Journal of Computational and Applied Mathematics 32 (1990) 47-57
North-Holland

A multivariate convergence theorem of the "de Montessus de Ballore" type

A. CUYT
Department of Mathematics and Computer Science, University of Antwerp (UIA), Universiteitsplein 1, B-2610 Antwerp (Wilrijk), Belgium

Received 24 September 1989
Revised 15 February 1990

Abstract: The univariate theorem of "de Montessus de Ballore" proves the convergence of column sequences of Padé approximants for functions $f(z)$ meromorphic in a disk, in case the number of poles of $f(z)$ and their multiplicity is known in advance. We prove here a multivariate analogon for the case of "simple" poles and for the general order Padé approximants as introduced by Cuyt and Verdonk (1984).

Keywords: Padé approximation, convergence, multivariate, meromorphic functions.

1. The univariate "de Montessus de Ballore" theorem

Let the polynomials $p(z)$ and $q(z)$, respectively of degree N and M, solve the (N, M) Padé approximation problem for the function $f(z)$. In other words, for

$$f(z) = \sum_{i=0}^{\infty} c_i z^i, \quad \text{with } c_i = \frac{f^{(i)}(0)}{i!},$$

the polynomials

$$p(z) = \sum_{i=0}^{N} a_i z^i, \qquad q(z) = \sum_{i=0}^{M} b_i z^i$$

are computed such that they satisfy

$$(fq - p)(z) = \sum_{i \geqslant N+M+1} d_i z^i.$$

We denote this Padé approximant by $[N/M]$. Usually these Padé approximants $[N/M] = p(z)/q(z)$ for different N and M are arranged in a table where the numerator degree is the row index and the denominator degree is the column index. The theorem of "de Montessus de Ballore" then proves convergence of column sequences of Padé approximants for functions $f(z)$ meromorphic in a disk, in case the number of poles of $f(z)$ is known a priori. The complete statement of the theorem allows the possibility of multiple poles for the function and is as follows.

Theorem 1. *Let $f(z)$ be a function which is meromorphic in the disk $B(0;\ R)=\{z\colon |z|\leqslant R\}$, with k poles at distinct points $z_1, z_2,\dots, z_k$ with*

$$0<|z_1|\leqslant|z_2|\leqslant\cdots\leqslant|z_k|<R.$$

Let the pole at z_i have multiplicity μ_i and let the total multiplicity be $M=\sum_{i=1}^k\mu_i$. Then

$$f(z)=\lim_{N\to\infty}[N/M]$$

uniformly on any compact subset of

$$B(0;\ R)\backslash\{z_1,\dots,z_k\}.$$

Proof. For the proof we refer to Saff's short and elegant proof, which can, for instance, be found in [1, pp.252–254]. The uniform convergence is based on the error formula

$$f(z)-[N/M]=\left(\frac{1}{2\pi i}\right)\frac{1}{(qR_M)(z)}\sum_{i>N+M}z^i\int_{|v|=R}\frac{(fqR_M)(v)}{v^{i+1}}\,\mathrm{d}v,$$

where

$$R_M(z)=(z-z_1)^{\mu_1}\cdots(z-z_k)^{\mu_k}.$$

This expression tends to zero for $|z|<R$. □

2. Multivariate Padé approximants

We restrict our description to the bivariate case, only for notational convenience, although we may use the term multivariate in the text.

Given a Taylor series expansion

$$f(x,\ y)=\sum_{(i,j)\in\mathbb{N}^2}c_{ij}x^iy^j,\quad\text{with } c_{ij}=\frac{1}{i!}\frac{1}{j!}\frac{\partial^{i+j}f}{\partial x^i\,\partial y^j}(0,\ 0),$$

we compute a Padé approximant $p(x,\ y)/q(x,\ y)$ to $f(x,\ y)$ where the polynomials $p(x,\ y)$ and $q(x,\ y)$ are given by

$$p(x,\ y)=\sum_{(i,j)\in N}a_{ij}x^iy^j,\quad N\subset\mathbb{N}^2,\qquad q(x,\ y)=\sum_{(i,j)\in M}b_{ij}x^iy^j,\quad M\subset\mathbb{N}^2,$$

