OPERATOR PADE APPROXIMANTS:
Some ideas behind the theory and a numerical illustration

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ABSTRACT

Section 1 will be devoted to the discussion of some generalizations of the concept of Pade-approximant for multivariate functions, based on the interpolation property of a Pade-approximant. Most of those generalizations preserve, under some conditions, a number of properties of the univariate Pade-approximant. In Section 2 we will repeat the recursive schemes used for the computation of the univariate Pade-approximant: the ε-algorithm and the qd-algorithm. We will also show that, if a generalizing definition is based on these recursive algorithms, then much more interesting properties remain valid for the generalization. In Section 3 we will illustrate the approximation power of this type of Pade approximants on a numerical example. Other applications are: the solution of nonlinear systems of equations [7], the solution of nonlinear differential and integral equations [3], the acceleration of convergence [4]. Since those applications have already been treated extensively, they will not be mentioned here; the interested reader is referred to the literature.

1. SOME DEFINITIONS FOR MULTIVARIATE PADE APPROXIMANTS

Let us first of all repeat the definition of univariate Pade approximant. Suppose we are given a function f(x) by its Taylor series expansion around the origin,

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \quad \text{with} \quad c_k = \frac{1}{k!} f^{(k)}(0)$$

S. P. Singh et al. (eds.), Approximation Theory and Spline Functions, 271–288.

In the Pade approximation problem of order (n,m) we look for polynomials

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

and

$$q(x) = \sum_{j=0}^{m} b_j x^j$$

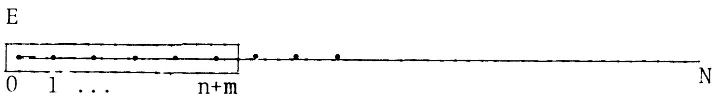
such that in the power series $(f \cdot q - p)(x)$ the first $n + m + 1$ terms disappear, i.e.

$$(f - q - p)(x) = \sum_{k=n+m+1}^{\infty} d_k x^k$$

It is well-known that this problem had indeed a nontrivial solution for the a_i and b_j and that the rational functions $(p/q)(x)$ satisfy a number of beautiful properties.

The fact that all terms of degree up to and including $n + m$ vanish in $(f \cdot q - p)(x)$, can be represented by means of an "interpolationset" E describing the fulfilled equations:

$$d_k = 0 \quad \text{for} \quad k \in E = \{0, \ldots, n + m\} \subset N$$



A natural generalization of this approximation problem to the multivariate case, is the following. We will describe the situation in the case of two variables, because the case of more than two variables is only notationally more difficult.

Suppose we know the Taylor series expansion (or at least part of it) of a bivariate function

$$f(x,y) = \sum_{i,j=0}^{\infty} c_{ij}\, x^i\, y^j \quad \text{with} \quad c_{ij} = \frac{1}{i!}\,\frac{1}{j!}\,\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0,0)$$

Let us try to calculate polynomials

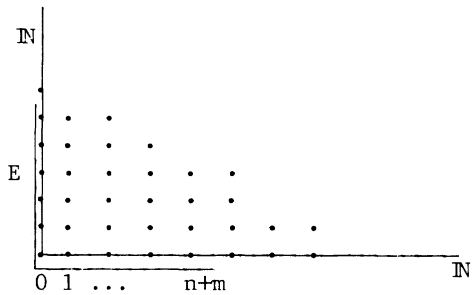
$$p(x,y) = \sum_{i+j=0}^{n} a_{ij}\, x^i\, y^j$$

$$q(x,y) = \sum_{i+j=0}^{m} b_{ij}\, x^i\, y^j$$

of total degree n and m respectively (a term $x^i y^j$ is said to be of total degree $i+j$), such that

$$(f \cdot q - p)(x,y) = \sum_{i+j=n+m+1}^{\infty} d_{ij}\, x^i y^j$$

If we represent this demand by an interpolation set E in $\mathbb{N}^2$, we have



Counting N_u, the number of unknown coefficients a_{ij} and b_{ij}, and N_e, the number of imposed equations, we remark that we have an overdetermined system of homogeneous equations and thus that we cannot guarantee the existence of a nontrivial solution:

$$N_u = \frac{1}{2}(n+1)(n+2) + \frac{1}{2}(m+1)(m+2)$$

$$N_e = \frac{1}{2}(n+m+1)(n+m+2)$$

The ideal situation would be

$$N_e = N_u - 1$$

because one unknown can always be determined by a normalization of the denominator $q(x,y)$.

