RATIONAL HERMITE INTERPOLATION IN ONE AND MORE VARIABLES.

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Abstract.

In the first 4 sections we discuss topics from univariate rational Hermite interpolation (URI). These topics include the structure of the table of URI, a recursive computation scheme and a continued fraction representation both in the normal case and the non-normal case and a convergence theorem for rational Hermite interpolants of meromorphic functions.

In the next 4 sections these items are generalized to the multivariate case. We first introduce multivariate rational Hermite interpolants (MRI) for data sets satisfying the inclusion property or rectangle rule and give a recursive computation scheme and a non-branched continued fraction representation, both for the non-degenerate and the degenerate case. For general data sets only results for ordinary rational interpolation in the case of non-degeneracy were obtained in [CUYTd].

1. Notations and definitions for URI.

Consider a function f defined in a sequence of distinct points $(y_i)_{i\in\mathbb{N}}$ of the complex plane and let the derivatives $f^{(\ell)}(y_i)$ of the function f be given for $\ell = 0,\ldots,s_i - 1$. We denote the exact degree of a polynomial p by ∂p and its order by ωp. The rational Hermite interpolation problem of order (n,m) for f consists in finding polynomials

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

and

$$q(x) = \sum_{i=0}^{m} b_i x^i$$

such that for a particular j

$$n + m + 1 = \sum_{i=0}^{j} s_i$$

and

$$f^{(\ell)}(y_i) = \left(\frac{p}{q}\right)^{(\ell)}(y_i) \qquad i = 0,\ldots,j \qquad \ell = 0,\ldots,s_i - 1 \tag{1}$$

S. P. Singh (ed.), Approximation Theory, Spline Functions and Applications, 69–103.

In this interpolation problem s_i interpolation points coincide with y_i, so s_i interpolation conditions must be fulfilled in y_i. Therefore this type of interpolation problem is also often referred to as the osculatory rational interpolation problem [WARN]. In case $s_i = 1$ for all $i \geq 0$ then the problem is identical to the ordinary rational interpolation problem. In case all the interpolation conditions must be satisfied in one single point y_0 then the osculatory rational interpolation problem is identical to the Padé approximation problem. Instead of solving problem (1) we consider the linear system of equations

$$(fq - p)^{(\ell)}(y_i) = 0 \qquad i = 0, \ldots, j \qquad \ell = 0, \ldots, s_i - 1 \tag{2}$$

Condition (2) is a homogeneous system of $n+m+1$ linear equations in the $n+m+2$ unknown coefficients a_i and b_i of p and q. Hence the system (2) always has at least one nontrivial solution. For different solutions of (2) the following equivalence can be proved. If the polynomials p_1, q_1 and p_2, q_2 both satisfy (2) then $p_1 q_2 = p_2 q_1$. Not all solutions of (2) also satisfy (1): it is very well possible that the polynomials p and q satisfying (2) are such that p/q is reducible. Nevertheless all solutions of (2) have the same irreducible form. For p and q satisfying (2) we shall denote by $r_{n,m}(x) = (p_{n,m}/q_{n,m})(x)$ the irreducible form of p/q where $q_{n,m}(x)$ is normalized such that $q_{n,m}(y_0) = 1$, and we shall call $r_{n,m}(x)$ the rational Hermite interpolant of order (n, m) for f. Although the terminology "interpolant" is used it may be that $r_{n,m}(x)$ does not satisfy the interpolation conditions (1) anymore [WUYT]. A simple example will illustrate this. Let $y_0 = 0$, $y_1 = 1$, $y_2 = 2$ and $f(y_0) = 0$, $f(y_1) = 3$, $f(y_2) = 3$. Take $n = m = 1$. A solution of this rational interpolation problem is $p(x) = 3x$ and $q(x) = x$. Thus $p_{1,1}(x) = 3$ and $q_{1,1}(x) = 1$. Clearly $(p_{1,1}/q_{1,1})(y_0) \neq f(y_0)$.

The problem of "unattainable" interpolation points is typical for the case of rational interpolation. Having computed the rational interpolant p/q from linear interpolation conditions, in other words conditions expressed for $fq - p$ instead of for $f - (p/q)$, it may occur that an interpolation point is also a common zero of p and q and hence that the rational function p/q is undefined in that interpolation point. Consequently the nonlinear interpolation condition cannot be satisfied in that interpolation point anymore, not even by the irreducible form of p/q. The interpolation point has become unattainable. As a conclusion we can say that the rational interpolation problem (1) has a solution if and only if $p_{n,m}(x)$ and $q_{n,m}(x)$ satisfy themselves the system of equations (2).

The rational Hermite interpolation problem can be reformulated as a Newton-Padé approximation problem. We introduce the following notations:

$$x_\ell = y_0 \qquad \ell = 0, \ldots, s_0 - 1$$
$$x_{d(i)+\ell} = y_i \qquad \ell = 0, \ldots, s_i - 1 \qquad d(i) = s_0 + s_1 + \ldots + s_{i-1} (i \geq 1)$$
$$c_{ij} = 0 \qquad i > j$$
$$c_{ij} = f[x_i, \ldots, x_j] \qquad i \leq j$$

with possible coalescence of points in the divided difference $f[x_i, \ldots, x_j]$. If we put

$$B_j(x) = \prod_{\ell=1}^{j}(x - x_{\ell-1})$$

with $B_0(x) = 1$ then formally

$$f(x) = \sum_{i=0}^{\infty} c_{0i} B_i(x)$$

This series is called the Newton series for f. Problem (2) is then equivalent [CLAEa] with the computation of polynomials

$$p(x) = \sum_{i=0}^{n} a_i B_i(x)$$

and

$$q(x) = \sum_{i=0}^{m} b_i B_i(x)$$

such that

$$(fq - p)(x) = \sum_{i \geq n+m+1} d_i B_i(x) \tag{3}$$

which is called the Newton-Padé approximation problem of order (n, m) for f. To determine solutions p and q of (3) the divided differences

$$d_i = (fq - p)[x_0, \ldots, x_i] \qquad i = 0, \ldots, n + m$$

must be calculated and put equal to zero. The following generalization of the Leibniz rule for differentiating a product of functions, is a useful tool to accomplish this [WARN]:

$$(fq)[x_0, \ldots, x_i] = \sum_{\ell=0}^{i} f[x_0, \ldots, x_\ell] q[x_\ell, \ldots, x_i]$$

Using this rule we can now write down the linear systems of equations that must be satisfied by the coefficients a_i and b_i in p and q:

$$\begin{cases} c_{00}b_0 = a_0 \\ c_{01}b_0 + c_{11}b_1 = a_1 \\ \quad \vdots \\ c_{0n}b_0 + c_{1n}b_1 + \ldots + c_{mn}b_m = a_n \end{cases}$$

$$\begin{cases} c_{0,n+1}b_0 + \ldots + c_{m,n+1}b_m = 0 \\ \vdots \\ c_{0,n+m}b_0 + \ldots + c_{m,n+m}b_m = 0 \end{cases} \tag{4}$$

Since the problems (2) and (3) are equivalent, the rational function $r_{n,m}$ can as well be called the Newton-Padé approximant of order (n,m) to f. We shall see that many properties and algorithms valid for Padé approximants can be generalized for Newton-Padé approximants or rational Hermite interpolants [WARN]. The rational Hermite interpolants of order (n,m) for f can be ordered in a table:

$$\begin{array}{cccc} r_{0,0} & r_{0,1} & r_{0,2} & \cdots \\ r_{1,0} & r_{1,1} & r_{1,2} & \cdots \\ r_{2,0} & r_{2,1} & \cdots & \\ r_{3,0} & r_{3,1} & \cdots & \\ \vdots & \vdots & & \end{array}$$

In the first column one finds the polynomial interpolants for f and in the first row the inverses of the polynomial interpolants for $(1/f)$. If we define $n' = \partial p_{n,m}$ and $m' = \partial q_{n,m}$ then it can be shown that at least $n'+m'+t+1$ points $z_0, \ldots, z_{n'+m'+t}$ with $t \geq 0$ exist in $\{x_0, \ldots, x_{n+m}\}$ such that $r_{n,m}(z_i) = f(z_i)$. Again we call an entry of the table normal if it occurs only once in that table. A necessary condition for the normality of the rational interpolant $r_{n,m}(x)$ is formulated in the following theorem [WUYT].

THEOREM 1:
If the rational Hermite interpolant $r_{n,m} = p_{n,m}/q_{n,m}$ is normal and if $(fq_{n,m} - p_{n,m})(x_i) = 0$ for $i = 0, \ldots, n'+m'$, then
(a) $n' = n$ and $m' = m$
(b) $(fq_{n,m} - p_{n,m})(x_i) \neq 0$ for $i = n+m+1, n+m+2$.

Conclusion (b) in theorem 1 does not imply that $(fq_{n,m} - p_{n,m})(x_i) \neq 0$ for $i \geq n+m+1$. That the conditions (a) and (b) are not sufficient to guarantee the normality of $r_{n,m}(x)$ is illustrated in the following example. Let $x_i = i$ for $i = 0, 1, 2, \ldots$ and $f(x_0) = 0$, $f(x_1) = 1$, $f(x_2) = 3$, $f(x_3) = 4$, $f(x_i) = i$ for $i = 4, 5, 6, \ldots$. For $n = 0$ and $m = 1$ we find $r_{n,m}(x) = x$ with (a) and (b) of theorem 1 satisfied. But $r_{n,m}$ is not normal because $r_{n,m} = r_{k,\ell}$ for $k \geq 3$ and $\ell \geq 2$. However, it is possible to formulate a sufficient condition for the normality of $r_{n,m}$ [WUYT].

THEOREM 2:
If $r_{n,m} = p_{n,m}/q_{n,m}$ with $n = n'$, $m = m'$ and $(fq_{n,m} - p_{n,m})(x_i) = 0$ for at most $n+m+1$ points from the sequence $(x_i)_{i\in\mathbb{N}}$, then $r_{n,m}$ is normal.

2. Methods to compute normal rational Hermite interpolants.

2.1. Determinant formulas.

First of all we give a determinant representation [CLAE] similar to that for the case of Padé approximation. Let us define

$$F_{i,j}(x) = \sum_{\ell=i}^{j} c_{i\ell} B_\ell(x) \qquad i \le j$$

with $F_{i,j}(x) = 0$ if $i > j$.

THEOREM 3:
If the rank of the system of equations (4) is maximal, then (up to a normalization) $r_{n,m} = p_{n,m}/q_{n,m}$ is given by

$$p_{n,m}(x) = \begin{vmatrix} F_{0,n}(x) & F_{1,n}(x) & \cdots & F_{m,n}(x) \\ c_{0,n+1} & c_{1,n+1} & \cdots & c_{m,n+1} \\ c_{0,n+2} & c_{1,n+2} & \cdots & c_{m,n+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{0,n+m} & c_{1,n+m} & \cdots & c_{m,n+m} \end{vmatrix}$$

and

$$q_{n,m}(x) = \begin{vmatrix} B_0(x) & B_1(x) & \cdots & B_m(x) \\ c_{0,n+1} & c_{1,n+1} & \cdots & c_{m,n+1} \\ c_{0,n+2} & c_{1,n+2} & \cdots & c_{m,n+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{0,n+m} & c_{1,n+m} & \cdots & c_{m,n+m} \end{vmatrix}$$

One can see that in case all the interpolation points coincide with one single point, these determinant formulas reduce to the ones given in [CUYT] since the divided differences reduce to Taylor coefficients.

In the sequel of this section we suppose that every rational interpolant $r_{n,m}(x)$ itself satisfies the interpolation conditions (1). This is for instance satisfied if $\min(n - n', m - m') = 0$. In discussing algorithms for the calculation of rational Hermite interpolants we restrict ourselves to those computation schemes that reduce to well-known algorithms for the calculation of Padé approximants in case all the interpolation points coincide. Exactly those algorithms will be generalized to the multivariate case in the following sections. We do not discuss the construction of Thiele interpolating continued fractions using inverse or reciprocal differences, nor methods that construct rational Hermite interpolants on paths in the table different from descending staircases.