and are determined by the following "accuracy-through-order" principle [6]. The finite sets N and M indicate the "degree" of $p(x,\ y)$ and $q(x,\ y)$. Let us denote

$$\#N=n+1,\qquad \#M=m+1.$$

It is possible to let $p(x,\ y)$ and $q(x,\ y)$ satisfy

$$(fq-p)(x,\ y)=\sum_{(i,j)\in\mathbb{N}^2\backslash E}d_{ij}x^iy^j, \tag{1}$$

if, in analogy with the univariate case, the index set $E\subset\mathbb{N}^2$ (Equations) is chosen such that

$$N\subseteq E, \tag{2a}$$

$$\#(E\backslash N)=m=\#M-1, \tag{2b}$$

E satisfies the inclusion property, (2c) meaning that if a point belongs to the index set E, then the rectangular subset of points emanating from the origin with the given point as its furthermost corner, also lies in E.

Condition (2a) enables us to split the system of equations

$$d_{ij}=0, \quad (i, j)\in E,$$

in an inhomogeneous part defining the numerator coefficients

$$\sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{\mu\nu}b_{i-\mu,j-\nu}=a_{ij}, \quad (i, j)\in N,$$

and a homogeneous part defining the denominator coefficients

$$\sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{\mu\nu}b_{i-\mu,j-\nu}=0, \quad (i, j)\in E\backslash N. \tag{3}$$

By convention

$$b_{kl}=0 \quad \text{if } (k, l)\notin M.$$

Condition (2b) guarantees the existence of a nontrivial denominator $q(x, y)$ because the homogeneous system has one equation less than the number of unknowns and so one unknown coefficient can be chosen freely.

Condition (2c) finally takes care of the Padé approximation property, namely if $q(0, 0)\neq 0$, then

$$\left(f-\frac{p}{q}\right)(x, y)=\sum_{(i,j)\in\mathbb{N}^2\backslash E} e_{ij}x^iy^j.$$

For more information we refer to [6,7]. We denote this multivariate Padé approximant by $[N/M]_E$ and we can arrange successive Padé approximants in a table after fixing an enumeration of the degree sets N and M and the equation set E. Numbering the points in $\mathbb{N}^2$, for instance, as (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), (3, 0),... and carrying this enumeration over to the index sets N, M and E, which are finite subsets of $\mathbb{N}^2$, provides us with an enumeration:

$$N=\{(i_0, j_0),\dots,(i_n, j_n)\}, \tag{4a}$$

$$M=\{(d_0, e_0),\dots,(d_m, e_m)\}, \tag{4b}$$

$$E=N\cup\{(i_{n+1}, j_{n+1}),\dots,(i_{n+m}, j_{n+m})\}. \tag{4c}$$

With this numbering, we can set up descending chains of index sets, defining bivariate polynomials of "lower degree" and bivariate Padé approximation problems of "lower order":

$$N=N_n\supset\cdots\supset N_k=\{(i_0, j_0),\dots,(i_k, j_k)\}\supset\cdots\supset N_0=\{(i_0, j_0)\}, \quad k=0,\dots,n, \tag{5a}$$

$$M=M_m\supset\cdots\supset M_l=\{(d_0, e_0),\dots,(d_l, e_l)\}\supset\cdots\supset M_0=\{(d_0, e_0)\},$$
$$l=0,\dots,m, \tag{5b}$$

$$E=E_{n+m}\supset\cdots\supset E_{k+l}=\{(i_0, j_0),\dots,(i_{k+l}, j_{k+l})\}\supset\cdots\supset E_0=\{(i_0, j_0)\},$$
$$k+l=0,\dots,n+m. \tag{5c}$$

With these subsets we can compute the following entries in a "table" of multivariate Padé approximants:

$$\begin{matrix} [N_0/M_0]_{E_0} & \cdots & [N_0/M_m]_{E_m} \\ \vdots & & \vdots \\ [N_n/M_0]_{E_n} & \cdots & [N_n/M_m]_{E_{n+m}} \end{matrix} \tag{6}$$

Remember that in order to set up (6), the enumeration of N and E should be such that all subsets E_{k+l} of E satisfy the inclusion property too. If we let n and m increase, infinite chains of index sets as in (5) can be constructed and an infinite table of multivariate Padé approximants results. Of course, in practice, only a finite number of entries will be computed.