Several authors have created this ideal situation by altering the form of the polynomials $p(x,y)$ and $q(x,y)$ and/or by choosing another interpolation set. The definitions which we shall compare are those of Chisholm and his group at the University of Kent in Canterbury, Lutterodt who introduces two types of approximants, Karlsson and Wallin who are primarily interested in convergence properties.

All those definitions are of the following kind. This very general setting has also been given by Levin [12].

Take a "numerator index set" N and a "denominator index set" D describing the form of the polynomials

$$p(x,y) = \sum_{(i,j)\in N \subseteq \mathbb{N}^2} a_{ij}\, x^i y^j$$

$$q(x,y) = \sum_{(i,j)\in D \subseteq \mathbb{N}^2} b_{ij}\, x^i y^j$$

and construct on interpolationset E such that

$$N \subset E$$

$$\#\,(E\backslash N) = \#\,D - 1$$

where $\#$ denotes the number of elements in a set.

Now one can assure the existence of nontrivial a_{ij} and b_{ij} when solving

$$(f \cdot q - p)(x,y) = \sum_{(i,j)\in \mathbb{N}^2\backslash E} d_{ij}\, x^i y^j$$

For Chisholm's bivariate Pade approximants

$$N = ([0,n_1] \times [0,n_2]) \cap \mathbb{N}^2 \quad \text{with} \quad n_1,n_2 \in \mathbb{N}$$

$$D = ([0,m_1] \times [0,m_2]) \cap \mathbb{N}^2 \quad \text{with} \quad m_1,m_2 \in \mathbb{N}$$

$$\begin{aligned} E = {} & \{(i,j) \mid 0 \le i \le \max(n_1,m_1),\ 0 \le j \le \min(n_2,m_2)\} \\ & \cup \{(i,j) \mid 0 \le i \le \min(m_1,m_1),\ 0 \le j \le \max(n_2,m_2)\} \\ & \cup \{(i,j) \mid \max(n_2,m_2) < j \le n_2+m_2,\ \max(n_2,m_2) < i+j \le n_2+m_2, \\ & \qquad 0 \le i \le \min(n_1,m_1)\} \end{aligned}$$

$$\cup\{(i,j)\,|\max(n_1,m_1) < i \leq n_1+m_1,\ \max(n_1,m_1) < i+j \leq n_1+m_1,$$
$$0 \leq j \leq \min(n_2,m_2)\}$$

$$d_{ij} = 0 \quad \text{for} \quad (i,j) \in E$$

$$d_{n_1+m_1+1-\ell_1\ell}+d_{\ell_1 n_2+m_2+1-\ell} = 0 \ \text{for}\ \ell=n,_{-1}\min(n_1,m_1,n_2,m_2)$$

For Lutterodt's approximants we have

$$N = ([0,n_1] \times [0,n_2]) \cap \mathbb{N}^2 \quad \text{with} \quad n_1,n_2 \in \mathbb{N}$$

$$D = ([0,m_1] \times [0,m_2]) \cap \mathbb{N}^2 \quad \text{with} \quad m_1,m_2 \quad \mathbb{N}$$

$$E \supseteq N$$

E satisfies the inclusion property, i.e. if $(i,j) \in E$ then $([0,i] \times [0,j]) \cap \mathbb{N}^2 \subset E$

$$\#E = (n_1+1)(n_2+1) + (m_1+1)(m_2+1) - 1$$

$$d_{ij} = 0 \quad \text{for} \quad (i,j) \in E$$

and for his Pade approximants of type B^1

$$N = ([0,n_1] \times [0,n_2]) \cap \mathbb{N}^2 \quad \text{with} \quad n_1,n_2 \in \mathbb{N}$$

$$D = ([0,m_1] \times [0,m_2]) \cap \mathbb{N}^2 \quad \text{with} \quad m_1,m_2 \in \mathbb{N}$$