2.2. The generalized ϵ-algorithm.

It was proved in [CLAEd] that

$$(x - x_{n+m})^{-1}(r_{n-1,m} - r_{n,m})^{-1} + (x - x_{n+m+1})^{-1}(r_{n+1,m} - r_{n,m})^{-1} = (x - x_{n+m})^{-1}(r_{n,m-1} - r_{n,m})^{-1} + (x - x_{n+m+1})^{-1}(r_{n,m+1} - r_{n,m})^{-1}$$

Using this result it is possible to set up the following generalized ϵ- algorithm [CLAEd], in the same way as the ϵ-algorithm for Padé approximants was constructed from the star identity:

$$\begin{aligned}
\epsilon_{-1}^{(n)} &= 0 \qquad n = 0,1,\dots \\
\epsilon_{2m}^{(-m-1)} &= 0 \qquad m = 0,1,\dots \\
\epsilon_{0}^{(n)} &= r_{n,0}(x) \qquad n = 0,1,\dots \\
\epsilon_{m+1}^{(n)} &= \epsilon_{m-1}^{(n+1)} + \frac{1}{(x - x_{m+n+1})(\epsilon_m^{(n+1)} - \epsilon_m^{(n)})} \\
&\qquad n = -\lfloor \tfrac{m}{2} \rfloor - 1, -\lfloor \tfrac{m}{2} \rfloor, \dots \qquad m = 0,1,\dots
\end{aligned}$$

Finally

$$\epsilon_{2m}^{(n-m)} = r_{n,m}(x)$$

2.3. A generalization of the qd-algorithm.

Consider descending staircases in the table of rational Hermite interpolants

$$T_k = \{r_{k,0}, r_{k+1,0}, r_{k+1,1}, r_{k+2,1}, \dots\} \qquad k \geq 0$$

and continued fractions of the form

$$g_k(x) = c_0 + \sum_{i=1}^{k} c_i(x - x_0)(x - x_1)\dots(x - x_{i-1}) + \frac{c_{k+1}(x - x_0)\dots(x - x_k)|}{|1} + \frac{-q_1^{(k+1)}(x - x_{k+1})|}{|1 + q_1^{(k+1)}(x_0 - x_{k+1})} + \frac{-e_1^{(k+1)}(x - x_{k+2})|}{|1 + e_1^{(k+1)}(x_0 - x_{k+2})} + \frac{-q_2^{(k+1)}(x - x_{k+3})|}{|1 + q_2^{(k+1)}(x_0 - x_{k+3})} + \frac{-e_2^{(k+1)}(x - x_{k+4})|}{|1 + e_2^{(k+1)}(x_0 - x_{k+4})} + \dots \qquad (5)$$

THEOREM 4:
If every three consecutive elements in T_k are different, then a continued fraction of the form (5) exists with $c_{k+1} \neq 0$, $q_i^{(k+1)} \neq 0$, $e_i^{(k+1)} \neq 0$, $1+q_i^{(k+1)}(x_0 - x_{k+2i-1}) \neq 0$, $1+e_i^{(k+1)}(x_0 - x_{k+2i}) \neq 0$ for $i \geq 1$ and such that the n^{th} convergent equals the $(n+1)^{th}$ element of T_k.

To calculate the coefficients $q_i^{(k+1)}$ and $e_i^{(k+1)}$ in (5) one can use the following recurrence relations. Compute the even part of the continued fraction $g_k(x)$ and the odd part of the continued fraction $g_{k-1}(x)$. These contractions have the same convergents $r_{k,0}, r_{k+1,1}, r_{k+2,2}, \ldots$ and they also have the same form. In this way one can check [CLAEc] that for $k \geq 1$

$$e_0^{(k)} = 0$$
$$q_1^{(k)} = \frac{f[x_0, \ldots, x_{k+1}]}{f[x_1, \ldots, x_{k+1}]}$$

and for $\ell \geq 1$ and $k \geq 1$

$$e_\ell^{(k)} = \frac{q_\ell^{(k+1)} - q_\ell^{(k)} + e_{\ell-1}^{(k+1)}\left[1 + q_\ell^{(k+1)}(x_0 - x_{k+2\ell-1})\right]}{1 + q_\ell^{(k)}(x_0 - x_{k+2\ell-1})}$$
$$q_{\ell+1}^{(k)} = \frac{e_\ell^{(k+1)} q_\ell^{(k+1)}\left[1 + e_\ell^{(k)}(x_0 - x_{k+2\ell})\right]}{e_\ell^{(k)}\left[1 + q_\ell^{(k+1)}(x_0 - x_{k+2\ell-1})\right] + e_\ell^{(k+1)}(e_\ell^{(k)} - q_\ell^{(k+1)})(x_0 - x_{k+2\ell+1})}$$

These coefficients are usually ordered as in the next table

$$\begin{array}{ccccccc}
e_0^{(1)} & & & & & & \\
 & q_1^{(1)} & & & & & \\
e_0^{(2)} & & e_1^{(1)} & & & & \\
 & q_1^{(2)} & & q_2^{(1)} & & & \\
e_0^{(3)} & & e_1^{(2)} & & e_2^{(1)} & \cdots & \\
 & q_1^{(3)} & & q_2^{(2)} & & & \\
e_0^{(4)} & & e_1^{(3)} & & e_2^{(2)} & \cdots & \\
 & \vdots & & \vdots & & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

where the superscript denotes a diagonal in the table and the subscript a column.

3. Structure of the table of rational Hermite interpolants.

We already mentioned in section 1 that at least $n'+m'+t+1$ points $z_0, \ldots, z_{n'+m'+t}$ with $t \geq 0$ exist among $\{x_0, \ldots, x_{n+m}\}$ such that $r_{n,m}(z_i) = f(z_i)$. On the basis of this conclusion a property comparable with the block structure of the Padé table can be formulated. It is based on the following property.

THEOREM 5:
If the rank of the linear system (4) is $m-t$ then (up to a normalization) a unique solution $\overline{p}$ and $\overline{q}$ of (4) exists with

$$\partial\overline{p} \leq n-t$$
$$\partial\overline{q} \leq m-t$$

where at least one of the upper bounds is attained. Every other solution $p(x)$ and $q(x)$ of (4) can be written in the form

$$p(x) = \overline{p}(x)s(x)$$
$$q(x) = \overline{q}(x)s(x)$$

where $\partial s \leq t$.

Before describing the shape of the sets in the table of rational Hermite interpolants that contain equal elements, it is important to emphasize that the structure of the table can only be studied if the ordering of the interpolation points $\{x_i\}_{i\in\mathbb{N}}$ remains fixed once it is chosen. Since the polynomials $\overline{p}$ and $\overline{q}$ constructed in the previous theorem have the property that their degrees cannot be lowered simultaneously anymore unless some interpolation conditions are lost, we shall call them a minimal solution. This does not imply that $\overline{p}/\overline{q}$ is irreducible. However we still have $p_{n,m}\overline{q} = \overline{p}q_{n,m}$.

THEOREM 6:
Let $\overline{p}(x)$ and $\overline{q}(x)$ be the minimal solution of the Newton-Padé approximation problem of order (n,m) for f and let the rank of the linear system (4) be $m-t$.

(a) If $\partial\overline{p} = n-t-t_1$ then all the minimal solutions lying in the triangle with corner elements $(n-t-t_1, m-t)$, $(n-t-t_1, m+t+t_1)$ and $(n+t, m-t)$ are equal to $\overline{p}(x)$ and $\overline{q}(x)$.

(b) If $\partial\overline{q} = m-t-t_2$ then all the minimal solutions lying in the triangle with corner elements $(n-t, m-t-t_2)$, $(n-t, m+t)$ and $(n+t+t_2, m-t-t_2)$ are equal to $\overline{p}(x)$ and $\overline{q}(x)$.

(c) If

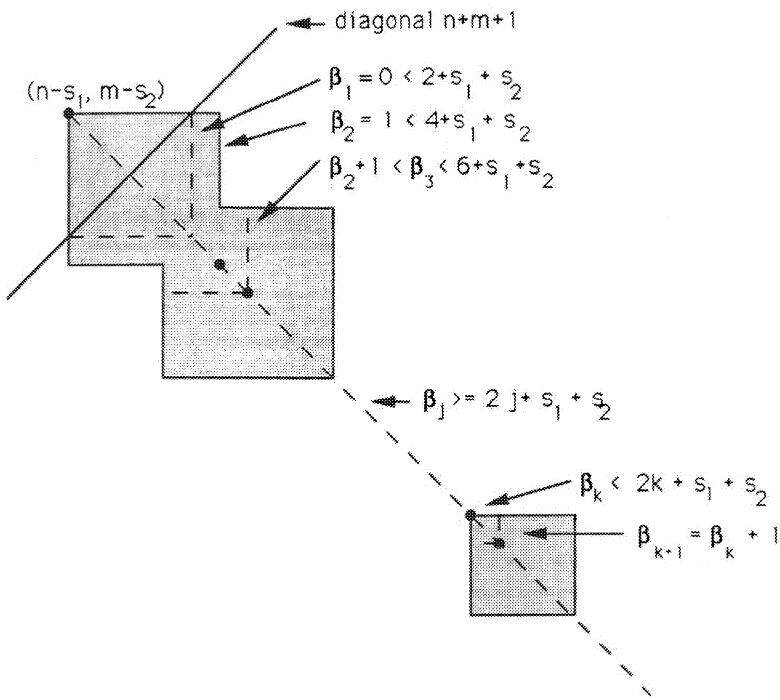
$$(f\overline{q}-\overline{p})(x) = \sum_{i\geq n+m-2t+t_3+1} d_i B_i(x)$$

with $d_{n+m-2t+t_3+1} \neq 0$ *then all the rational Hermite interpolants lying in the triangle with corner elements* $(n-t, m-t)$, $(n+t+t_3, m-t)$ *and* $(n-t, m+t+t_3)$ *are equal to* $\overline{p}(x)$ *and* $\overline{q}(x)$.

(d) *If* $\partial\overline{p} = n - s_1, \partial\overline{q} = m - s_2$ *and* $(f\overline{q} - \overline{p})(x) = \sum_{i \geq n+m+1+s_3} d_i B_i(x)$ *with* $d_{n+m+s_3+1} \neq 0$ *then all the rational interpolants lying in the square with corners* $(n - s_1, m - s_2)$ *and* $(n + s_2 + s_3, m + s_1 + s_3)$ *have the same irreducible form* $r_{n,m}(x)$.

(e) *If* $\partial\overline{p} = n-s_1, \partial\overline{q} = m-s_2, (f\overline{q}-\overline{p})(x) = \sum_{i \geq n+m+1} d_i B_i(x)$ *and* $r_{n-s_1,m-s_2}(x)$ *also satisfies the interpolation conditions in the points* $x_{n+m+1+\beta_j}$ *for* $j = 1, \ldots, s$ *and* $0 \leq \beta_1 < \ldots < \beta_s$, *then if* $\beta_j < 2j + s_1 + s_2$ *we have for* $\ell = \beta_j + 1, \ldots, 2j+s_1+s_2$: $r_{n+s_2+j,m-s_2+\ell-j}(x) = r_{n-s_1,m-s_2}(x) = r_{n-s_1+\ell-j,m+s_1+j}(x)$.

This theorem explains that the square block described in theorem 6c is only a starting point and that it can have a sort of tail concentrated along its main diagonal as illustrated in the next picture.



For the proof we refer to [CLAEf]. For a detailed study of the structure of the rational Hermite interpolation table we refer to [CLAEb]. Singular rules for the generalized ϵ- and qd-algorithm applicable in non-normal tables are under investigation.

