Proceedings of the Third CMFT Conference "Computational methods and function theory 1997", Nicosia, Cyprus. World Scientific, pp. 135–159.

Fernando, K. and Parlett, B. (1994): Accurate singular values and differential qd algorithms. Numer. Math. 67: 191–229.

Henrici, P. (1974): Applied and computational complex analysis I. John Wiley, New York.

Von Matt, U. (1997): The orthogonal QD-algorithm. SIAM J. Sci. Statist. Comput. 18: 1163–1186.

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* Research Director FWO-Vlaanderen