Rational Approximation Successes in Computational Science and Engineering

Annie Cuyt[*1]

[1] Department of Mathematics and Computer Science, University of Antwerp
Middelheimlaan 1, B2020 Antwerp, Belgium

1 Introduction

In recent years several highly technological problems could profit from some classical results in rational approximation theory, as can be seen from the existing literature. We discuss following selected problems :

1. The computation of the packet loss probability as a function of the buffer size in the context of multiplexing techniques, to support variable bit rate communication, can be realized in almost real-time making use of multipoint Padé-type approximants.
2. The reconstruction of general two- and three-dimensional shapes from indirect measurements such as bi- and trivariate moment information, is possible because of the relationship between several integral transforms and homogeneous multivariate Padé approximants.
3. Models describing complicated physical devices or extremely time-consuming simulations, can be highly simplified using adaptive scattered rational interpolation, while maintaining at the same time a required accuracy.
4. A large collection of special functions from science and engineering can be evaluated reliably and efficiently by means of modified continued fraction approximants, guaranteeing evaluations up to a user defined accuracy which can be chosen from a few digits to several hundreds or thousands, truncation and round-off error included.

2 Packet loss probability function

Variable bit rate (VBR) communications with real time constraints in general, and video communication services (video phone, video conferencing, television distribution) in particular, are expected to be a major class of services provided by the future Quality of Service (QoS) enabled Internet. The introduction of statistical multiplexing techniques offers the capability to efficiently support VBR connections by taking advantage of the variability of the bandwidth requirements of individual connections. These techniques handle a variety of traffic types such as video, voice, still images and data, each with their own QoS. Accurate traffic modelling and analysis of the QoS parameters in the multiplexer environment enable the admission controller to make decisions that ensure the integrity of the traffic sources and guarantee efficiency. An important QoS measure is the packet (or cell) loss probability (PLP).

To compute the PLP function $P_L(N)$ in terms of the buffer size N, several approaches have been developed in the past, based on exact analytical techniques, approximate techniques or simulation:

[*] Corresponding author: e-mail annie.cuyt@ua.ac.be

- In exact analysis, the traffic is described by Markovian arrival processes, leading to a Markov model of finite M/G/1-type [8, 10] and giving rise to finite embedded Markov chains whose transition matrices are upper block Hessenberg [9]. The complexity of the algorithms used to find the stationary probabilities of M/G/1-type queues is $O(cN_1^3N_2^2)$, where c is the number of servers, N_1 is the dimension of a block and N_2 is the number of block rows. This order of complexity does not allow one to compute $P_L(N)$ for large values of N in real time.

- In approximate analysis, models from fluid queues have been used. However, the computational requirements of the algorithms grow quite rapidly as the system's complexity grows.

- Monte Carlo simulation is also an option to compute the packet loss probabilities. However, if the desired probability is in the range of 10^{-6} to 10^{-12} (rare event probability), it is computationally impossible to use the conventional Monte Carlo simulation. A simulation technique called Importance Sampling (IS) can speed up simulations involving rare events. Because of the complicated nature of multiplexing queueing models, applying the IS technique is not straightforward.

The proposed rational approximation technique is a kind of divide and conquer technique, in the sense that:

- for small values of the buffer capacity N, the exact value $P_L(N)$ is computed;

- the function $\log P_L(N)$ is approximated by a rational function $r_n(N)$ of degree $n+1$ over n;

- and the approximate model is validated by a single simulation for one larger value of the buffer length.

Gong [7] and Yang [11] were the first to compute these probabilities for large buffer sizes, from sampled values of $\log P_L(N)$ for small buffer sizes, the decay rate and a rational model. The technique was first applied to multiplexer models with little or no correlation between the cells.