4. Convergence of rational Hermite interpolants.

We shall now mention some results for the convergence of columns in the table of rational Hermite interpolants. Broadly speaking, the convergence of an arbitrary series of interpolation does not depend on the entire sequence of interpolation points x_i (as defined in the Newton-Padé approximation problem) but merely on its asymptotic character, as can be seen in the next theorem.

THEOREM 7:
Let the sequence of interpolation points $\{x_0, x_1, x_2, \ldots\}$ *be asymptotic to the sequence*

$$\{w_0, w_1, \ldots, w_j, w_0, w_1, \ldots, w_j, w_0, w_1, \ldots, w_j, \ldots\}$$

in the sense that

$$lim_{k\to\infty} x_{k(j+1)+i} = w_i$$

for $i = 0, \ldots, j$. *If the function* $f(z)$ *is analytic throughout the interior of the lemniscate*

$$B(w_0, \ldots, w_j, r) = \{z \in \mathbb{C} : |(z - w_0)(z - w_1) \ldots (z - w_j)| = r\}$$

then the $r_{n,0}$ *converge to* f *on the interior of* $B(w_0, \ldots, w_j, r)$. *The convergence is uniform on every closed and bounded subset interior to* $B(w_0, \ldots, w_j, r)$.

For the proof we refer to [WALS p. 61] and [DAVI pp. 90-91]. Let us now turn to the case of a meromorphic function f with poles $z_1, \ldots, z_m$ (counted with their multiplicity). Let the table of minimal solutions for the Newton-Padé approximation problem be normal. According to theorem 1 we then have $\partial \overline{q}_{n,m} = m$. Let $z_i^{(n)}$ for $i = 1, \ldots, m$ be the zeros of $q_{n,m}$ for $n = 0, 1, 2, \ldots$ and let $\rho_i = |(z_i - w_0)(z_i - w_1) \ldots (z_i - w_j)|$ with $0 < \rho_1 \le \rho_2 \le \ldots \le \rho_m \le \alpha r < r$ for a positive constant α.

THEOREM 8:
If the sequence of interpolation points $\{x_0, x_1, x_2, \ldots\}$ *is asymptotic to the sequence* $\{w_0, w_1, \ldots, w_j, w_0, w_1, \ldots, w_j, \ldots\}$, *if* f *is meromorphic in the interior of the lemniscate* $B(w_0, \ldots, w_j, r)$ *with poles* $z_1, \ldots, z_m$ *counted with their multiplicity and if the table of minimal solutions for the Newton-Padé approximation problem is normal, then*

$$lim_{n\to\infty} q_{n,m}(z) = \prod_{i=1}^{m} \left(\frac{z - w_i}{x_0 - w_i} \right) \quad and \quad lim_{n\to\infty} r_{n,m}(z) = f(z)$$

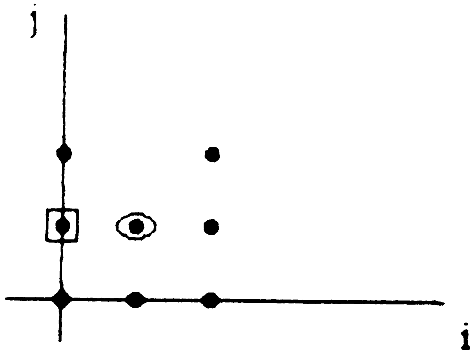
uniformly in every closed and bounded subset in the interior of $B(w_0, \ldots, w_j, r)$ *not containing the points* $z_1, \ldots, z_m$.

The proof is given in [CLAEb].

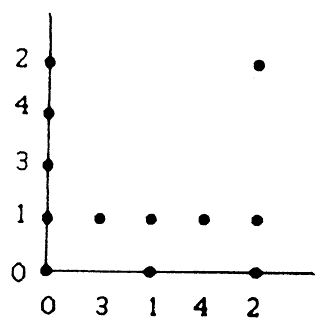
5. Multivariate rational Hermite interpolation problems.

For the sake of simplicity we restrict ourselves in the sequel of the text to the case of two variables. The generalization to the case of more than two variables will appear to be straightforward and only notationally more difficult. Let us first describe the conditions which have to be fulfilled by the multivariate data set before the interpolants can be constructed. Since we allow coalescence of interpolation points, we shall also point out how to deal with such a situation.

Consider for instance the following picture in $\mathbb{N}^2$ of the data set (x_i, y_j), where a circle indicates that in addition to $f_{ij} = f(x_i, y_j)$ also $\partial f/\partial x$ is given and a square indicates that also $\partial f/\partial x$, $\partial f/\partial y$ and $\partial^2 f/\partial y^2$ are provided.



This situation can be considered as the limit situation of a data set with non-coalescent interpolation points where we let $x_3 \to x_0$, $x_4 \to x_1$, $y_3 \to y_1$ and $y_4 \to y_1$.



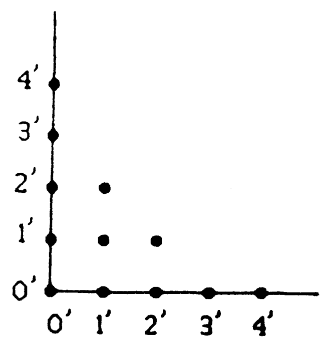
If we want to interpolate these (x_i, y_j, f_{ij}) by means of the techniques described below, then the data f_{ij} and the numbering of the x_i and y_j have to be such that

(a) x_0 is that x-coordinate for which the number of y-coordinates at which data are given is maximal, x_1 is the one of the leftover points for which the same is true, and so on

(b) y_0 is that y-coordinate for which the number of x-coordinates at which data are given is maximal, y_1 is the one of the leftover points for which the same is true, and so on

(c) the data set has the inclusion property, meaning that when a point belongs to the data set then the rectangular subset of points emanating from the origin with the given point as its furthermost corner also lies in the data set.

Note that (a) and (b) do not necessarily imply (c). We shall comment on the importance of condition (c) further on. For the picture above (c) is clearly not satisfied. So we try to renumber the interpolation points such that these three conditions are fulfilled. Let us introduce a new numbering (x'_i, y'_j) with $x'_0 = x_0, x'_1 = x_2, x'_2 = x_1, x'_3 = x_4, x'_4 = x_3$ and $y'_0 = y_1, y'_1 = y_0, y'_2 = y_2, y'_3 = y_4, y'_4 = y_3$. We then get the following picture in $I\!N^2$ of the data set.



The interpolation problems that can be reduced to this situation are of course not the most general ones but they already represent quite a number of situations that can be dealt with. In the sections 5–7 we assume that the given data set is structured such that the conditions (a–c) are fulfilled.

Let the complex function values f_{ij} be given in the complex points (x_i, y_j) with $(i,j) \in I \subseteq I\!N^2$, where I satisfies the inclusion property or rectangle rule, meaning that when (i,j) belongs to I then (k,ℓ) belongs to I for $k \leq i$ and $\ell \leq j$. We know from the pictures above that a data set with coalescent interpolation points can be replaced by an intermediate data set where only function values are given. When certain interpolation points coincide, we must bear in mind that due to the renumbering these coalescent x- and y-coordinates are not necessarily consecutive. With the given interpolation points we define the following polynomial basis functions:

$$B_{ij}(x,y) = \prod_{k=1}^{i}(x - x_{k-1}) \prod_{\ell=1}^{j}(y - y_{\ell-1})$$

These basis functions are bivariate polynomials of degree $i+j$. With

$$c_{0i,0j} = f[x_0,\ldots,x_i][y_0,\ldots,y_j]$$

where coalescence of points in the divided difference is admitted [CUYT], we can now write in a purely formal manner [BERE]

$$f(x,y) = \sum_{(i,j)\in \mathbb{N}^2} c_{0i,0j} B_{ij}(x,y) \tag{6}$$

Hence we have constructed with the data a bivariate Newton interpolating series and we can start approximating it using bivariate rational functions. For the bivariate divided differences a Leibniz type product rule remains valid and will prove to be useful in the sequel:

$$(fq)[x_0,\ldots,x_i][y_0,\ldots,y_j] = \sum_{\mu=0}^{i}\sum_{\nu=0}^{j} f[x_0,\ldots,x_\mu][y_0,\ldots,y_\nu]q[x_\mu,\ldots,x_i][y_\nu,\ldots,y_j]$$

The definition of multivariate Newton-Padé approximant which we shall give is a very general one. It includes the univariate definition and a lot of the definitions for multivariate Padé approximants as a special case. With any finite subset D of $\mathbb{N}^2$ we associate a polynomial of which the coefficients and the basisfunctions are indexed by the indices in D. Given the double Newton series, we choose three subsets N, D and I of $\mathbb{N}^2$ and construct an $[N/D]_I$ Newton-Padé approximant to $f(x,y)$ as follows:

$$p(x,y) = \sum_{(i,j)\in N} a_{ij} B_{ij}(x,y) \tag{7a}$$

$$q(x,y) = \sum_{(i,j)\in D} b_{ij} B_{ij}(x,y) \tag{7b}$$

$$(fq-p)(x,y) = \sum_{(i,j)\in \mathbb{N}^2\setminus I} d_{ij} B_{ij}(x,y) \tag{7c}$$

In analogy with the univariate case, we select N, D and I such that

D has $m+1$ elements, numbered $(d_0, e_0), \dots, (d_m, e_m)$
$N \subset I$
I satisfies the rectangle rule
$\#(I \backslash N) = m$.
We will denote $\partial p = N$ and $\partial q = D$. Clearly condition (7c) is equivalent with

$$d_{ij} = (fq - p)[x_0, \dots, x_i][y_0, \dots, y_j] = 0 \qquad (i,j) \in I \tag{8}$$

Because $N \subset I$, the system of equations (8) can be divided into a non-homogeneous and a homogeneous part:

$$(fq)[x_0, \dots, x_i][y_0, \dots, y_j] = p[x_0, \dots, x_i][y_0, \dots, y_j] = a_{ij} \qquad (i,j) \in N \tag{9a}$$

$$(fq)[x_0, \dots, x_i][y_0, \dots, y_j] = 0 \qquad (i,j) \in I \backslash N \tag{9b}$$

Let's take a look at the conditions (9b). Suppose that I is such that the m homogeneous equations in (9b) are linearly independent and let us number the m elements in $I \backslash N$ indexing these equations by $(i_{n+1}, j_{n+1}), \dots, (i_{n+m}, j_{n+m})$. By means of the Leibniz rule the homogeneous system (9b) of m equations in $m+1$ unknowns looks like

$$\begin{pmatrix} c_{d_0 i_{n+1}, e_0 j_{n+1}} & \cdots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \cdots & c_{d_m i_{n+m}, e_m j_{n+m}} \end{pmatrix} \begin{pmatrix} b_{d_0, e_0} \\ \vdots \\ b_{d_m, e_m} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{10}$$

As we suppose the rank of the coefficient matrix to be maximal, a solution $q(x,y)$ is given by

$$q(x,y) = \begin{vmatrix} B_{d_0 e_0}(x,y) & \cdots & B_{d_m e_m}(x,y) \\ c_{d_0 i_{n+1}, e_0 j_{n+1}} & \cdots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \cdots & c_{d_m i_{n+m}, e_m j_{n+m}} \end{vmatrix} \tag{11a}$$

By the conditions (9a) we find as determinant representation for $p(x,y)$

$$p(x,y) = \begin{vmatrix} \sum_{(i,j) \in N} c_{d_0 i, e_0 j} B_{ij}(x,y) & \cdots & \sum_{(i,j) \in N} c_{d_m i, e_m j} B_{ij}(x,y) \\ c_{d_0 i_{n+1}, e_0 j_{n+1}} & \cdots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \cdots & c_{d_m i_{n+m}, e_m j_{n+m}} \end{vmatrix} \tag{11b}$$

If for all $k, \ell \geq 0$ we have $q(x_k, y_\ell) \neq 0$ then with $e_{ij} = (1/q)[x_0, \dots, x_i][y_0, \dots, y_j]$ and I satisfying the inclusion property

$$(f - \frac{p}{q})(x,y) = \left[\frac{1}{q}(fq - p)\right](x,y) = \sum_{(i,j) \in \mathbb{N}^2 \backslash I} \tilde{d}_{ij} B_{ij}(x,y)$$

If I does not satisfy the inclusion property then

$$(fq-p)(x,y) = \sum_{(i,j)\in \mathbb{N}^2 \setminus I} d_{ij} B_{ij}(x,y)$$

does not imply

$$(f-\frac{p}{q})(x,y) = \sum_{(i,j)\in \mathbb{N}^2 \setminus I} \tilde{d}_{ij} B_{ij}(x,y)$$

since in that case $f - p/q$ also contains the terms that result from multiplying a "hole" in I by $(1/q)(x,y)$ [CUYTo]. From the determinant representations (11a) and (11b) we can easily obtain the determinant representation given in section 2 for univariate Newton-Padé approximants as a special case.