3. The multivariate "de Montessus de Ballore" theorem

The univariate theorem deals with the case of simple poles as well as with the case of multiple poles. The former means that we have information concerning the denominator of the meromorphic function while the latter means that we also have information on the derivatives of that denominator. We shall prove a multivariate analogon of the univariate "de Montessus de Ballore" theorem, for the case of "simple" poles. Each of the conditions of Theorem 1 shall have its multivariate counterpart in Theorem 2. Before stating the theorem we introduce some notations. By the set $N * M$ we denote the index set that results from the multiplication of a polynomial indexed by N with a polynomial indexed by M,

$$N * M = \{(i+k,\ j+l) \mid (i,\ j) \in N,\ (k,\ l) \in M\}.$$

Since the set E satisfies the inclusion property we can inscribe isosceles triangles in E, with top in (0, 0) and base along the antidiagonal. Let τ be the largest of these inscribed triangles.

On the other hand, because $N * M$ is a finite subset of $\mathbb{N}^2$, we can circumscribe it with such triangles. Let T be the smallest of these circumscribing triangles.

In Figs. 1 and 2. we call r_τ and r_T the "range" of the triangles τ and T, respectively.

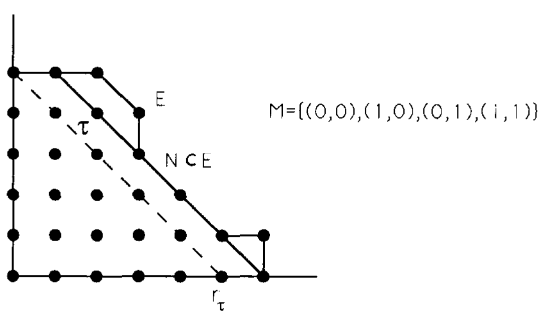

Fig. 1.

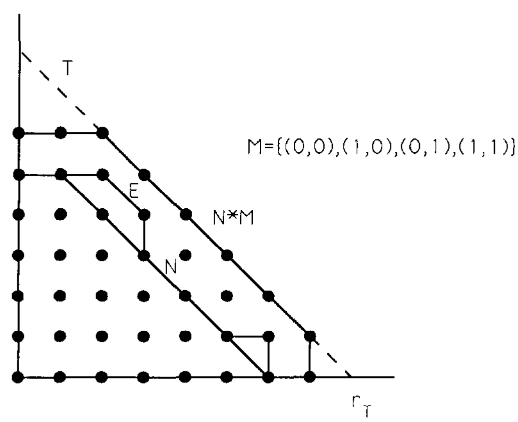

Fig. 2.

Theorem 2. *Let $f(x, y)$ be a function which is meromorphic in the polydisc $B(0; R_1, R_2) = \{(x, y): |x| < R_1, |y| < R_2\}$, meaning that there exists a polynomial*

$$R_M(x, y) = \sum_{(d, e) \in M \subseteq \mathbb{N}^2} r_{de} x^d y^e = \sum_{i=0}^{m} r_{d_i e_i} x^{d_i} y^{e_i},$$

such that $(fR_M)(x, y)$ is analytic in the polydisc above. Further, we assume that $R_M(0, 0) \neq 0$ so that necessarily $(0, 0) \in M$. Let there also exist m zeros $(x_h, y_h) \in B(0; R_1, R_2)$ of $R_M(x, y)$ satisfying

$$(fR_M)(x_h, y_h) \neq 0, \quad h = 1, \ldots, m, \tag{7a}$$

and

$$\begin{vmatrix} x_1^{d_1} y_1^{e_1} & \cdots & x_1^{d_m} y_1^{e_m} \\ \vdots & & \vdots \\ x_m^{d_1} y_m^{e_1} & \cdots & x_m^{d_m} y_m^{e_m} \end{vmatrix} \neq 0. \tag{7b}$$