$$E = N\cup\{(i,0)\,|n_1+1 \leq i \leq n_1+m_1\}\cup\{(0,j)\,|n_2+1 \leq j \leq n_2+m_2\}$$
$$\cup\{(i,j)\,|n_1+1 \leq i \leq n_1+m_1,\ n_2+1 \leq j \leq n_2+m_2\}$$

$$d_{ij} = 0 \quad \text{for} \quad (i,j) \in E$$

For the Karlsson-Wallin approximants

$$N = \{(i,j)\,|0 \leq i+j \leq n\} \quad \text{with} \quad n \in \mathbb{N}$$

$$D = \{(i,j)\,|0 \leq i+j \leq m\} \quad \text{with} \quad m \in \mathbb{N}$$

$$E \supseteq N$$

E satisfies the inclusion property

$$\#E \geq \frac{1}{2}(n+1)(n+2) + \frac{1}{2}(m+1)(m+2) - 1$$

None of those types of bivariate Pade approximants can be calculated recursively, and unicity of the approximant itself can only be guaranteed under certain conditions. By imposing restrictions on N,D and E one can preserve some of the univariate properties of Pade approximants. We will discuss this in detail at the end of the next section. Those who want to know more about some of these multivariate Pade approximants are referred to [1, 9, 14, 15, 11].

2. OPERATOR PADE APPROXIMANTS

Since the univariate ε- and qd-algorithm will serve as a motivation for the introduction of operator Pade approximants (multivariate Pade approximants turn out to be a special case), we will first repeat some facts about those recursive computation schemes.

Consider a series $\sum_{i=0}^{\infty} t_i$ in $\mathbb{R}$ and also the sequence $(s_i)_{i\in\mathbb{N}}$ of its partial sums; so $s_i = t_0 + \ldots + t_i$.

Input of the ε-algorithm are the elements s_i. We perform the following computations:

a) $\varepsilon_{-1}^{(i)} = 0 \qquad i = 0,1, \ldots$

$\varepsilon_{0}^{(i)} = s_i$

b) $\varepsilon_{2j}^{(-j-1)} = 0 \qquad j = 0,1, \ldots$

c) $$\varepsilon_{j+1}^{(i)} = \varepsilon_{j-1}^{(i+1)} + \frac{1}{\varepsilon_{j}^{(i+1)} - \varepsilon_{j}^{(i)}} \qquad \begin{array}{l} j = 0,1, \ldots \\ i = -j,-j+1, \ldots \end{array}$$

The index j refers to a column while i refers to a diagonal in the ε-table. If the algorithm does not break down the following property can be proved for the ε-algorithm. We denote by $\Delta s_k = s_{k+1} - s_k$.

Theorem 2.1:

With $s_i = 0$ for $i < 0$

$$\varepsilon_{2j}^{(i)} = \frac{\begin{vmatrix} s_{i+j} & \cdots & & s_i \\ \Delta s_{i+j} & \cdots & \Delta s_{i+1} & \Delta s_i \\ \vdots & & & \vdots \\ \Delta s_{i+2j-1} & & \Delta s_{i+j} & \Delta s_{i+j-1} \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 \\ \Delta s_{i+j} & \cdots & \Delta s_i \\ \vdots & & \vdots \\ \Delta s_{i+2j-1} & \cdots & \Delta s_{i+j-1} \end{vmatrix}}$$

Input of the qd-algorithm are the terms t_i. One performs the following calculations:

a) $e_0^{(i)} = 0$, $q_1^{(i)} = \frac{t_{i+1}}{t_i}$ $\qquad i = 0,1, \ldots$

b) $e_j^{(i)} = q_j^{(i+1)} + e_{j-1}^{(i+1)} - q_j^{(i)}$ $\qquad i = 0,1,2, \ldots \; j = 1,2, \ldots$

c) $q_{j+1}^{(i)} = q_j^{(i+1)} \cdot e_j^{(i+1)} / e_j^{(i)}$ $\qquad i = 0,1,2, \ldots \; j = 1,2, \ldots$

Again the index j refers to a column while i refers to a diagonal. If all the $q_j^{(i)}$ and $e_j^{(i)}$ exist, one can prove the following property.