In [6], an automatic procedure to select the sample points (also called support points) is proposed and used for the efficient computation of models $r_n(N)$ in case there is more correlation between the cells. The procedure selects the support points in a region which we determine from the system parameters, until the model $r_n(N)$ is sufficiently accurate, meaning that $|r_n(N) - r_{n+1}(N)|/|r_{n+1}(N)|$ does not exceed a prescribed error threshold.

In [5], the technique is perfected by making very good use of the knowledge of $P_L(N)$ for extremely small values of N, such as $N = 1, 2, 3, \ldots, M$, where M is the number of sources. The approach maximizes the information that can be extracted from the data, while minimizing the number of data samples to be collected. The modified technique allows to construct a model $r_n(N)$ which is free from positive real poles, and is extremely efficient, especially in the case of correlation between the packets (such as with more video sources), or high (but realistic) network load.

3 Shape reconstruction

In shape reconstruction, the celebrated Fourier slice theorem plays an essential role. It allows to reconstruct the shape of a quite general object from the knowledge of its Radon transform [15], in other words from the knowledge of projections of the object. In case the object is a polygon [13], or when it defines a quadrature domain in the complex plane [14], its shape can also be reconstructed from the knowledge of its moments. Essential tools in the solution of the latter inverse problem, are quadrature rules and formal orthogonal polynomials.

We show how shape reconstruction of general compact objects, in several dimensions, can be realized from the knowledge of the moments. To this end we use a less known homogeneous Padé slice property. Again integral transforms, in our case the multivariate Stieltjes transform and univariate Markov transform, formal orthogonal polynomials in the form of Padé denominators and multidimensional integration formulas or cubature rules play an essential role [1].

We emphasize that the new technique is applicable in all higher dimensions and illustrate it through the reconstruction of several two- and three-dimensional objects.

4 Metamodelling or model reduction

The behaviour of certain electromagnetic devices or components, for instance, can be simulated with great detail in software. A drawback of these simulation models is that they are very time-consuming. Since the required precision is usually in the order of 2 to 3 relative significant digits, an approximate analytic model for the computational electromagnetic analysis is sometimes used instead [12, 16]. The most complex form of this model is a multivariate rational function obeying some interpolation conditions. The number of simulation data from which the rational function is constructed, has to be kept to a minimum, because each evaluation of the simulation model is computationally costly.

The rational model we present here is:

(1) of the most general form, in the sense that the user can entirely freely choose the terms making up the numerator and denominator polynomial of the rational function, and this in any number of variables;

(2) of minimal complexity, because the data are sampled at optimally located points, without restriction to a grid-like structure, hence reducing the amount of data to be collected and the number of terms in the rational model;

(3) of minimal truncation error, since the algorithm constructs several models of the same complexity, by varying the numerator and denominator degree, and chooses the one with smaller truncation error upper bound.

We discuss the computation of the rational model using a fast technique that exploits the structured nature of the linear system of the interpolation conditions and the fact that the data points are added to the model one by one.

Step (2) translates to a data updating step, where an additional data point is added and a rational model of higher complexity is fitted, as long as the model is not sufficiently accurate. To this end an approximation of the truncation error is computed, and the data point at which the estimation of the truncation error is maximal is selected as additional data point.

Step (3) consists of a model updating step, where the degree of numerator and denominator in the rational model is varied, keeping the sum of the degrees constant, in order to find the best rational model. A rule of thumb from univariate approximation theory is that the best rational models are the ones with approximately equal numerator and denominator degree [4].

5 Multiprecision evaluation of special functions

The technique to provide a floating-point implementation of a function differs substantially when going from a fixed finite precision context to a finite multiprecision context. In the former, the aim is to provide an optimal mathematical model, valid on a reduced argument range and requiring as few operations as possible. Here optimal means that, in relation to the model's complexity, the truncation error is as small as it can get. The total relative error should not exceed a prescribed treshold, round-off error and argument reduction effect included. In the latter, the goal is to provide a more generic technique, from which an approximant yielding the user-defined accuracy, can be deduced at runtime. Hence best approximants are not an option since these models would have to be recomputed every time the precision is altered and a function evaluation is requested. At the same time the generic technique should generate an approximant of as low complexity as possible.