6. Methods for the computation of nondegenerate MRI.

In this section we continue to assume that the m equations in the homogeneous system (10) are linearly independent. Then the multivariate rational Hermite interpolation problem is called nondegenerate.

6.1. The E-algorithm.

Let us now introduce a numbering $r(i,j)$ of the points in $\mathbb{N}^2$, for instance the enumeration

$$(0,0),\ \underbrace{(1,0),(0,1),}_{\text{first diagonal}}\ \underbrace{(2,0),(1,1),(0,2),}_{\text{second diagonal}}\underbrace{(3,0),(2,1),(1,2),(0,3)}_{\text{third diagonal}},\ldots$$

and retain this order in N, D and I. If we denote $\#N = n+1$ then we can write

$$N = \bigcup_{k=0}^{n} N_k$$

with

$$\begin{gathered}\emptyset = N_{-1} \subset N_0 \subset N_1 \subset \ldots \subset N_{n-1} \subset N_n = N \\ \#N_k = k+1 \\ N_k \setminus N_{k-1} = \{(i_k,j_k)\} \qquad k = 0,1,\ldots,n \\ r(i_k,j_k) > r(i_s,j_s) \qquad k > s\end{gathered}$$

In other words, for each $k = 0,\ldots,n$ we add to N_{k-1} the point (i_k,j_k) which is the next in line in $N \cap \mathbb{N}^2$ according to the enumeration given above. Denote $\#D = m+1$ and proceed in the same way. Hence

$$D = \bigcup_{\ell=0}^{m} D_\ell$$

with

$$D_{-1} = \emptyset \qquad D_\ell \setminus D_{\ell-1} = \{(d_\ell, e_\ell)\} \qquad \ell = 0, \ldots, m$$

We have assumed that the interpolation set I is such that the m homogeneous equations are linearly independent and hence we write for $I \setminus N$

$$I \setminus N = I_{n+1,n+m} = \bigcup_{\ell=1}^{m} I_{n+1,n+\ell}$$

with

$$I_{n+1,n} = \emptyset \qquad I_{n+1,n+\ell} \setminus I_{n+1,n+\ell-1} = \{(i_{n+\ell}, j_{n+\ell})\} \qquad \ell = 1, \ldots, m$$

To obtain a recursive algorithm, the determinant formulas (11) for the polynomials $p(x,y)$ and $q(x,y)$ are rewritten as follows. Multiply the $(\ell+1)^{th}$ row in $p(x,y)$ and $q(x,y)$ by $B_{i_{n+\ell}j_{n+\ell}}(x,y)$ $(\ell = 1, \ldots, m)$, and then divide the $(\ell+1)^{th}$ column by $B_{d_\ell e_\ell}(x,y)$ $(\ell = 0, \ldots, m)$. This respectively results for numerator and denominator in

$$\begin{vmatrix} \sum_{(i,j)\in N} c_{d_0 i, e_0 j} B_{d_0 i, e_0 j}(x,y) & \ldots & \sum_{(i,j)\in N} c_{d_m i, e_m j} B_{d_m i, e_m j}(x,y) \\ c_{d_0 i_{n+1}, e_0 j_{n+1}} B_{d_0 i_{n+1}, e_0 j_{n+1}}(x,y) & \ldots & c_{d_m i_{n+1}, e_m j_{n+1}} B_{d_m i_{n+1}, e_m j_{n+1}}(x,y) \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} B_{d_0 i_{n+m}, e_0 j_{n+m}}(x,y) & \ldots & c_{d_m i_{n+m}, e_m j_{n+m}} B_{d_m i_{n+m}, e_m j_{n+m}}(x,y) \end{vmatrix}$$

and

$$\begin{vmatrix} 1 & \ldots & 1 \\ c_{d_0 i_{n+1}, e_0 j_{n+1}} B_{d_0 i_{n+1}, e_0 j_{n+1}}(x,y) & \ldots & c_{d_m i_{n+1}, e_m j_{n+1}} B_{d_m i_{n+1}, e_m j_{n+1}}(x,y) \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} B_{d_0 i_{n+m}, e_0 j_{n+m}}(x,y) & \ldots & c_{d_m i_{n+m}, e_m j_{n+m}} B_{d_m i_{n+m}, e_m j_{n+m}}(x,y) \end{vmatrix}$$

where for $k \le i$ and $\ell \le j$

$$B_{ki,\ell j}(x,y) = \frac{B_{ij}(x,y)}{B_{k\ell}(x,y)} = (x - x_k) \ldots (x - x_{i-1})(y - y_\ell) \ldots (y - y_{j-1})$$

and for $k > i$ or $\ell > j$, $c_{ki,\ell j} = 0$. We can now easily construct $(m+1)$ series of which the successive partial sums can be found in the columns of $p(x,y)$. Take

$$t_0(n) = \sum_{(i,j)\in N} c_{d_0 i, e_0 j} B_{d_0 i, e_0 j}(x,y)$$

and

$$\begin{aligned}\Delta t_0(n+\ell-1) &= t_0(n+\ell) - t_0(n+\ell-1)\\ &= c_{d_0 i_{n+\ell}, e_0 j_{n+\ell}} B_{d_0 i_{n+\ell}, e_0 j_{n+\ell}}(x,y) \qquad \ell = 1,\ldots,m\end{aligned}$$

for the first column of $p(x,y)$. Define for $r = 1,\ldots,m$

$$t_r(n) = \sum_{(i,j)\in N} c_{d_r i, e_r j} B_{d_r i, e_r j}(x,y)$$

and

$$\begin{aligned}\Delta t_r(n+\ell-1) &= t_r(n+\ell) - t_r(n+\ell-1)\\ &= c_{d_r i_{n+\ell}, e_r j_{n+\ell}} B_{d_r i_{n+\ell}, e_r j_{n+\ell}}(x,y) \qquad \ell = 1,\ldots,m\end{aligned}$$

for the $(r+1)^{th}$ column of $p(x,y)$. Consequently

$$p(x,y) = \begin{vmatrix} t_0(n) & \ldots & t_m(n) \\ \Delta t_0(n) & \ldots & \Delta t_m(n) \\ \vdots & & \vdots \\ \Delta t_0(n+m-1) & \ldots & \Delta t_m(n+m-1) \end{vmatrix} \tag{12a}$$

$$q(x,y) = \begin{vmatrix} 1 & \ldots & 1 \\ \Delta t_0(n) & \ldots & \Delta t_m(n) \\ \vdots & & \vdots \\ \Delta t_0(n+m-1) & \ldots & \Delta t_m(n+m-1) \end{vmatrix} \tag{12b}$$

This quotient of determinants can easily be computed using the E-algorithm [BREZb]:

$$\begin{aligned}
E_0^{(k)} &= t_0(k) \qquad k = 0,\ldots,n+m\\
g_{0,\ell}^{(k)} &= t_\ell(k) - t_{\ell-1}(k) \qquad \ell = 1,\ldots,m \qquad k = 0,\ldots,n+m\\
E_\ell^{(k)} &= \frac{E_{\ell-1}^{(k)} g_{\ell-1,\ell}^{(k+1)} - E_{\ell-1}^{(k+1)} g_{\ell-1,\ell}^{(k)}}{g_{\ell-1,\ell}^{(k+1)} - g_{\ell-1,\ell}^{(k)}} \qquad k = 0,1,\ldots,n \qquad \ell = 1,2,\ldots,m \qquad (13a)\\
g_{\ell,s}^{(k)} &= \frac{g_{\ell-1,s}^{(k)} g_{\ell-1,\ell}^{(k+1)} - g_{\ell-1,s}^{(k+1)} g_{\ell-1,\ell}^{(k)}}{g_{\ell-1,\ell}^{(k+1)} - g_{\ell-1,\ell}^{(k)}} \qquad s = \ell+1, \ell+2, \ldots \qquad (13b)
\end{aligned}$$

The values $E_\ell^{(k)}$ and $g_{\ell,s}^{(k)}$ are stored as in [BREZb]. We obtain $[N/D]_I = E_m^{(n)}$. Since the solution $q(x,y)$ of (7c) is unique, the value $E_m^{(n)}$ itself does not depend upon the numbering of the points within the sets N, D and H. But this numbering affects the interpolation conditions satisfied by the intermediate E-values [CUYTn].

THEOREM 10:
For $k = 0, \ldots, n$ and $\ell = 0, \ldots, m$

$$E_\ell^{(k)} = [N_k/D_\ell]_{N_k \cup I_{k+1,k+\ell}}$$

6.2. The qdg-algorithm.

If we suppose that the homogeneous system of equations (10) has maximal rank we can also write

$$I = \bigcup_{\ell=0}^{n+m} I_\ell$$

with

$$\begin{aligned} I_k &= N_k \qquad k = 0, \ldots, n \\ I_{n+\ell} \backslash I_{n+\ell-1} &= \{(i_{n+\ell}, j_{n+\ell})\} \qquad \ell = 1, \ldots, m \\ r(i_{n+\ell}, j_{n+\ell}) &> r(i_s, j_s) \qquad n + \ell > s \geq n + 1 \end{aligned}$$

With the subsets N_k, D_ℓ and $I_{k+\ell}$ rational interpolants $[N_k/D_\ell]_{I_{k+\ell}}$ can be constructed which satisfy only part of the interpolation conditions and which are of lower "degree". To this end we assume that the numbering $r(i_\ell, j_\ell)$ of the points in $\mathbb{N}^2$ is such that the inclusion property of the set I is carried over to the subsets I_ℓ. With these functions we can fill up a table of rational interpolants :

$$\begin{array}{cccc} [N_0/D_0]_{I_0} & [N_0/D_1]_{I_1} & [N_0/D_2]_{I_2} & \ldots \\ [N_1/D_0]_{I_1} & [N_1/D_1]_{I_2} & [N_1/D_2]_{I_3} & \ldots \\ [N_2/D_0]_{I_2} & [N_2/D_1]_{I_3} & [N_2/D_2]_{I_4} & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

where $[N/D]_I = [N_n/D_m]_{I_{n+m}}$. Our aim is to consider descending staircases of multivariate rational interpolants

$$\begin{array}{llll} [N_s/D_0]_{I_s} & & & \\ [N_{s+1}/D_0]_{I_{s+1}} & [N_{s+1}/D_1]_{I_{s+2}} & & \\ & [N_{s+2}/D_1]_{I_{s+3}} & [N_{s+2}/D_2]_{I_{s+4}} & \\ & & \vdots & \ldots \end{array} \tag{14}$$

and construct continued fractions of which the successive convergents equal the successive interpolants on the staircase. We restrict ourselves to the case where every three subsequent elements on the staircase are different. It was proved in [CUYTm] that given such a descending staircase, it is possible to construct a continued fraction of the form

