To my dear mother and brother.

NUMERICAL COMPARISON OF ABSTRACT PADE-APPROXIMANTS AND ABSTRACT RATIONAL APPROXIMANTS WITH OTHER GENERALIZATIONS OF THE CLASSICAL PADE-APPROXIMANT **

by

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For the introduction of abstract Padé-approximants we refer to (I)*and (II). Now we want to consider an interesting numerical example that can teach us something about the location of zeros and singularities of a nonlinear operator and its different approximations (sections 1-2-3).
Also we shall compare abstract Padé-approximants for a nonlinear operator $\mathbb{R}^2 \to \mathbb{R}$ with other types of 2-variable rational approximants (sections 4-5).

* Roman figures between brackets refer to a work in the reference list.

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1. Nonlinear operator

$$\text{Let } F: \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \to \begin{pmatrix} \dfrac{\dfrac{e^x}{2(1-10y)} - \dfrac{1.05-y}{1-10x}}{\sin(\frac{\pi}{2} + 0.05 + x - y)} \\ \\ \cos(\frac{\pi}{2} - 0.05 - x + y) \end{pmatrix} = \begin{pmatrix} F_1(x,y) \\ \\ F_2(x,y) \end{pmatrix}$$

The operator F is singular for $x = 0.1$

or $y = 0.1$

or $y = x + (2k+1)\frac{\pi}{2} + 0.05 \quad (k \in \mathbb{Z})$.

The second component F_2 vanishes on $y = x + k\pi + 0.05$ $(k \in \mathbb{Z})$.

For $k = o$: $F = 0$ in $\binom{0}{0.05}$ and $\binom{0.37981434..}{0.42981434..}$.

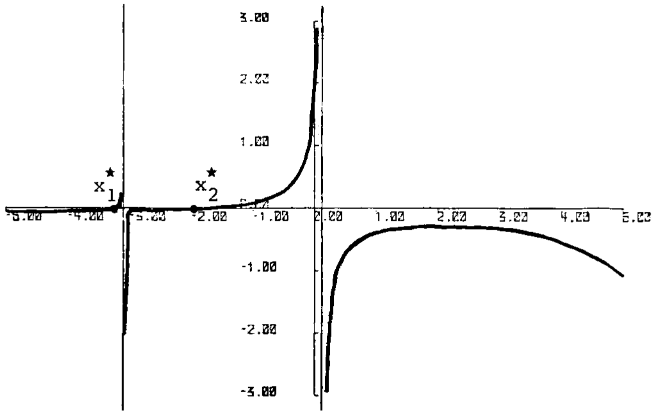
For $k < o$: the first component F_1 does not vanish on $y = x + k\pi + 0.05$.

For $k > o$: F has two zeros $x_1^\star$ and $x_2^\star$ on $y = x + k\pi + 0.05$.

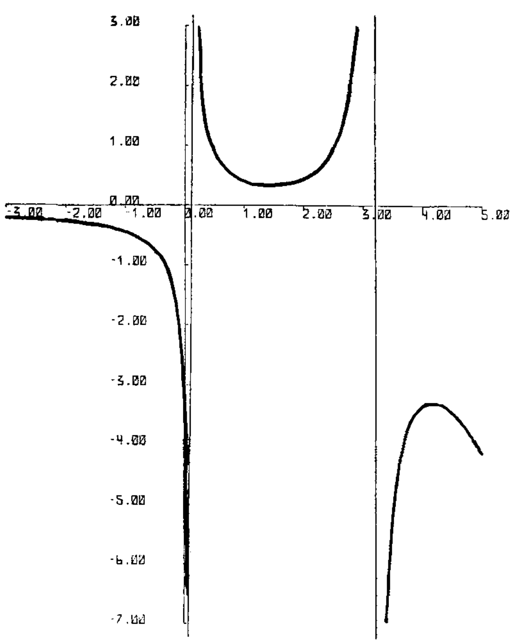
On $y = x + k\pi + 0.05$ the operator F has two poles, namely in $x_1^\blacksquare = 0.05 - k\pi$ and $x_2^\blacksquare = 0.01$.

A characteristic behaviour of F on $y = x + k\pi + 0.05$ for $k > o$ and $k < o$ is respectively shown in F1.1 and F1.2, while F1.3 shows the behaviour of F on $y = x + 0.05$ ($k=o$).

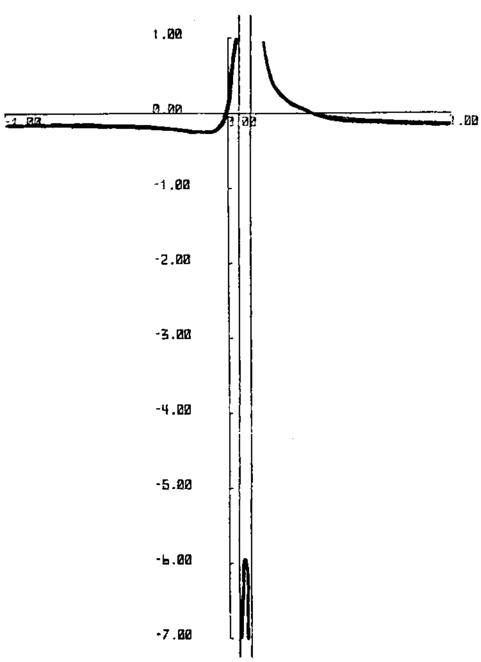
The fact that for $k > o$: $|x_1^\star - x_1^\blacksquare|$ decreases for increasing k, complicates the calculation of the root $x_1^\star$ of $F(x,y) = 0$.