Theorem 2.2

For $s_i = t_o + \ldots + t_i$,

$$\varepsilon_{2j}^{(i)} = s_i + \frac{t_{i+1}|}{|1} - \frac{q_1^{(i+1)}|}{|1} - \frac{e_1^{(i+1)}|}{|1} - \frac{q_2^{(i+1)}|}{|1} - \frac{e_2^{(i+1)}|}{|1} - \ldots - \frac{q_j^{(i+1)}|}{|1}$$

For $t_i = c_i x^i$ (the i^{th} term in the Taylor series of f), thus for $s_i = \sum_{k=0}^{i} c_k x^k$, it is well-known that $\varepsilon_{2m}^{(n-m)}$ is the Pade approximant of order (n,m) for f. This can easily be seen as follows. The imposed conditions

$$(f \cdot q - p)(x) = \sum_{k=n+m+1} d_k x^k$$

result in two linear systems of equations in the unknown coefficients a_i and b_j of the polynomials p and q :

$$\begin{cases} c_o b_o = a_o \\ c_1 b_o + c_o b_1 = a_1 \\ \vdots \\ c_n b_o + \ldots + c_o b_n = a_n \end{cases} \qquad \begin{cases} c_{n+1} b_o + \ldots + c_{n+1-m} b_m = 0 \\ \vdots \\ c_{n+m} b_o + \ldots + c_n b_m = 0 \end{cases}$$

with $c_k = 0$ for $k < 0$ and $b_j = 0$ for $j > m$.

A solution of the homogeneous system of equations is given by the following determinants:

$$b_o = \begin{vmatrix} c_n & c_{n-1} & \cdots & c_{n+1-m} \\ c_{n+1} & c_{n-1} & & \\ \vdots & & & \\ c_{n+m-1} & c_{n+1} & c_n & \end{vmatrix} \qquad b_j = \begin{vmatrix} c_n & \boxed{\begin{matrix} -c_{n+1} \\ \vdots \\ -c_{n+m} \end{matrix}} & c_{n+1-m} \\ c_{n+1} & & \vdots \\ \vdots & & \\ c_{n+m-1} & & c_n \end{vmatrix}$$

j^{th} column in b_o replaced by this column

Hence an explicit formula for $\frac{p(x)}{q(x)}$ is

$$\frac{\begin{vmatrix} \sum_{k=0}^{n} c_k x^k & \sum_{k=0}^{n-1} c_k x^k & \cdots & \sum_{k=0}^{n-m} c_k x_k \\ c_{n+1} & c_n & \cdots & c_{n+1-m} \\ \vdots & \vdots & & \vdots \\ c_{n+m} & c_{n+m-1} & \cdots & c_n \end{vmatrix}}{\begin{vmatrix} 1 & x & \cdots & x^m \\ c_{n+1} & c_n & \cdots & c_{n+1-m} \\ \vdots & \vdots & & \vdots \\ c_{n+m} & c_{n+m-1} & & c_n \end{vmatrix}}$$

$$\frac{\begin{vmatrix} \sum_{k=0}^{n} c_k x^k & \sum_{k=0}^{n-1} c_k x^k & \cdots & \sum_{k=0}^{n-m} c_k x^k \\ c_{n+1}x^{n+1} & c_n x^n & \cdots & c_{n+1-m}x^{n+1-m} \\ \vdots & \vdots & & \\ c_{n+m}x^{n+m} & c_{n+m-1}x^{n+m-1} & \cdots & c_n x^n \end{vmatrix}}{\begin{vmatrix} 1 & x & \cdots & x^m \\ c_{n+1}x^{n+1} & c_n x^n & & c_{n+1-m}x^{n+1-m} \\ \vdots & \vdots & & \vdots \\ c_{n+m}x^{n+m} & c_{n+m-1}x^{n+m-1} & & c_n x^n \end{vmatrix}} \tag{1}$$

$$= \varepsilon_{2m}^{(n-m)}$$

Remark that formula (1) is a rational function of the form

$$\frac{\sum_{i=0}^{n} a_i x^{i+nm}}{\sum_{j=0}^{m} b_j x^{j+nm}}$$

where

$$f(x) \sum_{j=0}^{m} b_j x^{j+nm} - \sum_{i=0}^{n} a_i x^{i+nm} = \sum_{k=n+m+1} d_k x^{k+nm}$$

This shift of the degrees over nm, which results from the determinantal expression for $\varepsilon_{2m}^{(n-m)}$, does not bother us in the univariate case because we can divide it out and it will serve as a useful tool in the operator case.