$$C_s(x,y) = [N_s/D_0]_{I_s} + \frac{[N_{s+1}/D_0]_{I_{s+1}} - [N_s/D_0]_{I_s} \rvert}{\lvert 1} + \frac{-q_1^{(s+1)} \rvert}{\lvert 1+q_1^{(s+1)}} + \frac{-e_1^{(s+1)} \rvert}{\lvert 1+e_1^{(s+1)}} + \frac{-q_2^{(s+1)} \rvert}{\lvert 1+q_2^{(s+1)}} + \frac{-e_2^{(s+1)} \rvert}{\lvert 1+e_2^{(s+1)}} + \dots \tag{15}$$

with this property. Here

$$[N_s/D_0]_{I_s} = \sum_{(i,j)\in N_s} c_{d_0 i, e_0 j} B_{d_0 i, e_0 j}(x,y)$$

$$[N_{s+1}/D_0]_{I_{s+1}} = \sum_{(i,j)\in N_{s+1}} c_{d_0 i, e_0 j} B_{d_0 i, e_0 j}(x,y)$$

and the coefficients $q_\ell^{(s+1)}$ and $e_\ell^{(s+1)}$ are computed using the following rules: for $\ell \geq 2$

$$q_\ell^{(s+1)} = \frac{e_{\ell-1}^{(s+2)} q_{\ell-1}^{(s+2)}}{e_{\ell-1}^{(s+1)}} \quad \frac{g_{\ell-2,\ell-1}^{(s+\ell-1)} - g_{\ell-2,\ell-1}^{(s+\ell)}}{g_{\ell-2,\ell-1}^{(s+\ell-1)}} \quad \frac{g_{\ell-1,\ell}^{(s+\ell)}}{g_{\ell-1,\ell}^{(s+\ell)} - g_{\ell-1,\ell}^{(s+\ell+1)}} \tag{16a}$$

and for $\ell \geq 1$

$$e_\ell^{(s+1)} + 1 = \frac{g_{\ell-1,\ell}^{(s+\ell)} - g_{\ell-1,\ell}^{(s+\ell+1)}}{g_{\ell-1,\ell}^{(s+\ell)}} \left(q_\ell^{(s+2)} + 1\right) \tag{16b}$$

If we arrange the values $q_\ell^{(s+1)}$ and $e_\ell^{(s+1)}$ in a table as follows

$$\begin{array}{ccccc}
q_1^{(1)} & & & & \\
 & e_1^{(1)} & & & \\
q_1^{(2)} & & q_2^{(1)} & & \\
 & e_1^{(2)} & & e_2^{(1)} & \\
q_1^{(3)} & & q_2^{(2)} & & \ddots \\
 & e_1^{(3)} & & e_2^{(2)} & \\
q_1^{(4)} & & q_2^{(3)} & & \ddots \\
\vdots & e_1^{(4)} & \vdots & e_2^{(3)} & \\
 & \vdots & & \vdots &
\end{array}$$

where subscripts indicate columns and superscripts indicate downward sloping diagonals, then (16a) links the elements in the rhombus

$$\begin{array}{ccc} & e_{\ell-1}^{(s+1)} & \\ q_{\ell-1}^{(s+2)} & & q_{\ell}^{(s+1)} \\ & e_{\ell-1}^{(s+2)} & \end{array}$$

and (16b) links two elements on an upward sloping diagonal

$$\begin{array}{cc} & e_{\ell}^{(s+1)} \\ q_{\ell}^{(s+2)} & \end{array}$$

If starting values for $q_{\ell}^{(s+1)}$ were known, all the values could be computed. These starting values are given by

$$q_1^{(s+1)} = \frac{E_1^{(s+1)} - E_0^{(s+1)}}{E_0^{(s+1)} - E_0^{(s)}} = \frac{\Delta t_0(s+1)}{\Delta t_0(s)} \quad \frac{g_{0,1}^{(s+1)}}{g_{0,1}^{(s+1)} - g_{0,1}^{(s+2)}} \tag{16c}$$

7. Structure of a degenerate table of MRI.

If the rank of the defining system of equations (10) is not maximal, we should look at $[N/D]_I$ as being a set of rational functions of which the numerator and denominator are given by (7a-b) and are satisfying (7c). A solution $[N/D]_I$ containing numerators and denominators of different "degrees" is called "degenerate". Let us denote the coefficient matrix of (10) by $C_{n+1,n+m}$. Note that the rows in $C_{n+1,n+m}$ are indexed by $I_{n+1,n+m}$.

In the univariate case and under certain conditions, the table of minimal solutions of the rational interpolation problem consists of triangles, once the numbering of the interpolation points is fixed [CLAEf]. The size of the triangles, as pointed out in section 3, is related to the rank deficiency of the interpolation problem. We shall now give a similar multivariate theorem and point out the differences with the univariate version. From this discussion it will also become clear why different solutions of the same rational interpolation problem are not necessarily equivalent anymore and hence not providing a unique irreducible form.

Theorem 11:
Let $p(x,y)$ and $q(x,y)$ be defined by (7). Let the rank of $C_{n+1,n+m}$ in (10) be given by $m-t$. Then for each pair (k,ℓ) with $0 \le k \le t$, $0 \le \ell \le t$, $k+\ell = t$ and the rank of $C_{n-k+1,n+m-t}$ equal to $m-\ell$ the following holds.

(a) *For $0 \le i$, $0 \le j$ and $i+j \le t$, $[N_{n-k}/D_{m-\ell}]_{I_{n+m-t}}$ belongs to the solution set $[N_{n-k+i}/D_{m-\ell+j}]_{I_{n+m-t+i+j}}$, meaning that the (up to a multiplicative constant factor) unique rational function $[N_{n-k}/D_{m-\ell}]_{I_{n+m-t}}$ also solves the interpolation problems posed in $[N_{n-k+i}/D_{m-\ell+j}]_{I_{n+m-t+i+j}}$ where the solution set $[N_{n-k+i}/D_{m-\ell+j}]_{I_{n+m-t+i+j}}$ lies in the triangle of the table of rational interpolants with corner elements $[N_{n-k}/D_{m-\ell}]_{I_{n+m-t}}$, $[N_{n-k}/D_{m+k}]_{I_{n+m}}$ and $[N_{n+\ell}/D_{m-\ell}]_{I_{n+m}}$.*

(b) *If the solution $[N_{n-k}/D_{m-\ell}]_{I_{n+m-t}} = (p/q)(x,y)$ is such that $\partial p = N_{n-k-s_1}$ with $s_1 > 0$, then under the condition that the rank of $C_{n-k-s_1+1,n+m-t-s_1}$ is $m-\ell$, $[N_{n-k-s_1}/D_{m-\ell}]_{I_{n+m-t-s_1}}$ also solves $[N_{n-k-s_1+i}/D_{m-\ell+j}]_{I_{n+m-t-s_1+i+j}}$ for $0 \le i$, $0 \le j$ and $i+j \le t+s_1$.*

(c) *If the solution $[N_{n-k}/D_{m-\ell}]_{I_{n+m-t}} = (p/q)(x,y)$ is such that $\partial q = D_{m-\ell-s_2}$ with $s_2 > 0$, then under the condition that the rank of $C_{n-k+1,n+m-t-s_2}$ is $m-\ell-s_2$, $[N_{n-k}/D_{m-\ell-s_2}]_{I_{n+m-t-s_2}}$ also solves $[N_{n-k+i}/D_{m-\ell-s_2+j}]_{I_{n+m-t-s_2+i+j}}$ for $0 \le i$, $0 \le j$ and $i+j \le t+s_2$.*

(d) *If the solution $[N_{n-k}/D_{m-\ell}]_{I_{n+m-t}} = (p/q)(x,y)$ is such that*

$$(fq-p)(x,y) = \sum_{(i,j)\in \mathbb{N}^2 \setminus I_{n+m+t_3}} d_{ij} B_{ij}(x,y)$$

with $t_3 > 0$, then $[N_{n-k}/D_{m-\ell}]_{I_{n+m-t}}$ also solves $[N_{n-k+i}/D_{m-\ell+j}]_{I_{n+m-t+i+j}}$ where $0 \le i$, $0 \le j$ and $i+j \le t+t_3$.

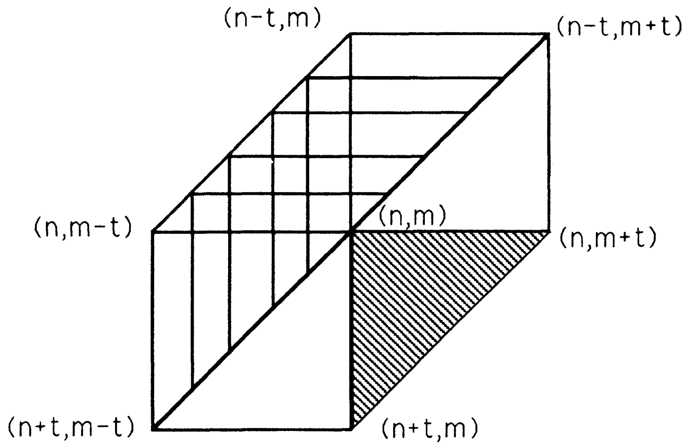
(e) *If the solution $[N_{n-k}/D_{m-\ell}]_{I_{n+m-t}} = (p/q)(x,y)$ is such that $\partial p = N_{n-k}$, $\partial q = D_{m-\ell}$ and*

$$(fq-p)(x,y) = \sum_{(i,j)\in \mathbb{N}^2 \setminus I_{n+m}} d_{ij} B_{ij}(x,y)$$

with $d_{i_{n+m+1}j_{n+m+1}} \ne 0$ then $[N_{n-k}/D_{m-\ell}]_{I_{n+m-t}} \in [N_i/D_j]_{I_{i+j}}$ if and only if (i,j) belongs to the triangle with corner elements $(n-k, m-\ell)$, $(n+\ell, m-\ell)$ and $(n-k, m+k)$.

(f) *For $i \ge 0$, $j \ge 0$ and $i+j \le t$:*

$$\bigcap_{(i,j)} [N_{n+i}/D_{m+j}]_{I_{n+m+i+j}} \ne \emptyset$$



(g) Let $0 \leq k \leq t$ and the rank of $C_{n-k+1,n+m-t}$ be equal to $m-t+k$:

$$\bigcap_{j=0}^{t} [N_{n-k}/D_{m+j}]_{I_{n-k+m+j}} \neq \emptyset$$

$$\bigcap_{i=0}^{t} [N_{n+i}/D_{m-k}]_{I_{n+i+m-k}} \neq \emptyset$$

Let us now point out some differences between this theorem and its univariate counterpart in [CLAEf]. First of all, it is important to note that both the univariate and the multivariate theorem are proved under the same conditions. With the rank of $C_{n+1,n+m}$ equal to $m-t$, we are able in both cases to construct solutions p_1, q_1 of $[N_{n-t}/D_m]_{I_{n+m-t}}$ and p_2, q_2 of $[N_n/D_{m-t}]_{I_{n+m-t}}$ that are also contained in $[N_n/D_m]_{I_{n+m}}$. We have

$$\begin{aligned}(p_1q_2 - p_2q_1)(x,y) &= [q_1(fq_2 - p_2) - q_2(fq_1 - p_1)](x,y) \\ &= q_1(x,y) \sum_{(i,j)\in \mathbb{N}^2 \setminus I_{n+m}} d_{ij}^{(2)} B_{ij}(x,y) - q_2(x,y) \sum_{(i,j)\in \mathbb{N}^2 \setminus I_{n+m}} d_{ij}^{(1)} B_{ij}(x,y)\end{aligned}$$

from which we can conclude that $(p_1q_2 - q_1p_2)(x_i, y_j) = 0$ for all $(i,j) \in I_{n+m}$ with I_{n+m} satisfying the inclusion property. We also have

$$\partial(p_1q_2 - q_1p_2) = \{(i,j) = (r,s) + (t,u) \mid (r,s) \in N_n, (t,u) \in D_m\}$$