F1.1 (k=1)



F1.2 (k=-1)



F1.3 (k=o)

2. (1,1) Abstract Padé Approximant (APA)

Let us now approximate F by a rational operator R and study the location of the zeros and the poles of this approximation. We perform the necessary calculations (as described in (I)) to obtain the (1,1)-APA in $\binom{0}{0}$ and have to conclude that its first component is undefined in $\binom{0}{0}$. But the second component is the (1,1) Abstract Padé Approximant to the second component of F.

$$R: \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \to \begin{pmatrix} \dfrac{ax+by+cx^2+dxy+ey^2}{a'x+b'y+(c'x^2+d'xy+e'y^2)\ \sin(\frac{\pi}{2}+0.05)} \\[2ex] \dfrac{\cos(\frac{\pi}{2}-0.05)+(x-y)\ [\sin(\frac{\pi}{2}-0.05)+0.5\operatorname{cotg}(\frac{\pi}{2}-0.05)\cos(\frac{\pi}{2}-0.05)]}{1+0.5\ (x-y)\ \operatorname{cotg}(\frac{\pi}{2}-0.05)} \end{pmatrix}$$

$$= \begin{pmatrix} R_1(x,y) \\ R_2(x,y) \end{pmatrix}$$

with $a = -0.3025\ \operatorname{cotg}(\frac{\pi}{2}+0.05) + 5.5$

$b = 0.3025\ \operatorname{cotg}(\frac{\pi}{2}+0.05) - 3.3$

$c = 42.3875 - 5.5\ \operatorname{cotg}(\frac{\pi}{2}+0.05) - 0.3025/\sin^2(\frac{\pi}{2}+0.05)$

$d = -111.75 + 9.6\ \operatorname{cotg}(\frac{\pi}{2}+0.05) + 0.605/\sin^2(\frac{\pi}{2}+0.05)$

$e = 63.5 - 3.3\ \operatorname{cotg}(\frac{\pi}{2}+0.05) - 0.3025/\sin^2(\frac{\pi}{2}+0.05)$

$a' = 0.55\ \cos(\frac{\pi}{2}+0.05) - 10\sin(\frac{\pi}{2}+0.05)$

$b' = -0.55\ \cos(\frac{\pi}{2}+0.05) + 6\sin(\frac{\pi}{2}+0.05)$

$c' = 0.55\ \operatorname{cotg}^2(\frac{\pi}{2}+0.05) - 10\operatorname{cotg}(\frac{\pi}{2}+0.05) + 104.75 + 0.55/\sin^2(\frac{\pi}{2}+0.05)$

$d' = -1.1\operatorname{cotg}^2(\frac{\pi}{2}+0.05) + 16\operatorname{cotg}(\frac{\pi}{2}+0.05) - 15 - 1.1/\sin^2(\frac{\pi}{2}+0.05)$

$e' = 0.55\ \operatorname{cotg}^2(\frac{\pi}{2}+0.05) - 6\operatorname{cotg}(\frac{\pi}{2}+0.05) - 50 + 0.55/\sin^2(\frac{\pi}{2}+0.05).$

The second component R_2 vanishes on

$$y = x + \frac{\cos(\frac{\pi}{2} - 0.05)}{\sin(\frac{\pi}{2} - 0.05) + 0.5\mathrm{cotg}(\frac{\pi}{2} - 0.05)\cos(\frac{\pi}{2} - 0.05)}$$

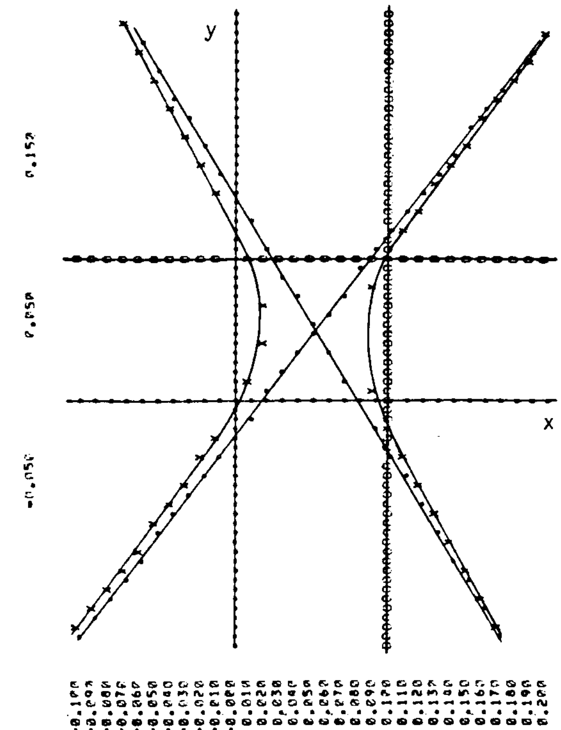
$$= x + 0.0499...$$

R has two zeros, near to the zeros of F on $y = x+0.05$ (k=0), namely in $\binom{0.00252235...}{0.05250148...}$ and $\binom{0.49805568...}{0.54803481...}$.