Let us now turn to the definition of Pade approximants for nonlinear operators $F:X \to Y$ where X is a Banach space and Y is a commutative Banach algebra. We shall denote elements in the space X by the symbol z.

Suppose the nonlinear operator F is abstract analytic in 0, i.e.

$$F(z) = \sum_{k=0}^{\infty} c_k z^k$$

where now $c_k = \frac{1}{k!} F^{(k)}(0)$ with $F^{(k)}(0)$ the k^{th} Frechet derivative of F at the origin, and thus a k-linear bounded operator.

In the case $X = \mathbb{R}^2$ and $Y = \mathbb{R}$, $F(z)$ is merely a bivariate Taylor series expansion with $c_k z^k = \sum_{i+j=k} c_{ij} x^i y^j$.

For a given operator F we can now construct $\frac{P(z)}{Q(z)}$, the operator Pade approximant of order (n,m), as

$$\frac{P(z)}{Q(z)} = \frac{\begin{vmatrix} \sum_{k=0}^{n} c_k z^k & \sum_{k=0}^{n-1} c_k z^k & \cdots & \sum_{k=0}^{n-m} c_k z^k \\ c_{n+1} z^{n+1} & c_n z^n & \cdots & c_{m+1-m} z^{n+1-m} \\ \vdots & \vdots & & \vdots \\ c_{n+m} z^{n+m} & c_{n+m-1} z^{n+m-1} & \cdots & c_n z^n \end{vmatrix}}{\begin{vmatrix} I & I & \cdots & I \\ c_{n+1} z^{n+1} & c_n z^n & \cdots & c_{n+1-m} z^{n+1-m} \\ \vdots & \vdots & & \vdots \\ c_{n+m} z^{n+m} & c_{n+m-1} z^{n+m-1} & \cdots & c_n z^n \end{vmatrix}}$$

where I is the unit element for the multiplication in Y and division in Y is multiplication by the inverse element for the multiplicative operator defined in the Banach algebra Y.

It is easy to see [5] that $P(z)$ and $Q(z)$ are respectively of the form

$$P(z) = \sum_{i=0}^{n} A_i z^{i+nm}$$

$$Q(z) = \sum_{j=0}^{m} B_j z^{j+nm}$$

where A_i and B_j are $(i+nm)$- and $(j+nm)$- linear operators, and that

$$(F \cdot Q - P)(z) = \sum_{k=n+m+1} D_k \, z^{k+nm} \tag{2}$$

We want to emphasize here that originally these operator Pade approximants were not introduced in this way [2]. They were defined by means of the set of equations (2) and the validity of the ε-algorithm was only proved afterwards.

One of the immediate results of this definition is that the operator Pade approximants can be calculated recursively by means of the ε-algorithm and that there is also a connection with the theory of continued fractions by means of the qd-algorithm (see [6]).

But many other properties are satisfied. We give here a list of desirable properties and will compare the multivariate Pade approximants introduced by means of the ε-algorithm with the multivariate Pade approximants introduced via the interpolationsets:

a) unicity of the solution

b) reciprocal covariance: if $f(x,y)$ is replaced by $\frac{1}{f}(x,y)$ and if $\frac{p}{q}(x,y)$ is the Pade approximant of order (n,m) for $f(x,y)$, is then $\frac{q}{p}(x,y)$ the Pade approximant of order (m,n) for $\frac{1}{f}(x,y)$?

c) homografic covariance: if $f(x,y)$ is replaced by $\frac{a \cdot f + b}{c \cdot f + d}(x,y)$ with a,b,c,d in $\mathbb{R}$, and if $\frac{p}{q}(x,y)$ is the Pade approximant of order (n,n) for $f(x,y)$, is then $\frac{a \cdot p + b \cdot q}{c \cdot p + d \cdot q}(x,y)$ the Pade approximant of order (n,n) for $\frac{a \cdot f + b}{c \cdot f + d}(x,y)$?