However, since we do not always have that

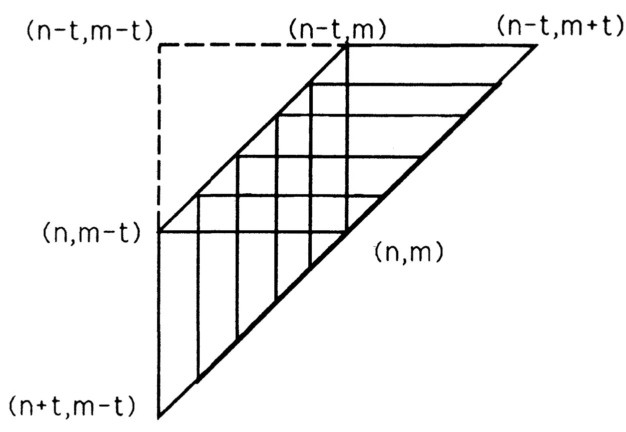
$$\partial(p_1q_2 - q_1p_2) \subset I_{n+m}$$

we cannot conclude that $(p_1 q_2 - q_1 p_2)(x, y)=0$ and hence we cannot prove as in [CLAEf] that it is also possible to construct a solution p_3, q_3 of $[N_n/D_m]_{I_{n+m}}$ with $\partial p_3 \subset N_{n-t}$ ánd $\partial q_3 \subset D_{m-t}$. In the univariate case however

$$N_n = \{(i, 0) \mid 0 \leq i \leq n\}$$
$$D_m = \{(j, 0) \mid 0 \leq j \leq m\}$$
$$\{(i + j, 0) \mid i \in N_n, j \in D_m\} \subseteq I_{n+m} = \{(k, 0) \mid 0 \leq k \leq n + m\}$$

and hence $p_1 q_2 = p_2 q_1$. Consequently in the univariate case the configuration described in theorem 11 can be enlarged with the triangle with corner elements $[N_{n-t}/D_{m-t}]_{I_{n+m-2t}}$, $[N_{n-1}/D_{m-t}]_{I_{n+m-t-1}}$ and $[N_{n-t}/D_{m-1}]_{I_{n+m-t-1}}$ resulting in the configuration described in section 3.

How is theorem 11 to be understood as a generalization of theorem 6? Clearly minimal solutions aren't uniquely determined anymore. In theorem 11 all solutions of the $(n - k, m - \ell)$ rational Hermite interpolation problem with $k + \ell = t$ are "minimal" in the sense that they use a minimal number of parameters and data to solve the (n, m) rational interpolation problem. Now each of the minimal solutions on the $(n + m - t)^{th}$ diagonal (with the restriction that the numerator and denominator "degree" must be less than or equal to n and m respectively) give rise to a triangular structure in the table. There's a whole triangle of rational interpolation problems that is solved by each minimal solution from the $(n + m - t)^{th}$ diagonal.



What's more, in the multivariate case a rational Hermite interpolation problem can have both a true irreducible minimal solution, a reducible minimal solution and a minimal solution with unattainable points. Note that in the multivariate case the solution must not be reducible in order to have unattainable interpolation points. This is a situation which is essentially different from the univariate one. In the univariate case theorem 6a and 6b never apply simultaneously [CLAEf] while this can be true in the multivariate case. The solution of $[N_n/D_m]$ common to all solution sets $[N_{n+i}/D_{m+j}]_{I_{n+m+i+j}}$ as described in theorem 11f could be called the "optimal solution" in the sense that it satisfies as many conditions as possible. If the rank of $C_{n+1,n+m+t}$ is still not maximal even more conditions can be added. So in the rational interpolation table a triangle emanating from $[N_n/D_m]_{I_{n+m}}$ can be filled with the optimal solution, while triangles emanating from $[N_{n-k}/D_{m-\ell}]_{I_{n+m-t}}$ with $k+\ell=t$ can be filled with minimal solutions. The rest of the hexagon is filled with the solutions constructed in the proof of theorem 11g.

7.1. Singular rules for the E-algorithm.

Let us introduce some new ratios of determinants. Let $E_{\ell,t}^{(k,u)}$ denote

$$E_{\ell,t}^{(k,u)} = \frac{\begin{vmatrix} t_0(k) & \dots & t_0(u-t) & t_0(u+1) & \dots & t_0(k+\ell+t) \\ \delta t_0(k) & \dots & & & & \\ \vdots & & & & & \\ \delta t_{\ell-1}(k) & \dots & & & & \end{vmatrix}}{\begin{vmatrix} 1 & \dots & 1 \\ \delta t_0(k) & \dots & \\ \vdots & & \\ \delta t_{\ell-1}(k) & \dots & \end{vmatrix}} \tag{17}$$

with

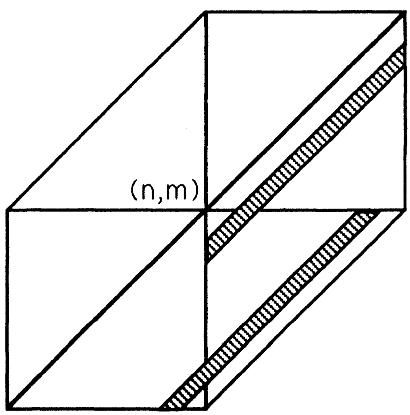
$$\delta t_j(i) = t_{j+1}(i) - t_j(i) \qquad j \geq 0 \qquad\qquad t_j(i) = 0 \qquad i < 0$$

These $E_{\ell,t}^{(k,u)}$ strongly resemble the E-values of the previous section (the classical values are obtained for $t=0$ and for $u \geq k+\ell+t$) and for fixed t and u they can be calculated recursively like the E-values [CUYTb] but now using help-entries

$$g_{h,\ell,t}^{(k,u)} = \frac{\begin{vmatrix} \delta t_\ell(k) & \dots & \delta t_\ell(u-t) & \delta t_\ell(u+1) & \dots & \delta t_\ell(k+h+t) \\ \delta t_0(k) & \dots & & & & \\ \vdots & & & & & \\ \delta t_{h-1}(k) & \dots & & & & \end{vmatrix}}{\begin{vmatrix} 1 & \dots & 1 \\ \delta t_0(k) & \dots & \\ \vdots & & \\ \delta t_{h-1}(k) & \dots & \end{vmatrix}} \tag{18}$$

THEOREM 12:
Let $p(x,y)$ and $q(x,y)$ be defined by (7) with I satisfying the inclusion property. Let the rank of the coefficient matrix $C_{n+1,n+m}$ in (10) be given by $m-t$. Let for each pair (k,ℓ) with $0 \le k \le t$, $0 \le \ell \le t$, $k+\ell = t$, the rank of $C_{n-k+1,n+m-t}$ equal its maximal rank $m-\ell$. Let the hexagonal block of degenerate solutions be isolated, which means that for $0 \le k \le t$ the coefficient matrices $C_{n-t+1,n+m-t+k}$ (top row), $C_{n-k+1,n+m-t}$ (leftmost antidiagonal), $C_{n+k+1,n+m-t+k}$ (leftmost column), $C_{n+t+1,n+m+k}$ (bottom row), $C_{n+t-k+1,n+m+t}$ (rightmost antidiagonal) and finally $C_{n-t+k+1,n+m+k}$ (rightmost column) all have maximal rank.
Then for $i = 1,\dots,t-1$ the following can be proved.

(a) *$E_{m,t}^{(n-t+i,n+m)}$ is well-defined and solves $[N_{n-t+i}/D_{m+t}]_{I_{n+m+i}}$.*
It also belongs to $[N_{n-t+i+k}/D_{m+t-k}]_{I_{n+m+i}}$ with $k = 0,\dots,t$, meaning that $E_{m,t}^{(n-t+i,n+m)}$ solving $[N_{n-t+i}/D_{m+t}]_{I_{n+m+i}}$ can be shifted downwards in the hexagonal block in the direction of the antidiagonal because it also solves the interpolation problems posed in $[N_{n-t+i+k}/D_{m+t-k}]_{I_{n+m+i}}$.



(b) *$E_{m-t+i,t}^{(n,n+m)}$ is well-defined and solves $[N_{n+t}/D_{m-t+i}]_{I_{n+m+i}}$.*

It also belongs to $[N_{n+t-k}/D_{m-t+i+k}]_{I_{n+m+i}}$ *with* $k = 0, \ldots, t$, *meaning that* $E^{(n,n+m)}_{m-t+i,t}$ *solving* $[N_{n+t}/D_{m-t+i}]_{I_{n+m+i}}$ *can be shifted upwards in the hexagonal block in the direction of the antidiagonal because it also solves the interpolation problems posed in* $[N_{n+t-k}/D_{m-t+i+k}]_{I_{n+m+i}}$.

(c) *On the rightmost upward sloping diagonal we have for* $i = 0, \ldots, t$: $[N_{n+t-i}/D_{m+i}]_{I_{n+m+t}} = E^{(n,n+m)}_{m,t}$.

Note that the theorem provides us with a solution in the rightmost column of the isolated hexagonal block, column $m + t$, in the form of a ratio of determinants of size $m + 1$, while the coefficient matrix $C_{n-t+i,n+m+i}$ is regular because the block is isolated, implying that its unique solution (up to a multiplicative constant) can also be represented as a ratio of determinants of size $m + t + 1$. From this we can conclude that $E^{(n-t+i,n+m)}_{m,t}$ and $E^{(n-t+i)}_{m+t}$ differ only in a common multiplicative factor in numerator and denominator. When we run across such an isolated singular hexagonal block we want to know the values on the edges of the block, because from there on we can take up the nonsingular rules again and proceed with our recursive scheme. Let's walk around the block and try to identify the rational interpolants on all the edges. Remember that $[N_\ell/D_k]_{I_{\ell+k}}$ denotes the complete set of solutions while $E^{(\ell)}_k$ or $E^{(\ell,u)}_{k,t}$ denote a particular value from that set.

First there's the upward sloping diagonal with regular entries $[N_{n-t+i}/D_{m-i}]_{I_{n+m-t}}$ because $C_{n-t+i+1,n+m-t}$ has maximal rank for all $i = 0, \ldots, t$. Then we proved in [ALLO] that for $i = 0, \ldots, t$ the value $E^{(n-t)}_m$ also solves the rational interpolation problems posed in $[N_{n-t}/D_{m+i}]_{I_{n+m-t+i}}$ and analogously for $E^{(n)}_{m-t}$ and $[N_{n+i}/D_{m-t}]_{I_{n+m-t+i}}$. So this deals with the top row and leftmost column of our isolated block. The values in the rightmost column and on the bottom line of the hexagonal block were just respectively identified as $E^{(n-t+i,n+m)}_{m,t}$ and $E^{(n,n+m)}_{m-t+i,t}$ with $i = 1, \ldots, t-1$. The closing rightmost upward sloping diagonal is filled with $E^{(n,n+m)}_{m,t}$.