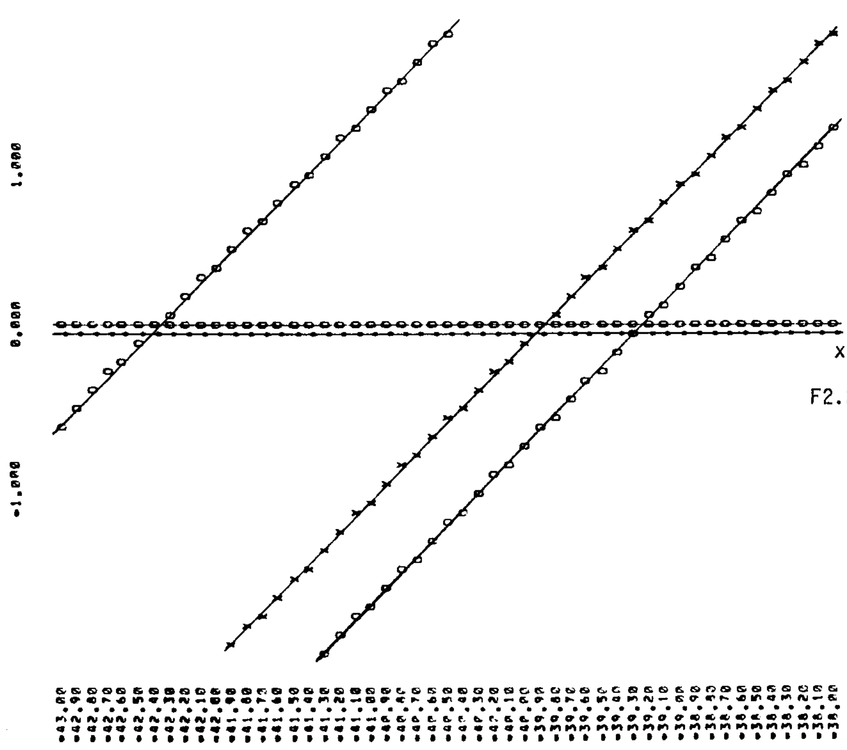
Because numerator and denominator of R are polynomial operators we lose the periodicity of F (no infinite number of zeros). The abstract rational approximant has distributed its poles in a very interesting manner.

Looking at F2.1 and F2.2 which show the poles of F(plotted as 000-lines) and those of R(plotted as XXX-lines) in the considered area, we remark that the dominating direction of the first bisector for the poles of F is somewhat found back in the asymptotic behaviour of the poles of R_1 (hyperbola) and in the situation of the poles of R_2 on y=x+39.966...
X-axis and Y-axis are marked by dots (...) as well as the asymptotes for the poles of R_1: $y = 1.305x - 0.015$

$$y = -1.649x + 0.138$$



F2.1



F2.2

3. Taylor series expansion

Since for functions $f : \mathbb{R} \to \mathbb{R}$ we can compare the curve-fitting ability of a rational function of degree n in the numerator and degree m in the denominator with that of a polynomial of degree n+m, we also calculated the Taylor series expansion T in $\binom{0}{0}$ up to and including 2^{nd} order terms.

$$T : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \to \begin{pmatrix} \dfrac{-0.55}{\sin(\frac{\pi}{2}+0.05)} + ax + by + cx^2 + dxy + ey^2 \\ \cos(\frac{\pi}{2}-0.05) + (x-y)\sin(\frac{\pi}{2}-0.05) - 0.5\,(x-y)^2\cos(\frac{\pi}{2}-0.05) \end{pmatrix}$$

$$= \begin{pmatrix} T_1(x,y) \\ T_2(x,y) \end{pmatrix}$$

with $a = \dfrac{1}{\sin(\frac{\pi}{2}+0.05)}\,(0.55\ \mathrm{cotg}(\frac{\pi}{2}+0.05)-10)$