Then the $[N/M]_E = (p/q)(x, y)$ Padé approximant with M fixed as given above and N and E growing, converges to $f(x, y)$ uniformly on compact subsets of

$$\{(x, y): |x| < R_1, |y| < R_2, R_M(x, y) \neq 0\},$$

and its denominator

$$q(x, y) = \sum_{i=0}^{m} b_{d_i e_i} x^{d_i} y^{e_i}$$

converges to $R_M(x, y)$ under the following conditions for N and E: the range of the largest inscribed triangle in E and the range of the smallest triangle circumscribing $N * M$ should both tend to infinity as the sets N and E grow along a column in the multivariate Padé table.

Proof. Let the polynomials $p(x, y)$ and $q(x, y)$, respectively of "degree" N and M, satisfy the Padé conditions (1). We also assume that the sets N, M and E are enumerated as in (4). Since the function fR_M is an analytic function we can write, using Cauchy's integral representation [10]

$$(fqR_M)(x, y) = \left(\frac{1}{2\pi i}\right)^2 \sum_{(i,j) \in \mathbb{N}^2} x^i y^j \int_{|t|=R_1} \int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1} u^{j+1}} \mathrm{d}t \, \mathrm{d}u. \tag{8}$$

The partial sum of this series containing the terms indexed by T circumscribing $N * M$, will be denoted by Π_T and because of the Padé conditions it is given by

$$\Pi_T(x, y) = (pR_M)(x, y) + \left(\frac{1}{2\pi i}\right)^2 \sum_{(i,j) \in T \setminus E} x^i y^j \int_{|t|=R_1} \int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1} u^{j+1}} \mathrm{d}t \, \mathrm{d}u. \tag{9}$$

Let us write

$$q(x, y) = R_M(x, y) + \Delta(x, y), \quad \text{with } \Delta(x, y) = \sum_{(i, j) \in M} \delta_{ij} x^i y^j.$$

We know that the coefficients in $R_M(x, y)$ are determined by m of its zeros (x_h, y_h) satisfying (7). We study $\Pi_T(x_h, y_h)$ and see how this affects $\Delta(x, y)$. On the one hand,

$$\begin{aligned}\Pi_T(x_h, y_h) &= (pR_M)(x_h, y_h) \\ &\quad + \left(\frac{1}{2\pi i}\right)^2 \sum_{(i,j)\in T\setminus E} x_h^i y_h^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1}u^{j+1}} \,dt\,du \\ &= \left(\frac{1}{2\pi i}\right)^2 \sum_{(i,j)\in T\setminus E} x_h^i y_h^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1}u^{j+1}} \,dt\,du,\end{aligned}$$

and on the other hand,

$$\begin{aligned}\Pi_T(x_h, y_h) &= \left(\frac{1}{2\pi i}\right)^2 \sum_{(i,j)\in T} x_h^i y_h^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1}u^{j+1}} \,dt\,du \\ &= \left(\frac{1}{2\pi i}\right)^2 \sum_{(i,j)\in T} x_h^i y_h^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fR_M^2 + f\Delta R_M)(t, u)}{t^{i+1}u^{j+1}} \,dt\,du.\end{aligned}$$

From this we conclude for $\Delta(x, y)$:

$$\begin{aligned}&\left(\frac{1}{2\pi i}\right)^2 \sum_{(k,l)\in M} \int_{|t|=R_1}\int_{|u|=R_2} \sum_{(i,j)\in T} \frac{x_h^i y_h^j (fR_M)(t, u)\delta_{kl}}{t^{i+1-k}u^{j+1-l}} \,dt\,du \\ &\quad= \left(\frac{1}{2\pi i}\right)^2 \sum_{(i,j)\in T\setminus E} x_h^i y_h^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1}u^{j+1}} \,dt\,du \\ &\qquad - \left(\frac{1}{2\pi i}\right)^2 \sum_{(i,j)\in T} x_h^i y_h^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fR_M^2)(t, u)}{t^{i+1}u^{j+1}} \,dt\,du, \quad h=1,\ldots,m.\end{aligned}$$