In the current approach we point out how continued fraction representations of functions can be helpful in the multiprecision context. The newly developed generic technique is mainly based on the use of new a priori truncation error estimates for modified approximants of a continued fraction representation of the function [3]. The technique is even quite competitive when compared to the traditional fixed precision implementations. The implementation is reliable in the sense that it allows to return a sharp interval enclosure for the requested function evaluation, at the same cost [2].

References

[1] A. Cuyt, G. Golub, P. Milanfar, and B. Verdonk. Multidimensional integral inversion with applications in shape reconstruction. *SIAM J. Sci. Comput.*, 2004. Submitted for publication.

[2] A. Cuyt, B. Verdonk, H. Waadeland, F. Backeljauw, S. Becuwe, and J. Vervloet. Fast and reliable evaluation of some real- and complex-valued special functions. 2004. Submitted for publication.

[3] A. Cuyt, B. Verdonk, H. Waadeland, and J. Vervloet. Efficient and reliable multiprecision implementation of some elementary and special functions. 2004. In preparation.

[4] J. Gilewicz and A.P. Magnus. Optimal inequalities of Padé approximant errors in the Stieltjes case: closing result. *Integral Transforms and Special Functions*, 1(1):9–18, 1993.

[5] A. Cuyt, R.B. Lenin, Computing packet loss probabilities in multiplexer models using adaptive rational interpolation with optimal pole placement, submitted.

[6] A. Cuyt, R.B. Lenin, G. Willems, C. Blondia, and P. Rousseeuw, Rational approximation technique to compute cell loss probabilities in multiplexer models, IEEE Trans. on Computers, **52**, 1–12 (2003).

[7] W. Gong et al., On the convergence of global rational approximants for stochastic discrete event systems, Discrete Event Dynamic Systems - Theory and Applications, **7**, 93–116 (1997).

[8] A.E. Kamal, Efficient solution of multiple server queues with application to the modeling of ATM concentrators, in: Proceedings of IEEE INFOCOM '96, San Francisco, CA, March 1996, pp. 248–254.

[9] G. Latouche and G. Stewart, Numerical methods for M/G/1 type queues, in: Proceedings of the 1995 International Meeting on Computations with Markov Chains, edited by W. Stewart Kluwer Academic, 1995, pp. 571–581.

[10] K. Wuyts and R. Boel, Efficient matrix geometric methods for B-ISDN by using the spectral decomposition of the rate matrices, in: Proceedings of the 2nd conference on Matrix-Analytic Methods in stochastic models, edited by S. Chakravarthy and A. Alfa (Notable Publications, 1998), pp. 341–359.

[11] H. Yang, Global rational approximation for computer systems and communication networks, Ph.D. thesis, University of Massachusetts, Amherst, MA 01003 (1996).

[12] J. De Geest, T. Dhaene, and N. Faché. Adaptive CAD-model building algorithm for general planar microwave structures. *IEEE Trans. Microwave Theory Tech.*, 47:1801 – 1809, 1999.

[13] G.H. Golub, P. Milanfar, and J. Varah. A stable numerical method for inverting shape from moments. *SIAM J. Sci. Comput.*, 21:1222–1243, 1999.

[14] B. Gustafsson, C. He, P. Milanfar, and M. Putinar. Reconstructing planar domains from their moments. *Inverse Problems*, 16:1053–1070, 2000.

[15] S. Helgason. *The Radon transform*. Birkhäuser, Boston, 1980.

[16] R. Lehmensiek and P. Meyer. Creating accurate multivariate rational interpolation models of microwave circuits by using efficient adaptive sampling to minimize the number of computational electromagnetic analyses. *IEEE Trans. Microwave Theory Tech.*, 49(8):1419–1430, 2001.