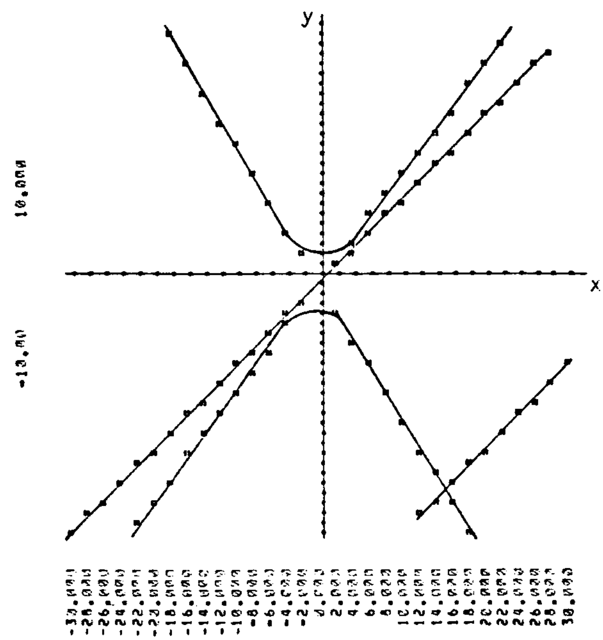
$$b = \frac{1}{\sin(\frac{\pi}{2}+0.05)}\,(-0.55\ \mathrm{cotg}(\frac{\pi}{2}+0.05)+6)$$

$$c = \frac{1}{\sin(\frac{\pi}{2}+0.05)}\,\left(-0.55\,\frac{(1+\cos^2(\frac{\pi}{2}+0.05))}{\sin^2(\frac{\pi}{2}+0.05)} + 10\ \mathrm{cotg}(\frac{\pi}{2}+0.05) - 104.75\right)$$

$$d = \frac{1}{\sin(\frac{\pi}{2}+0.05)}\,\left(1.1\,\frac{(1+\cos^2(\frac{\pi}{2}+0.05))}{\sin^2(\frac{\pi}{2}+0.05)} - 16\ \mathrm{cotg}(\frac{\pi}{2}+0.05) + 15\right)$$

$$e = \frac{1}{\sin(\frac{\pi}{2}+0.05)}\,\left(-0.55\,\frac{(1+\cos^2(\frac{\pi}{2}+0.05))}{\sin^2(\frac{\pi}{2}+0.05)} + 6\ \mathrm{cotg}(\frac{\pi}{2}+0.05) + 50\right)$$

F3.1 which shows the zeros of T_1 (hyperbola) and T_2 (straight lines) demonstrates that we do not have to look for zeros of T near the origin. The singularities of F are the cause of this bad behaviour of T.



F3.1

But T_2 does also preserve the dominant direction of the first bisector for the zeros of F.

Assume that we got $\binom{0}{0}$, the point in which the approximations were calculated, from a previous iteration-step in a procedure that calculates the root $\binom{0}{0.05}$ of $F(x,y) = 0$.

Calculating the approximation R and equating its numerator to zero would supply a good estimate of $\binom{0}{0.05}$, while the approximation T cannot be used to obtain an estimate of the root in $\binom{0}{0.05}$.

4. Different Padé-type 2-variable rational approximants

We are going to compare Abstract Padé Approximants (APA) or Abstract Rational Approximants (ARA) for F with Chisholm diagonal (III) approximants (CA) or Hughes Jones off-diagonal (IV,V) approximants (HJA), Lutterodt (VII)-approximants (LA), Lutterodt-approximants of type B^1 (VIII) (LAB^1), Karlsson and Wallin-approximants (VI) (KWA) and partial sums of the abstract (IX) Taylor-series development (PS), all in $\binom{n}{0}$.

The calculation of each type of approximant $\frac{P}{Q}:\mathbb{R}^2 \to \mathbb{R}$ is based on :

$$(FQ-P)(x,y) = \sum_{i,j=o}^{\infty} d_{ij}\, x^i\, y^j \text{ with } d_{ij}=0 \text{ for } (i,j) \in S \subset \mathbb{N}^2.$$

We call S the interpolationset; the choice of S determines the type of approximant. The KWA is unique when the interpolationset S contains in addition to $\{(i,j)\,|\,i+j \leq n\}$, as many points as possible in a given enumeration in $\mathbb{N}^2$ (we have used the diagonal enumeration (0,0), (1,0), (0,1),(2,0),(1,1),(0,2),(3,0), ...).

The LA need not be unique with respect to the chosen interpolation set (we shall give the interpolationset together with the calculated approximant). For the CA, HJA and LA we denote by $(n_1,n_2)/(m_1,m_2)$ a rational approximant of degree n_1 in x and n_2 in y in the numerator and of degree m_1 in x and m_2 in y in the denominator. For the APA and KWA we denote by n/m a rational approximant where the sum of the degrees in x and y is at most n in the numerator and at most m in the denominator. The n^{th} partial sum of the Taylor series development is indicated by PSn.

Let N be the amount of (unknown) coefficients in the approximant (for rational approximants 1 coefficient can always be determined by a normalisation).

We consider N-1 to be a measure for the "operator-fitting" ability of the calculated rational approximant, and N to be a measure for the "operator-fitting" ability of the considered partial sum of the Taylor-series development.

For CA, HJA and LA : $N=(n_1+1)(n_2+1)+(m_1+1)(m_2+1)$.