Let us now discuss some particular solutions at the interior of the hexagon. It is essential when identifying certain rational interpolants that we present solutions which are well-defined, in other words which can be represented as definite E-values with nonzero denominator determinants. In theorem 11a we mentioned how to fill the left upper half of the hexagonal block with nonsingular E-values. Using theorem 12c the triangle emanating from (n, m) in the hexagon can be filled with the regular values from the rightmost upward sloping diagonal, which are all equal and which have the correct degrees N_n and D_m. We just learned how the rest of the right lower half of the hexagonal structure can be filled with regular E-values. From theorem 12a and 12b we see that well-defined solutions for the rational Hermite interpolation problems posed in this half of the hexagon come from copies of the rightmost column or copies of the bottom line. Essentially this leaves us with the problem of computing these new E-values $E^{(n-t+i,n+m)}_{m,t}$ and $E^{(n,n+m)}_{m-t+i,t}$. When trying to provide a coherent computation scheme we must be careful not to involve intermediate singular values. Using the initialisation

$$\begin{aligned}
E_{m-i,t}^{(n-t+i,n+m)} &= E_{m-i}^{(n-t+i)} && i = 1,\dots,t-1 \\
E_{m-i,t}^{(n,n+m)} &= E_{m-i}^{(n+t)} && i = 1,\dots,t-1 \\
g_{m-i,r,t}^{(n-t+i,n+m)} &= g_{m-i,r}^{(n-t+i)} && i = 0,\dots,t \\
g_{m-t+i,r,t}^{(n,n+m)} &= g_{m-t+i,r}^{(n+t)} && i = 1,\dots,t
\end{aligned} \tag{19a}$$

and the rules

$$E_{\ell,t}^{(k,n+m)} = \frac{E_{\ell-1,t}^{(k,n+m)} g_{\ell-1,\ell,t}^{(k+1,n+m)} - E_{\ell-1,t}^{(k+1,n+m)} g_{\ell-1,\ell,t}^{(k,n+m)}}{g_{\ell-1,\ell,t}^{(k+1,n+m)} - g_{\ell-1,\ell,t}^{(k,n+m)}}$$

$$k = 0,1,\dots,n \qquad \ell = 1,2,\dots,m$$

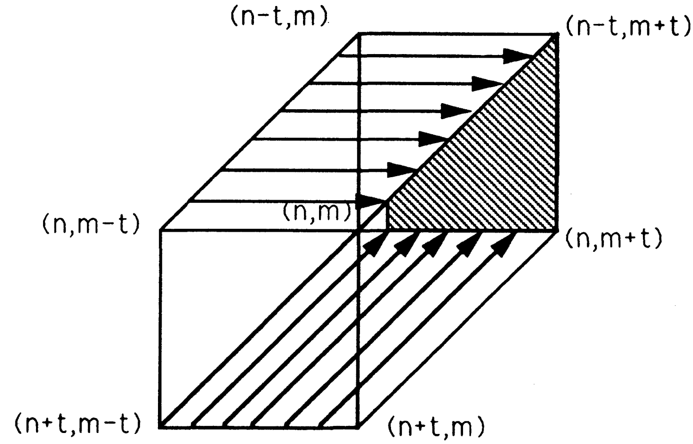
$$g_{h,\ell,t}^{(k,n+m)} = \frac{g_{h-1,\ell,t}^{(k,n+m)} g_{h-1,h,t}^{(k+1,n+m)} - g_{h-1,\ell,t}^{(k+1,n+m)} g_{h-1,h,t}^{(k,n+m)}}{g_{h-1,h,t}^{(k+1,n+m)} - g_{h-1,h,t}^{(k,n+m)}}$$

$$\ell = h+1, h+2, \dots$$

we can fill the following quasi-triangular table of values:

$$\begin{array}{ccccc}
 & & & E_{m-1,t}^{(n-t+1,n+m)} & E_{m,t}^{(n-t+1,n+m)} \\
 & & \cdot^{\cdot^{\cdot}} & E_{m-1,t}^{(n-t+1,n+m)} & E_{m,t}^{(n-t+1,n+m)} \\
 & E_{m-t+2,t}^{(n-2,n+m)} & & \vdots & \vdots \\
E_{m-t+1,t}^{(n-1,n+m)} & E_{m-t+2,t}^{(n-1,n+m)} & & & \\
E_{m-t+1,t}^{(n,n+m)} & E_{m-t+2,t}^{(n,n+m)} & \cdots & E_{m-1,t}^{(n,n+m)} & E_{m,t}^{(n,n+m)}
\end{array} \tag{19b}$$

where the bottom row and rightmost column of (19b) respectively solve the interpolation problems in the bottom row and rightmost column of our hexagon. A proof for these rules can be constructed as in [CUYTn] for the case $t = 0$. The quasi-triangular table (19b) can on its turn only be filled completely if we do not encounter indefinite values at the interior of (19b). But even if we are unfortunate we should not despair. In [ALLO] we describe a more general iterative procedure to deal with singularities. This section in fact only describes the first iteration step, which may suffice if the hexagonal block is isolated and everything goes well. The initialisations in (19a) are easy to understand. The first is merely by notation: from the determinant representations for $E_{m-i}^{(n-t+i)}$ and $E_{m-i,t}^{(n-t+i,n+m)}$ one can see that these expressions are equal. The second initialisation follows from theorem 12b. The initialisations for the g-values are analogous. The first is by notation, the other by theorem 12b. How do we now get the starting E- and g-values on the bottom row of the hexagon? The bottom row of (19b) is computed from the nondegenerate rules with input values $E_{m-t}^{(n+t)}$ and $E_{m-t}^{(n+t+1)}$. The newly described extension together with its initialisations can then best be understood from the picture below:



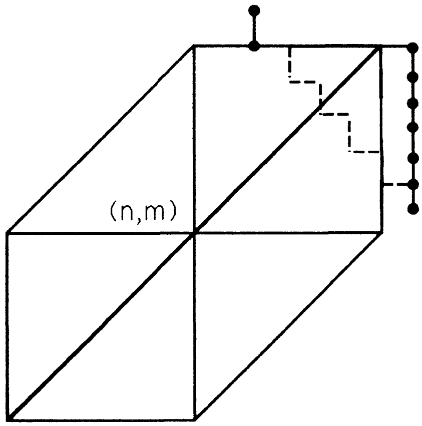
7.2. Singular rules for the qdg-algorithm.

It is clear that in the degenerate case, with the conditions of theorem 11 fullfilled, some of the continued fractions (15) are perturbed because the elements on the staircase (14) may not all be different. For a rank deficiency of t in $C_{n+1,n+m}$ it concerns the continued fractions $C_s(x,y)$ for $s = n-m-2t, \ldots, n-m+2t-1$. In this presentation the rank-deficiency t will be fixed because we focus our attention on a particular singular interpolation problem, namely $[N_n/D_m]_{I_{n+m}}$ with an isolated hexagonal block built around it. Hence we shall drop the index t in the notation of both $E_{k,t}^{(\ell,u)}$ and $g_{h,k,t}^{(\ell,u)}$. From now on when we refer to a particular element $[N_\ell/D_k]_{I_{\ell+k}}$ in the table, we only take the regular solutions in consideration. The hexagonal block of "size" t described above perturbs the partial numerators and denominators inside and on the border of the following octagonal structure from the qd-table:

$$
\begin{array}{ccccccc}
 & & & q_{m+1}^{(n-m-t)} \cdots q_{m+t}^{(n-m-2t+1)} & & & \\
 & & e_m^{(n-m-t+1)} & & e_{m+t}^{(n-m-2t+1)} & & \\
 & & & & & q_{m+t+1}^{(n-m-2t+1)} & \\
 & & & & & \vdots & \\
 & & \cdot^{\cdot^{\cdot}} & & & & \\
 & q_{m-t+1}^{(n-m+t)} & & & & & \\
e_{m-t}^{(n-m+t+1)} & & & & & q_{m+t+1}^{(n-m-t)} & \\
\vdots & & & & e_{m+t}^{(n-m-t+1)} & & \\
 & & & & & & \\
e_{m-t}^{(n-m+2t)} & & & & \cdot^{\cdot^{\cdot}} & & \\
 & q_{m-t+1}^{(n-m+2t)} & & q_{m+1}^{(n-m+t)} & & & \\
 & & e_{m-t+1}^{(n-m+2t)} \cdots e_m^{(n-m+t+1)} & & & &
\end{array}
\tag{20}
$$

In order to cope with the rank-deficient situation described in the theorem above, we introduce staircases T_s^* that coincide with T_s before entering the hexagonal block and after leaving it. The degenerate elements of T_s within the block are deleted and replaced by a number of nondegenerate elements, mostly from around the block. In general T_s^* is given by the following staircase. We distinguish between three cases. Let us use the shorter notation $[\frac{n}{m}]$ instead of $[N_n/D_m]_{I_{n+m}}$. The first case is where T_s enters the hexagonal block through the square in the right upper corner of the hexagon, in other words when s ranges from $n-m-2t$ to $n-m-t-1$:

$$
\begin{array}{llllll}
T_s^* = \{ & \left[\frac{s}{0}\right] & & & & \\
 & \left[\frac{s+1}{0}\right] & \cdots & & & \\
 & & \vdots & & & \\
 & & \left[\frac{n-t}{n-s-t-1}\right] & \left[\frac{n-t}{m+t+1}\right] & & \\
 & & & \left[\frac{n-t+1}{m+t+1}\right] & & \\
 & & & \vdots & & \\
 & & & \left[\frac{m+t+s+1}{m+t+1}\right] & & \\
 & & & \left[\frac{m+t+s+2}{m+t+1}\right] & \left[\frac{m+t+s+2}{m+t+2}\right] & \\
 & & & & \vdots & \}
\end{array}
\tag{21}
$$



The second case is where T_s enters the hexagonal block by crossing the upward sloping diagonal, in other words when s ranges from $n-m-t$ to $n-m+t-1$. The third case is where T_s enters the hexagonal block through the square in the left bottom corner of the hexagon, in other words when s ranges from $n-m+t$ to $n-m+2t-1$. Because of the similarities between the different cases only the first case will be treated in detail here. If ℓ new elements are introduced by working with T_s^*, $\ell+2$ coefficients in $C_s(x,y)$ are perturbed and new rules must be given for the partial numerators and denominators in the new continued fraction $C_s^*(x,y)$ associated with T_s^*. Before proceeding to the continued fraction representation $C_s^*(x,y)$, we introduce some new quantities. We define

$$v_{m+1}^{(n-m)} = \frac{E_m^{(n+1)} - E_m^{(n)}}{E_m^{(n)} - E_m^{(n-1)}}$$

which links 3 consecutive elements in a column of the E-table, and

$$h_m^{(n-m+1)} = \frac{E_m^{(n)} - E_{m-1}^{(n)}}{E_{m-1}^{(n)} - E_{m-2}^{(n)}}$$

which links 3 consecutive elements in a row of the E-table. We also define for $n-t+1 \le s+\ell \le n, m-t+1 \le \ell \le m, n+m-t+1 \le s+2\ell$

$$q_\ell^{*(s+1)} = \frac{E_\ell^{(s+\ell,n+m)} - E_{\ell-1}^{(s+\ell,n+m)}}{E_{\ell-1}^{(s+\ell,n+m)} - E_{\ell-1}^{(s+\ell-1,n+m)}}$$

for $n-t+1 \le s+\ell+1 \le n, m-t+1 \le \ell \le m, n+m-t+1 \le s+2\ell+1$

$$e_\ell^{*(s+1)} = \frac{E_\ell^{(s+\ell+1,n+m)} - E_\ell^{(s+\ell,n+m)}}{E_\ell^{(s+\ell,n+m)} - E_{\ell-1}^{(s+\ell,n+m)}}$$

for $n-t+1 \leq s+\ell \leq n-1, m-t \leq \ell \leq m, s+2\ell-1 \geq n-m-t$

$$v_{\ell+1}^{*(s)} = \frac{E_\ell^{(s+\ell+1,n+m)} - E_\ell^{(s+\ell,n+m)}}{E_\ell^{(s+\ell,n+m)} - E_\ell^{(s+\ell-1,n+m)}}$$

and for $n-t \leq s+\ell \leq n, m-t+2 \leq \ell \leq m, s+2\ell-2 \geq n+m-t$

$$h_\ell^{*(s+1)} = \frac{E_\ell^{(s+\ell,n+m)} - E_{\ell-1}^{(s+\ell,n+m)}}{E_{\ell-1}^{(s+\ell,n+m)} - E_{\ell-2}^{(s+\ell,n+m)}}$$

It is clear that for definitions linking 3 elements from the E-table, certain transition rules apply when entering and leaving the hexagonal singular block. For $q_\ell^{*(s+1)}$, $e_\ell^{*(s+1)}$, $v_\ell^{*(s+1)}$ and $h_\ell^{*(s+1)}$ this transition is described in [ALLOb].