d) projection property: if $\frac{p}{q}(x,y)$ is the Pade approximant of order (n,m) for $f(x,y)$, are then $\frac{p}{q}(x,0)$ and $\frac{p}{q}(0,y)$ the Pade approximants of order (n,m) for $f(x,0)$ and $f(0,y)$ respectively?

e) symmetry: if $f(x,y) = f(y,x)$, is then the Pade approximant symmetric too?

f) consistency property: if $f(x,y)$ is a rational function itself, do we then find f back as its Pade approximant by choosing the degrees of p and q appropriately?

g) recursive computation

h) block-structure: if the multivariate Pade approximants of order (n,m) are ordered in a two-dimensional array for increasing n and m, does the Pade-table then consist of square blocks containing equal Pade-approximants?

i) continued fractions: can the multivariate Pade approximant also be obtained as the convergent of a multivariate continued fraction?

The following review gives an answer to those questions. Chisholm's and Lutterodt's approximants shall be denoted by $(n_1,n_2)/(m_1,m_2)$ while the Karlsson-Wallin and the operator Pade approximants are indicated by n/m.

	Chisholm	Lutterodt		Karlsson-Wallin	Operator
Unicity	only under certain conditions on c_{ij}	only with respect to a given E, if the homogeneous system has a unique solution		only if E contains as many points as possible	yes
Recipr. Cov.	yes	yes	no	yes	yes
Homogr. Cov.	yes	yes	no	yes	yes
Projection Pr.	yes	only if E contains the univariate inter-polation-sets	yes	only if E contains the univariate inter-polationsets	yes

	Chisholm	Lutterodt		Karlsson-Wallin	Operator
Symmetry	only for (n,n)/(m,m)	only for symmetric E and for (n,n)/(m,m)	only for (n,n)/(m,m)	only for symmetric E	yes
Consistency Pr.	no	no	no	no	yes
ε-algorithm	no	no	no	no	yes
Block-structure	no	no	no	no	yes
qd-algorithm	no	no	no	no	yes

3 NUMERICAL APPROXIMATION OF THE BETA FUNCTION

The Beta function is an example which has also been studied by the Canterbury group [8] and by Levin [13]. We will compare our results with theirs. The Beta function may be defined by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

where Γ is the Gamma function. Singularities occur for $x = -k$ and $y = -k$ $(k = 0,1,2,...)$ and zeros for $y = -x-k$ $(k=0,1,2,...)$. We write

$$B(x,y) = \frac{A(x-1,y-1)}{xy}$$

with

$$A(u,v) = 1 + uv\, f(u,v)$$

The coefficients in the Taylor series expansion of $f(u,v)$ have been calculated by the first method suggested in [8].

We will calculate some bivariate Pade approximants $\frac{p}{q}(u,v)$ of order (n,m) for $f(u,v)$ and compute

$$\frac{1 + (x-1)(y-1)\frac{p}{q}(x-1,y-1)}{xy}$$

as an approximation for $B(x,y)$.

Let us first take a look at the computational effort it takes for the calculation of a certain approximant. We denote by N_f the number of coefficients in the Taylor series of $f(x,y)$ which we shall need for the computation of the approximant. Since the coefficients a_{ij} can be calculated by substitution of the b_{ij} in the left hand sides of the equations, N_u will now denote the number of unknown coefficients b_{ij} in the homogeneous system.