For KWA and APA : $N=\frac{1}{2}(n+1)(n+2)+\frac{1}{2}(m+1)(m+2)$.

For PSn : $N=(n+1)(n+2)/2$.

For increasing N we expect increasing accuracy.

We do remind that for all the types of rational approximants considered, except LAB^1, a lot of classical properties of Padé-approximants for analytic functions: $\mathbb{R} \to \mathbb{R}$ remain valid, now for analytic operators: $\mathbb{R}^2 \to \mathbb{R}$, such as:

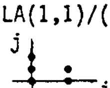
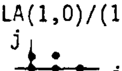
a) reciprocal covariance: if $F(0,0)\neq 0$ and P/Q is the Padé-type approximant for F with interpolationset S, then Q/P is the Padé-type approximant for $\frac{1}{F}$ with the same interpolationset.

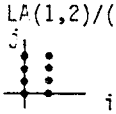
b) if P/Q is a diagonal Padé-type approximant $((n_1,n_2) = (m_1,m_2)$ or $n = m)$ and for $a,b,c,d \in \mathbb{R}$: $ad-bc\neq 0$, $cF(0,0)+d\neq 0$, then $(aP+bQ)/(cP+dQ)$ is the Padé-type approximant $((n_1,n_2)/(m_1,m_2)$ or $n/m)$ for $\frac{aF+b}{cF+d}$ with the same interpolationset.

But only the CA, HJA, LA type B^1 and APA have the projection property: equating in the Padé-type approximant P/Q a variable to zero, supplies the Padé-type approximant in the remaining variables.

5. Examples and conclusions

a) Let us consider $F : \mathbb{R}^2 \to \mathbb{R} : (x,y) \to 1 + \frac{x}{0.1-y} + \sin(xy)$.

type	approximant	N	exact order of FQ-P
(1) PS 2	$1+10x+101xy$	6	$O(xy^2)$
(2) HJA(1,1)/(0,1)	degenerate $\frac{1+10x+\alpha y+(101+10\alpha)xy}{1+\alpha y}$	6	$O(xy^2)$ $\alpha = \frac{-1000}{101} \Rightarrow O(xy^3)$
(3) HJA(1,0)/(1,1)	$\frac{1+10x}{1-101xy}$	6	$O(x^2y)$
(4) HJA(0,1)/(1,1)	degenerate $\frac{1-(\frac{101+\alpha}{10})y}{1-10x-(\frac{101+\alpha}{10})y+\alpha xy}$	6	$O(x^2)\ \forall\alpha$
(5) HJA(1,1)/(1,0)	$1+10x+101xy$	6	$O(xy^2)$
(6) KWA 1/1	$\frac{1+10x-10.1y}{1-10.1y}$	6	$O(xy^2)$
(7) LA(1,1)/(0,1) 	$\frac{1+10x+\alpha y+(101+10\alpha)xy}{1+\alpha y}$	6	see (2)
(8) LA(1,0)/(1,1) 	$\frac{1+10x}{1-101xy}$	6	$O(x^2y)$
(9) APA 1/1	$\frac{1+10x-10.1y}{1-10.1y}$	6	$O(xy^2)$
(10) CA(1,1)/(1,1)	degenerate $\frac{1+10x+(10-10\alpha)y+\alpha xy}{1+(10-10\alpha)y+(101\alpha-201)xy}$	8	$O(x^2y\ ,\ xy^2)$ $\alpha = \frac{201}{101} \Rightarrow O(xy^3)$

(11) HJA(1,2)/(0,2)	degenerate $\dfrac{1+10x+\alpha y+(101+10\alpha)xy+\beta y^2+(10\beta+101\alpha+1000)xy^2}{1+\alpha y+\beta y^2}$	9	$O(xy^3)$ $\alpha=-10$ and $\beta=0$ $\Rightarrow O(x^3y^3)$
(12) HJA(2,1)/(0,2)	degenerate $\dfrac{1+10x+\alpha y+(101+10\alpha)xy}{1+\alpha y}$	9	$O(xy^2)$ $\alpha=\frac{-1000}{101}\Rightarrow O(xy^3)$
(13) LA(1,2)/(0,2) 	$\dfrac{1+10x+\alpha y+(101+10\alpha)xy+\beta y^2+(10\beta+101\alpha+1000)xy^2}{1+\alpha y+\beta y^2}$ with $\beta=-10^3(10+\alpha)/101$	9	see (11)
(14) LA(2,1)/(0,2) 	$\dfrac{1+10x-\frac{1000}{101}y+\frac{201}{101}xy}{1-\frac{1000}{101}y}$	9	$O(xy^3)$
(15) KWA 1/2	$\dfrac{1+10x-10.1y}{1-10.1y}$	9	$O(xy^2)$
(16) ARA 1/2	$\dfrac{x-1.01y+10y^2+10x^2-20.2xy}{x-1.01y+10y^2-10.1xy+2.01xy^2}$	14	$O(xy^3)$
(17) APA 2/1	$\dfrac{1+10x-\frac{1000}{101}y+\frac{201}{101}xy}{1-\frac{1000}{101}y}$	9	$O(xy^3)$
(18) KWA 2/1	$\dfrac{1+10x-\frac{1000}{101}y+\frac{201}{101}xy}{1-\frac{1000}{101}y}$	9	$O(xy^3)$
(19) PS 3	$1+10x+101xy+1000xy^2$	10	$O(xy^3)$
(20) CA(2,2)/(1,1)	degenerate $\dfrac{1+(10+\alpha)x-10y+(\beta+1)xy+10\alpha x^2-10xy^2+(10\beta+101\alpha)x^2y+(101\beta+1000\alpha)x^2y^2}{1+\alpha x-10y+\beta xy}$	13	$O(x^2y^3)$ $\alpha=0$ and $\beta=0$ $\Rightarrow O(x^3y^3)$