THEOREM 13:
The continued fraction representation $C_s^*(x,y)$ associated with T_s^* as in (21) is given by the following formula. The first and last line of the expression for C_s^* coincide with that for C_s while the middle part deals with the discrepancy between T_s^* and T_s:

$$C_s^*(x,y) = E_0^{(s)} + \left\lfloor\frac{E_0^{(s+1)} - E_0^{(s)}}{1}\right\rfloor + \sum_{i=1}^{n-t-s-1}\left(\left\lfloor\frac{-q_i^{(s+1)}}{1+q_i^{(s+1)}}\right\rfloor + \left\lfloor\frac{-e_i^{(s+1)}}{1+e_i^{(s+1)}}\right\rfloor\right)$$
$$+ \left\lfloor\frac{-q_{n-t-s}^{*(s+1)}}{1+q_{n-t-s}^{*(s+1)}}\right\rfloor + \left\lfloor\frac{-e_{m+t+1}^{(n-m-2t)}}{1+e_{m+t+1}^{(n-m-2t)}}\right\rfloor + \sum_{i=1}^{m-n+2t+s+1}\left\lfloor\frac{-v_{m+t+2}^{(n-m-2t+i-1)}}{1+v_{m+t+2}^{(n-m-2t+i-1)}}\right\rfloor$$
$$+ \sum_{i=m+t+2}^{\infty}\left(\left\lfloor\frac{-q_i^{(s+1)}}{1+q_i^{(s+1)}}\right\rfloor + \left\lfloor\frac{-e_i^{(s+1)}}{1+e_i^{(s+1)}}\right\rfloor\right)$$

To be able to use the continued fraction representation obtained in the previous theorem, we must find a coherent computation scheme for its partial numerators and denominators. Let us first introduce the following notations:

$$G_\ell^{(s)} = \frac{g_{\ell,\ell+1}^{(s)} - g_{\ell,\ell+1}^{(s+1)}}{g_{\ell,\ell+1}^{(s)}}$$

and

$$G_\ell^{(s,n+m)} = \frac{g_{\ell,\ell+1}^{(s,n+m)} - g_{\ell,\ell+1}^{(s+1,n+m)}}{g_{\ell,\ell+1}^{(s,n+m)}}$$

The next theorem lists the singular rules in the order they have to be implemented. This set of rules is complete and provides the partial numerators and denominators for all continued fractions disturbed by the degeneracy, not only for (21).

THEOREM 14:
We first concentrate on the values $v_{\ell+1}^{(s)}$, $v_{\ell+1}^{*(s)}$, $h_\ell^{(s+1)}$ and $h_\ell^{*(s+1)}$.

(a)

$$v_{\ell+1}^{(s)} = \frac{g_{\ell-1,\ell}^{(s+\ell-1)} - g_{\ell-1,\ell}^{(s+\ell)}}{g_{\ell-1,\ell}^{(s+\ell-1)}} \frac{e_\ell^{(s+1)}}{e_\ell^{(s)}} q_\ell^{(s+1)}$$

$$h_\ell^{(s+1)} = \frac{g_{\ell-1,\ell}^{(s+\ell)}}{g_{\ell-1,\ell}^{(s+\ell)} - g_{\ell-1,\ell}^{(s+\ell+1)}} e_{\ell-1}^{(s+2)}$$

(b)

$$v_{\ell+1}^{*(s)} = G_{\ell-1}^{(s+\ell,n+m)} \frac{e_\ell^{*(s+1)}}{e_\ell^{*(s)}} q_\ell^{*(s+1)}$$

$$h_\ell^{*(s+1)} = \frac{1}{G_{\ell-1}^{(s+\ell,n+m)}} e_{\ell-1}^{*(s+2)} \qquad s+\ell < n$$

$$h_\ell^{*(s+1)} = \frac{1}{G_{\ell-1}^{(n+s)}} e_{\ell-1}^{(n+s-\ell+2)} \qquad s+\ell = n$$

(c)

$$h_{m+1}^{*(n-m+t)} = \frac{1}{G_m^{(n+t)}} e_m^{(n-m+t+1)}$$

$$h_{m+2}^{*(n-m+t-2)} = \frac{G_{m-1}^{(n-1,n+m)}}{G_{m+1}^{(n+t-1)}} \frac{h_{m+1}^{*(n-m+t)} q_m^{*(n-m+1)}}{e_m^{*(n-m)}}$$

$$h_{m+t-u}^{*(n-m-t+2u+2)} = \frac{1}{G_{m+t-u-1}^{(n+u+1)}} h_{m+t-u-1}^{*(n-m-t+2u+4)} \qquad u = 0,\ldots,t-3$$

$$v_{m+2}^{*(n-m+t-2)} = \frac{1}{G_m^{(n+t)}} e_m^{(n-m+t+1)}$$

$$v_{m+t-u+1}^{*(n-m-t+2u)} = \frac{1}{G_{m+t-u-1}^{(n+u+1)}} v_{m+t-u}^{*(n-m-t+2u+2)} \qquad u = t-2,\ldots,1$$

$$v_{m+t+1}^{*(n-m-t)} = \frac{G_{m-1}^{(n-1,n+m)}}{G_{m+t-1}^{(n+1)}} \frac{v_{m+t}^{*(n-m-t+2)} q_m^{*(n-m+1)}}{e_m^{*(n-m)}}$$

We shall now concentrate on the octagonal gap in the qd-table due to the hexagonal block in the table of rational Hermite interpolants.

(d) To fill the leftmost column of the octagonal singular block (20) in the qd-table in a bottom-up way, we compute for $k = 1,\ldots,t$

$$e_{m-t}^{*(n-m+t+k)} = G_{m-t-1}^{(n+k-1)} q_{m-t}^{(n-m+t+k+1)} e_{m-t}^{*(n-m+t+k+1)}$$

with $e^{*(n-m+2t+1)}_{m-t} = e^{(n-m+2t+1)}_{m-t}$.

(e) *To fill the leftmost upward sloping diagonal of (20), we compute for* $k = t, \dots, 1$

$$q^{*(n-m-t+2k)}_{m-k+1} = \frac{G^{(n-t+k-1)}_{m-k-1}}{G^{(n-t+k,n+m)}_{m-k}} \frac{q^{(n-m-t+2k+1)}_{m-k} e^{*(n-m-t+2k+1)}_{m-k}}{e^{(n-m-t+2k)}_{m-k}}$$

and for $k = t-1, \dots, 0$

$$e^{*(n-m-t+2k+1)}_{m-k} + 1 = G^{(n-t+k)}_{m-k-1} \left(q^{*(n-m-t+2k+2)}_{m-k} + 1 \right)$$

with

$$q^{*(n-m-t)}_{m+1} = \frac{G^{(n-t-1)}_{m}}{G^{(n-t)}_{m+t}} \frac{q^{(n-m-t+1)}_{m} e^{*(n-m-t+1)}_{m}}{e^{(n-m-t)}_{m}}$$

(f) *To fill the top row of (20) together with the row immediately above the block since this uses "degenerate" values, we compute for* $k = 2, \dots, t+1$

$$q^{*(n-m-t-k+1)}_{m+k} = \frac{-g^{(n-t-1)}_{m+k-1,m+k}}{g^{(n-t)}_{m+k-1,m+k}} G^{(n-t-1)}_{m+k-2} q^{*(n-m-t-k+2)}_{m+k-1}$$

with $q^{*(n-m-2t)}_{m+t+1} = q^{(n-m-2t)}_{m+t+1}$, *and for* $k = 1, \dots, t$

$$e^{(n-m-t-k)}_{m+k} = \frac{-g^{(n-t)}_{m+k-1,m+k}}{g^{(n-t-1)}_{m+k-1,m+k}}$$

(g) *Column* $m+t+2$ *of q-values is the first to reappear in the continued fraction representations* $C^*_s(x,y)$. *It can be computed from the q-values and e-values with column index* $m+t+1$ *using the well-known non-singular rules. Column* $m+t+1$ *of e-values depends solely on column* $m+t+1$ *of q-values, so we focus on this last one. For* $k = 1, \dots, t-1$

$$q^{(n-m-2t+k)}_{m+t+1} = \frac{G^{(n-t+k-1,n+m)}_{m-1}}{G^{(n-t+k)}_{m+t}} \frac{e^{*(n-m-t+k+1)}_{m} q^{*(n-m-t+k+1)}_{m}}{e^{*(n-m-t+k)}_{m}}$$

where a band of q^**-values and* e^**-values is filled using rules constructed from the classical ones:*

$$q^{*(s+1)}_{\ell} = \frac{G^{(s+\ell-1,n+m)}_{\ell-2}}{G^{(s+\ell,n+m)}_{\ell-1}} \frac{e^{*(s+2)}_{\ell-1} q^{*(s+2)}_{\ell-1}}{e^{*(s+1)}_{\ell-1}}$$

$$\ell = m-t+1, \dots, m \quad s = n+m-t-2\ell+3, \dots, n+m-2\ell+1$$

and

$$e_\ell^{*(s+1)} + 1 = G_{\ell-1}^{(s+\ell,n+m)}\left(q_\ell^{*(s+2)} + 1\right)$$
$$\ell = m-t+1,\ldots,m \quad s = n+m-t-2\ell+2,\ldots,n+m-2\ell$$

For $k = t$

$$q_{m+t+1}^{(n-m-t)} = \frac{G_{m-1}^{(n-1,n+m)}}{G_{m+t}^{(n)}} \prod_{i=1}^{t} \frac{1}{G_{m+t-i}^{(n+i)}} \frac{e_m^{(n-m+t+1)} q_m^{*(n-m+1)}}{e_m^{(n-m)}}$$

(h) To close the octagonal gap in the qd-table we now calculate the remaining elements. For $k = t,\ldots,1$

$$q_{m+t-k+2}^{(n-m-t+2k-1)} = \frac{1}{G_{m+t-k+1}^{(n+k)}} e_{m+t-k+1}^{(n-m-t+2k)}$$

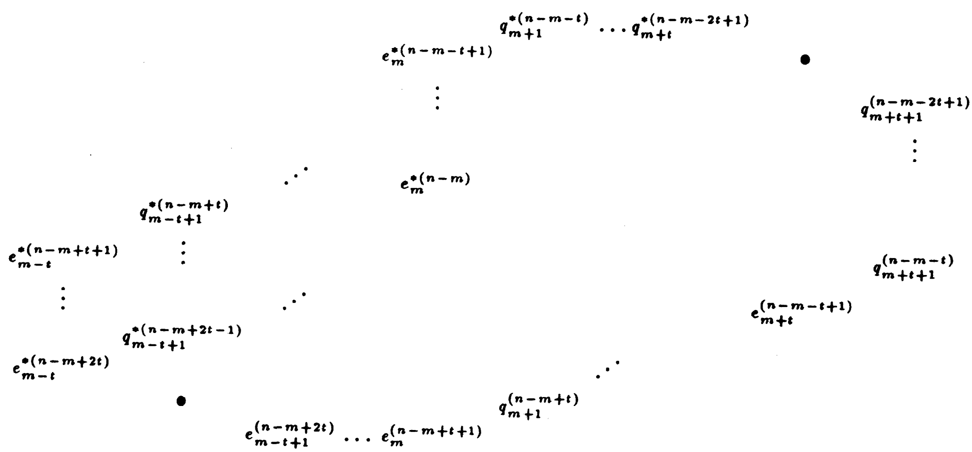
and inbetween

$$e_{m+t-k+2}^{(n-m-t+2k-2)} + 1 = G_{m+t-k+1}^{(n+k-1)}\left(q_{m+t-k+2}^{(n-m-t+2k-1)} + 1\right)$$

On the bottom line we have for $k = 1,\ldots,t$

$$e_{m-t+k}^{(n-m+2t-k+1)} + 1 = G_{m-t+k-1}^{(n+t)}\left(q_{m-t+k}^{(n-m+2t-k+2)} + 1\right)$$

Using theorem 14 the gap bordered in (20) involves the computation of the elements listed in the octagon below.



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