For Chisholm's approximants with $n_1 = n_2 = n$ and $m_1 = m_2 = m$, we have

$$N_u = (m+1)^2$$

$$N_f = (m+1)^2 + (n+1)^2 + 2\min(n,m) - 1$$

For the operator Pade approximants

$$N_u = \frac{1}{2}[(nm+m+1)(nm+m+2) - nm(nm+1)] \quad \text{if } nm > 0$$

$$\frac{1}{2}(m+1)(m+2) \quad \text{if } nm = 0$$

$$N_f = \frac{1}{2}(n+m+1)(n+m+2)$$

The rational functions which Levin used for the approximation of the Beta function, were of the following type

$$\frac{\displaystyle\sum_{j=0}^{n_1} x^j \frac{\sum_{i=0}^{n_2} \alpha_{ij} y^i}{\sum_{i=0}^{n_2} \beta_{ij} y^i} + \sum_{j=0}^{n_1} y^j \frac{\sum_{i=0}^{n_2} p_{ij} x^i}{\sum_{i=0}^{n_2} q_{ij} x^i}}{\displaystyle\sum_{i=0}^{m} \sum_{j=0}^{m} \alpha_{ij} x^i y^j}$$

and we shall denote them by $[(n_1;n_2)/m]_r$ because for their computation:

$$N_u = (m+1)^2 + (n_2+1)(n_1+1)$$

$$N_f = 2(2n_2+1)(n_1+1) - (n_1+1)^2 + [\max(0,m+r-n_1)]^2 - 1$$

(for more details see [13]).

Using the prong method [10] the homogeneous system of equations for the calculations of Chisholm's approximants can be solved in $0[m^2(2m^2+2m-1)]$ operations. The calculation of a function value of an operator Pade approximant can be performed via the ε-algorithm in $O[m^2(n+m)^2]$ operations and we prefer this method to the solution of the system of equations (2).

The solution of the homogeneous system for the calculation of $[n_1;n_2)/m]_r$ involves $O[(m+1)^6 + (n_2+1)^2(n_1+1)]$ operations.

After comparison of the N_f, N_u and the computational effort, we decided to compare the numerical values of the bivariate Pade approximants given in the table below. Chisholm's approximants are of the type $(n,n/(m,m)$; the operator Pade approximants are still indicated by n/m.

For the different groups (I), (II) and (III) we have N_f approximately equal to 87, 40 and 71 respectively.

It is easy to see that the operator Pade approximants can produce better results than the Chisholm approximants, e.g. for $(x,y) = (-0.75),-0.75)$, and that they also can be better than the approximants Levin used, e.g. for $(x,y) = (0.50,0.50)$. They are most accurate for $(x-1,y-1)$ not too far from the origin.

(x,y)	(-0.75,-0.75)	(-0.50,-0.50)	(-0.25,-0.25)	(0.25,0.25)	(0.50,0.50)	(0.75,0.75)	(-1.75,1.75)
B(x,y)	9.88829829	0.	-6.77770467	7.41629871	3.14159265	1.694426166	0.
$[(4;5)/2]_3$	9.888	-0.00021	-6.777755	7.41629594	3.14159248	1.69442616	0.0186
(7;7)/(3,3)	9.820	-0.0010	-6.77774	7.41629871	3.14159265	1.69442617	0.0016
8/4	9.884	-0.00006	-6.777705	7.41629871	3.14159265	1.69442617	-0.0351

(x,y)	(-0.75,-0.75)	(-0.50,-0.50)	(-0.25,-0.25)	(0.25,0.25)	(0.50,0.50)	(0.75,0.75)	(0.75,0.25)
B(x,y)	9.88839829	0.	-6.77770467	7.41629871	3.14159265	1.694426166	4.44288293
$[(3;3)/1]_1$	9.94	-0.03	-6.794	7.416229	3.14159242	1.69442617	4.442883
(3;3)/(3,3)	7.0	-0.14	-6.787	7.416310	3.14159269	1.69442617	4.442883
4/4	8.38	-0.13	-6.802	7.416281	3.14159263	1.69442617	4.442883

(x,y)	(-0.75,-0.75)	(-0.50,-0.50)	(-0.25,-0.25)	(0.25,0.25)	(0.50,0.50)	(0.75,0.75)	(1.75,-.75)
B(x,y)	9.88839829	0.	-6.7777-467	7.41629871	3.14159265	1.694426166	-4.44288293
$[2;3)/2]_2$	9.86	-0.003	-6.7783	7.41629639	3.14159252	1.69442617	-4.4428
(7,7)/(2,2)	9.3	-0.014	-6.7783	7.41629881	3.14159265	1.69442617	-4.4421
8/3	9.74	-0.006	-6.7783	7.41629862	3.14159265	1.69442617	-4.4442

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