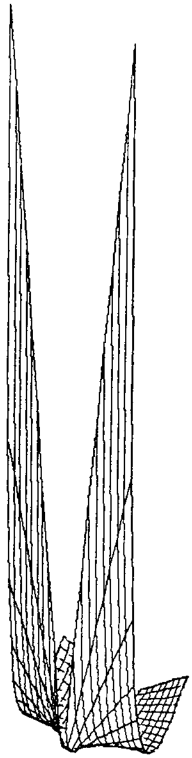
(21) KWA 3/1	$\dfrac{1+10x-10y+xy-10xy^2}{1-10y}$	13	$0(x^3y^3)$
(22) LA(2,2)/(1,1)	no interpolationset satisfying the description in (VII)	13	
(23) APA 3/1	$\dfrac{1+10x-10y+xy-10xy^2}{1-10y}$	13	$0(x^3y^3)$
(24) PS 4	$1+10x+101xy+1000xy^2+10000xy^3$	15	$0(xy^4)$
(25) $LAB^1(1,1)/(0,1)$	the prescribed interpolationset supplies a system of linearly dependent equations	6	
(26) $LAB^1(1,0)/(1,1)$		6	
(27) $LAB^1(1,2)/(0,2)$		9	
(28) $LAB^1(2,1)/(0,2)$		9	
(29) $LAB^1(2,2)/(1,1)$		13	

For each of the mentioned approximations of the first example we have plotted the surface $|F(x,y)\text{-approximation }(x,y)|$ on $A = [-0.09,0.09] \times [-0.09,0.09]$ (F6.1 - F6.10), nearly all from the same viewpoint.

We have also calculated an estimate ε_r of $\dfrac{\sup\limits_A |F(x,y)\text{-approximation}(x,y)|}{\sup\limits_A |F(x,y)|}$

which is a measure for the relative error made by approximating ($\sup\limits_A |F(x,y)| \cong 10$).

We remark that we may call the APA accurate.



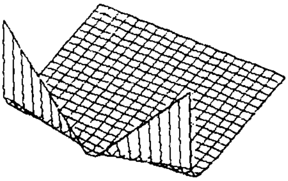
F6.1

$|F(x,y) - (1+10x+101xy)|$

$\varepsilon_r = 0.73$

viewpoint (1,2,10)

① ⑤

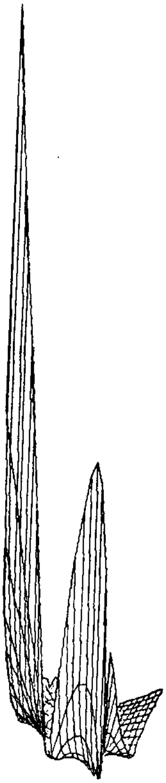


F6.2

$$\left| F(x,y) - \frac{1+10x - \frac{1000}{101} y + \frac{201}{101} xy}{1 - \frac{1000}{101} y} \right|$$

$\varepsilon_r = 0.06$

viewpoint (1,2,10) ② ⑩ ⑫ ⑭ ⑰ ⑱



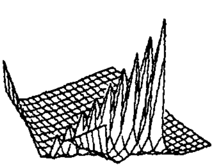
F6.3

$$\left| F(x,y) - \frac{1+10x}{1-101xy} \right|$$

$\varepsilon_r = 0.81$

viewpoint (1,2,10)

③ ⑧



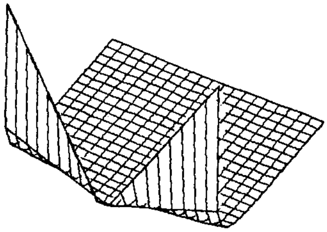
F6.4

$$\left| F(x,y) - \frac{1-10.1y}{1-10x-10.1y} \right|$$

$\varepsilon_r = 90.1$

viewpoint (1,1,100) because of steepness

④



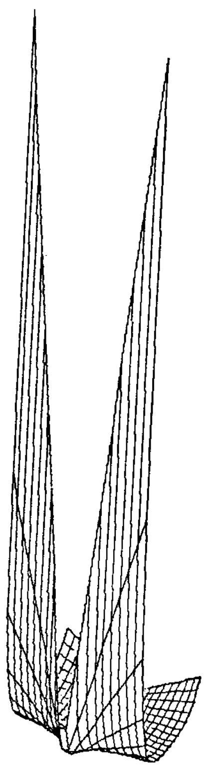
F6.5

$$\left| F(x,y) - \frac{1+10x-10.1y}{1-10.1y} \right|$$

$\epsilon_r = 0.09$

viewpoint (1,2,10)

