J. OPERATOR THEORY
10(1983), 127–131

A PROJECTION-PROPERTY FOR ABSTRACT RATIONAL (1-POINT) APPROXIMANTS

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1. NOTATION AND DEFINITIONS

Consider the operator $F: X \to Y$, analytic in 0 [2, pp. 113] where $X \supsetneqq \{0\}$ is a Banach space and $Y \supsetneqq \{0, I\}$ is a commutative Banach algebra without nilpotent elements (0 is the unit for addition and I is the unit for multiplication). The scalar field is $\mathbf{R}$ or $\mathbf{C}$.

A nonlinear operator $P: X \to Y$ such that $P(x) = A_n x^n + \ldots + A_0$ with $A_i: X^i \to Y$ a symmetric and bounded i-linear operator $(i = 0, \ldots, n)$ is called an abstract polynomial [2, pp. 111]. The degree of $P(x)$ is n. The notation for the exact degree of $P(x)$ is ∂P (the largest integer k with $A_k x^k \not\equiv 0$) and the notation for the order of $P(x)$ is $\partial_0 P$ (the smallest integer k with $A_k x^k \not\equiv 0$).

Write $D(F) = \{x \in X \mid F(x)$ is regular in Y, i.e. there exists $y \in Y: F(x)\cdot y = I\}$. Since F is analytic in 0, there exists $r > 0$ such that

$$F(x) = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(0) x^k \quad \text{for } \|x\| < r.$$

We say that $F(x) = O(x^j)$ for $j \in \mathbf{N}$ if there exist $J \in \mathbf{R}_0^+$ and $0 < r < 1$ such that

$$\|F(x)\| \leqslant J \|x\|^j.$$

DEFINITION 1.1. The couple of abstract polynomials

$$(P(x), Q(x)) = (A_{nm+n} x^{nm+n} + \ldots + A_{nm} x^{nm},\ B_{nm+m} x^{nm+m} + \ldots + B_{nm} x^{nm})$$

is called *a solution of the Padé approximation problem of order (n, m) for F* if the abstract power series

$$(F \cdot Q - P)(x) = O(x^{nm+n+m+1}).$$

We define the operator $\frac{1}{Q}: D(Q) \to Y$ by $\frac{1}{Q}(x) = [Q(x)]^{-1} =$ the inverse element of $Q(x)$ for the multiplication in Y. We call the abstract rational operator $\frac{1}{Q} \cdot P$, the quotient of two abstract polynomials, reducible if there exist abstract polynomials T, R and S such that $P = T \cdot R$, $Q = T \cdot S$ and $\partial T \geqslant 1$.

Let us assume that the Banach space X and the Banach algebra Y are such that the irreducible form of an abstract rational operator is unique and that the abstract rational approximant of order (n, m) for F (see Definition 1.2) is unique. The matter was discussed in [1].

DEFINITION 1.2. Let (P, Q) be a couple of abstract polynomials satisfying Definition 1.1, with $D(P) \cup D(Q) \neq \emptyset$. The irreducible form $\frac{1}{Q_*} \cdot P_*$ of $\frac{1}{Q} \cdot P$ is called the *abstract rational approximant of order* (n, m) *for* F (abbreviated (n, m)-ARA).

To prove our projection-property we shall need the condition numbered (1). Let $T(x) = \sum_{k=\partial_0 T}^{\partial T} T_k x^k$ be the abstract polynomial such that $P = P_* \cdot T$ and $Q = Q_* \cdot T$. Because $D(P) \cup D(Q) \neq \emptyset$ we have $D(T) \neq \emptyset$. If

$$D(T_{\partial_0 T}) \neq \emptyset \tag{1}$$

then we have $t \geqslant 0$ such that

$$(F \cdot Q_* - P_*)(x) = O(x^{\partial_1 P_* + \partial_1 Q_* + \partial_0 Q_* + t + 1})$$

$$\partial_1 P_* \leqslant n \leqslant \partial_1 P_* + t$$

$$\partial_1 Q_* \leqslant m \leqslant \partial_1 Q_* + t$$

where $\partial_1 P_* = \partial P_* - \partial_0 Q_*$ and $\partial_1 Q_* = \partial Q_* - \partial_0 Q_*$ [1, pp. 208].

2. PROJECTION-PROPERTY

Consider Banach spaces X_i $(i = 1, \ldots, p)$. The space $\prod_{i=1}^{p} X_i$ normed by one of the following Minkowski norms

$$\|x\|_q = \left(\sum_{i=1}^{p} \|x_i\|_{(i)}^q \right)^{1/q}$$

or

$$\|x\|_1 = \sum_{i=1}^{p} \|x_i\|_{(i)}$$

or

$$\|x\|_\infty = \max(\|x_1\|_{(1)}, \ldots, \|x_p\|_{(p)})$$

where $\|x_i\|_{(i)}$ is the norm of x_i in X_i and $x = (x_1, \ldots, x_p)$, is also a Banach space. We introduce the notations

$$x_{(j)} = (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_p)$$

$$\hat{x}_{(j)} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p).$$

THEOREM 2.1. *Let* $X = \prod_{i=1}^{p} X_i$ *and* $\left(\frac{1}{Q_*} \cdot P_*\right)(x)$ *be the* (n, m)-ARA *for* F *and* $j \in \{1, \ldots, p\}$.