⑥⑨⑮



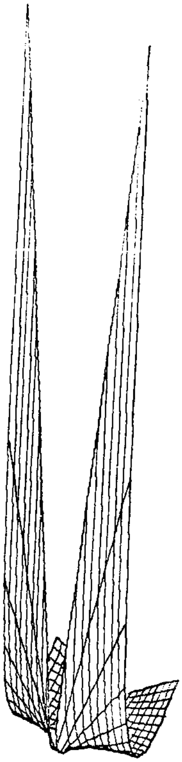
F6.8

$$\left| F(x,y) - (1 + 10x + 101xy + 1000xy^2) \right|$$

$\epsilon_r = 0.66$

viewpoint (1,2,10)

⑬ ⑲



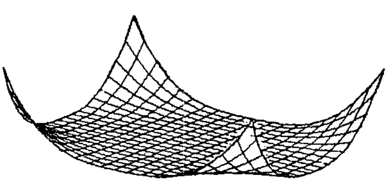
F6.6

$$\left| F(x,y) - \frac{1 + 10x - \frac{1}{101}\, y + \frac{10191}{101}\, xy}{1 - \frac{1}{101}\, y} \right|$$

$\epsilon_r = 0.73$

viewpoint (1,2,10)

⑦



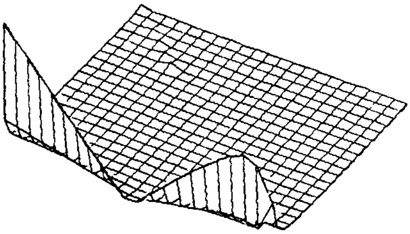
F6.7

$$\left| F(x,y) - \frac{1 + 10x - 10y + xy - 10xy^2}{1-10y} \right| * 10^5$$

$\epsilon_r = 0.9 \times 10^{-7}$ for $\left| F(x,y) - \frac{1+10x-10y+xy-10xy^2}{1-10y} \right|$

viewpoint (1,2,1) because of flatness

⑪ ⑳ ㉑ ㉓



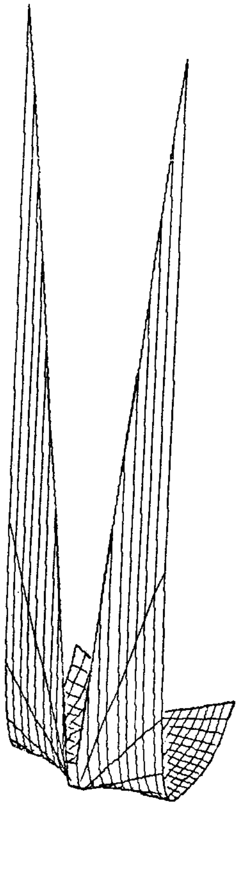
F6.9

$$\left| F(x,y) - \frac{x - 1.01y + 10y^2 + 10x^2 - 20.2xy}{x - 1.01y + 10y^2 - 10.1xy + 2.01x^2y} \right|$$

$\epsilon_r = 0.07$ (equating $|F - \text{ARA } 1/2| \binom{0}{0}$ to 0)

viewpoint (1,2,10)

(16)



F6.10

$$\left| F(x,y) - (1 + 10x + 101xy + 1000xy^2 + 10000xy^3) \right|$$

$\epsilon_r = 0.6$

viewpoint (1,2,10)

(24)

We merely have to compare : ε_r for (1) - (9) and remark that at F6.2 and F6.5 the most accurate approximations are gathered; HJA(1,1)/(0,1) is a bit more accurate than KWA 1/1 and APA 1/1 because a rational function (1,1)/(0,1) fits very well the behaviour of F; however sometimes the approximation cannot be adjusted to F in this way (more complicated operators F) and we can as well at random have chosen worse approximants without knowing it (e.g. (1,0)/(1,1) or (0,1)/(1,1) or (1,1)/(1,0) in this case)

ε_r for (10) - (19) and remark that F6.2 and F6.9 gather very good approximations; only HJA(1,2)/(0,2) is better, partly because of the very degenerate solution and partly because the denominator $1+\alpha y+\beta y^2$ can fit F very well

ε_r for (20) - (24) and remark that PS 4 is very bad in comparison with all the rational approximations, what was to be expected.

We compare the different types of approximants on two other examples.

b) Let us consider $F : \mathbb{R}^2 \to \mathbb{R} : (x,y) \to \dfrac{xe^x - ye^y}{x-y} = \sum\limits_{i,j=o}^{\infty} \dfrac{1}{(i+j)!} x^i y^j$.

Here we have in the Taylor series expansion of F a term in every power $x^i y^j$.

We compare the function values in some points.

		N					
x			0.05	0.25	0.25	0.65	0.65
y			0.25	0.05	0.45	0.45	0.85
F(x,y)	$\frac{xe^x - ye^y}{x-y}$		1.342	1.342	1.924	2.697	3.718
PS 2	$1+x+y+\frac{1}{2}(x^2+xy+y^2)$	6	1.339	1.339	1.889	2.559	3.349
$LAB^1(1,1)/(1,0)$	$\frac{1+\frac{1}{2}x+y}{1-\frac{1}{2}x}$	6	1.308	1.343	1.800	2.630	3.222
$LAB^1(1,1)/(1,1)$	$\frac{1+\frac{1}{2}(x+y)-\frac{1}{4}xy}{1-\frac{1}{2}(x+y)+\frac{1}{4}xy}$	8	1.328	1.328	2.032	2.109	4.153
ARA 1/1	$\frac{x+y+\frac{1}{2}(x^2+3xy+y^2)}{x+y-\frac{1}{2}(x^2+xy+y^2)}$	10	1.344	1.344	1.958	2.887	4.455
CA(1,1)/(1,1)	$\frac{1+\frac{1}{2}(x+y)-\frac{1}{6}xy}{1-\frac{1}{2}(x+y)+\frac{1}{3}xy}$	8	1.344	1.344	1.936	2.742	3.819
HJA(1,1)/(0,1)	$\frac{1+x+\frac{1}{2}y}{1-\frac{1}{2}y}$	6	1.343	1.308	1.903	2.419	3.609
KWA 1/1	$\frac{1+\frac{1}{2}x+y}{1-\frac{1}{2}x}$	6	1.308	1.343	1.800	2.630	3.222

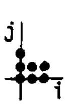
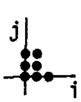
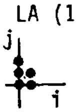
We see that ARA is good as well for $x>y$ as for $x<y$ (on a not too large neighbourhood), while the other approximations, except CA(1,1)/(1,1), are not. The reason is still the same as in section 5a: (1,1)/(1,0) fits the behaviour of F if $x>y$ and (1,1)/(0,1) fits the behaviour of F if $y>x$. What's more: $F(x,y)=F(y,x)$ and APA and ARA always conserve this property, while the other types of approximants do not.

c) Now consider $F:\{(x,y) \mid y \geq -x-1\} \subset \mathbb{R}^2 \to \mathbb{R} : (x,y) \to \sqrt{1+x+y} =$

$$1 + \frac{x+y}{2} + \sum_{k=2}^{\infty} (-1)^{k-1} \frac{(x+y)^k}{k!} \frac{(2k-3)!!}{2^k}$$

where $(2k-3)!! = (2k-3)(2k-1) \ldots 5.3.1$

We calculate some approximants:

APA 1/1	$\dfrac{1 + 0.75\ (x+y)}{1 + 0.25\ (x+y)}$
CA (1,1)/(1,1)	$\dfrac{1 + 0.75\ (x+y) - 0.1875\ xy}{1 + 0.25\ (x+y) - 0.1875\ xy}$
HJA (1,1)/(1,0)	$\dfrac{1 + 0.75x + 0.5y - 0.125\ xy}{1 + 0.25\ x}$
HJA (1,1)/(0,1)	$\dfrac{1 + 0.5x + 0.75y - 0.125\ xy}{1 + 0.25\ y}$
KWA 1/1	$\dfrac{1 + 0.75\ (x+y)}{1 + 0.25\ (x+y)}$
LA (1,1)/(1,1)  or 	$\dfrac{1 + 0.75\ (x+y) - 0.1875\ xy}{1 + 0.25\ (x+y) - 0.1875\ xy}$
LA (1,1)/(1,0) 	$\dfrac{1 + 0.75x + 0.5y - 0.125\ xy}{1 + 0.25\ x}$
LA (1,1)/(0,1) 	$\dfrac{1 + 0.5x + 0.75y - 0.125\ xy}{1 + 0.25\ y}$

The border of the domain of F is nicely simulated by the poles of the APA k/1 : $y = -x - \frac{2k+2}{2k-1}$ with $\lim_{k\to\infty} \frac{-2k-2}{2k-1} = -1$

We also compare the function-values in different points:

	(x,y)=(2,-1)	(x,y)=(-0.4,-0.5)	(x,y)=(2,-2)
F	1.4142	0.3162	1.0000
APA 1/1	1.4000	0.4194	1.0000
CA (1,1)/(1,1)	1.3077	0.3898	1.0000
HJA (1,1)/(1,0)	1.5000	0.4722	1.3333
HJA (1,1)/(0,1)	2.0000	0.4571	2.0000

When we compare the approximations that have the same "operator-fitting" ability (as defined earlier), we see that APA 1/1 and KWA 1/1 are much more accurate than the other types.

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