Let (1) *be satisfied. If*

$$S(\hat{x}_{(j)}) := Q_*(x_{(j)})$$

$$R(\hat{x}_{(j)}) := P_*(x_{(j)})$$

$$D(S) \cup D(R) \neq \emptyset$$

$$G_j(\hat{x}_{(j)}) := F(x_{(j)})$$

then the irreducible form $\left(\frac{1}{S_*} \cdot R_*\right)(\hat{x}_{(j)})$ of $\left(\frac{1}{S} \cdot R\right)(\hat{x}_{(j)})$ *is the* (n, m)-ARA *for* G_j.

Proof. First we remark that if $L: X^k \to Y$ is a bounded k-linear operator, then the operator $M : \left(\prod_{\substack{i=1 \\ i \neq j}}^{p} X_i\right)^k \to Y$ defined by $M\hat{x}_{(j)}^k = Lx_{(j)}^k$ is also bounded and k-linear.

Since $\left(\frac{1}{Q_*} \cdot P_*\right)(x)$ is the (n, m)-ARA for F and since (1) is satisfied, we have $t \geqslant 0$ such that

$$(F \cdot Q_* - P_*)(x) = O(x^{\partial_1 P_* + \partial_1 Q_* + \partial_0 Q_* + t + 1})$$

$$\partial_1 P_* \leqslant n \leqslant \partial_1 P_* + t$$

$$\partial_1 Q_* \leqslant m \leqslant \partial_1 Q_* + t.$$

Using one of the Minkowski norms $\|\cdot\|_q$ $(1 \leqslant q \leqslant \infty)$, $\|x_{(j)}\|_q = \|(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_p)\|_q$ in $\prod_{i=1}^{p} X_i$ equals $\|\hat{x}_{(j)}\|_q = \|(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p)\|_q$ in $\prod_{\substack{i=1\\ i\neq j}}^{p} X_i$.

Thus

$$(F\cdot Q_* - P_*)(x_{(j)}) = (G_j \cdot S - R)(\hat{x}_{(j)}) =$$

$$= O(\hat{x}_{(j)}^{\partial_1 P_* + \partial_1 Q_* + \partial_0 Q_* + t + 1}).$$

Now $\partial P_* = \partial_0 Q_* + \partial_1 P_* \leqslant \partial P - \partial_0 T \leqslant nm + n$ [1, pp. 199], and $\partial Q_* = \partial_0 Q_* + \partial_1 Q_* \leqslant \partial Q - \partial_0 T \leqslant nm + m$ [1, pp. 199]. So $s = nm - \partial_0 Q_* + \min(n - \partial_1 P_*, m - \partial_1 Q_*) \geqslant 0$.

Take a bounded s-linear operator $D_s \colon \left(\prod_{\substack{i=1\\ i\neq j}}^{p} X_i\right)^s \to Y$ with $D(D_s) \cap [D(S) \cup D(R)] \neq \emptyset$.

Then

$$\partial_0(S\cdot D_s) \geqslant nm$$

$$\partial_0(R\cdot D_s) \geqslant nm$$

$$\partial(S\cdot D_s) \leqslant \partial_0 Q_* + \partial_1 Q_* + nm - \partial_0 Q_* + \min(n - \partial_1 P_*, m - \partial_1 Q_*) \leqslant nm + m$$

$$\partial(R\cdot D_s) \leqslant \partial_0 Q_* + \partial_1 P_* + nm - \partial_0 Q_* + \min(n - \partial_1 P_*, m - \partial_1 Q_*) \leqslant nm + n$$

$$[(G_j\cdot S - R)\cdot D_s](\hat{x}_{(j)}) = O(\hat{x}_{(j)}^{\partial_1 P_* + \partial_1 Q_* + nm + t + \min(n - \partial_1 P_*, m - \partial_1 Q_*) + 1}) =$$

$$= O(\hat{x}_{(j)}^{nm + n + m + 1})$$

since $m \leqslant \partial_1 Q_* + t$ and $n \leqslant \partial_1 P_* + t$. The irreducible form of $\frac{1}{S\cdot D_s}\cdot R\cdot D_s$ is the irreducible form of $\frac{1}{S}\cdot R$.

We give a simple example to illustrate the theorem. Take

$$G\colon \mathbf{R}^2 \to \mathbf{R}\colon (x, y) \to \frac{x\exp(x) - y\exp(y)}{x - y}.$$

The (1,1)-ARA for G is

$$\frac{x + y + 0.5(x^2 + 3xy + y^2)}{x + y - 0.5(x^2 + xy + y^2)}.$$

For $j = 1: x = 0$

$$G_1: \mathbf{R} \to \mathbf{R}: y \to \exp(y)$$

and for $j = 2: y = 0$

$$G_2: \mathbf{R} \to \mathbf{R}: x \to \exp(x).$$

Indeed the (1,1)-ARA for G_1 equals $\frac{1 + 0.5y}{1 - 0.5y}$ and for G_2 equals $\frac{1 + 0.5x}{1 - 0.5x}$.

The author is Aspirant Belgisch National Fonds voor Wetenschappelijk Onderzoek.

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Received April 13, 1982.