The above equations are a linear system of m equations in the $m+1$ unknown coefficients δ_{kl} with $(k, l)\in M$. Knowing that the set T satisfies the inclusion property and is triangular in structure, let us say of range r, we can write for the coefficient of δ_{kl} in the hth equation of this linear system

$$\begin{aligned}&\left(\frac{1}{2\pi i}\right)^2 \int_{|t|=R_1}\int_{|u|=R_2} \sum_{(i,\,j)\in T} \frac{x_h^i y_h^j (fR_M)(t, u)}{t^{i+1-k}u^{j+1-l}} \,dt\,du \\ &\quad= \left(\frac{1}{2\pi i}\right)^2 \int_{|t|=R_1}\int_{|u|=R_2} t^k u^l \sum_{i=0}^{r}\sum_{j=0}^{r-i} \frac{x_h^i}{t^{i+1}} \frac{y_h^j}{u^{j+1}} (fR_M)(t, u) \,dt\,du \\ &\quad= \left(\frac{1}{2\pi i}\right)^2 \int_{|t|=R_1}\int_{|u|=R_2} t^k u^l \left[\frac{1-(x_h/t)^{r+1}}{(t-x_h)(u-y_h)} - \left(\frac{y_h}{tu}\right)\frac{\sum_{i=0}^{r}(y_h/u)^i (x_h/t)^{r-i}}{(u-y_h)}\right](fR_M)(t, u) \,dt\,du,\end{aligned}$$

which for $r\to\infty$ and for (x_h, y_h) satisfying (7a), converges to

$$\left(\frac{1}{2\pi i}\right)^2 \int_{|t|=R_1}\int_{|u|=R_2} \frac{t^k u^l (fR_M)(t, u)}{(t-x_h)(u-y_h)} \,dt\,du = x_h^k y_h^l (fR_M)(x_h, y_h).$$

Similar computations can be made for the right-hand side of the hth equation in the linear system for the δ_{kl}. We have

$$\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \sum_{(i,j)\in T\setminus E} x_h^i y_h^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t,u)}{t^{i+1}u^{j+1}}\mathrm{d}t\,\mathrm{d}u$$

$$-\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \sum_{(i,j)\in T} x_h^i y_h^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fR_M^2)(t,u)}{t^{i+1}u^{j+1}}\mathrm{d}t\,\mathrm{d}u$$

$$=\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \sum_{(i,j)\in T\setminus E} x_h^i y_h^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t,u)}{t^{i+1}u^{j+1}}\mathrm{d}t\,\mathrm{d}u$$

$$-\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_{|t|=R_1}\int_{|u|=R_2} \left[\frac{1-(x_h/t)^{r+1}}{(t-x_h)(u-y_h)} - \left(\frac{y_h}{tu}\right)\frac{\sum_{i=0}^{r}(y_h/u)^i(x_h/t)^{r-i}}{(u-y_h)}(fR_M^2)(t,u)\right]\mathrm{d}t\,\mathrm{d}u,$$

which, for $r\to\infty$ and the range of τ inscribed in E tending to infinity, converges to

$$-\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_{|t|=R_1}\int_{|u|=R_2} \frac{fR_M^2(t,u)}{(t-x_h)(u-y_h)}\mathrm{d}t\,\mathrm{d}u = (fR_M^2)(x_h,y_h)=0.$$

Hence for $n\to\infty$ we have in the limit a homogeneous system of m equations in the unknowns $\delta_{d_0e_0},\dots,\delta_{d_me_m}$, with coefficient matrix

$$\begin{pmatrix} x_1^{d_0}y_1^{e_0}(fR_M)(x_1,y_1) & \cdots & x_1^{d_m}y_1^{e_m}(fR_M)(x_1,y_1)\\ \vdots & & \vdots\\ x_m^{d_0}y_m^{e_0}(fR_M)(x_m,y_m) & \cdots & x_m^{d_m}y_m^{e_m}(fR_M)(x_m,y_m)\end{pmatrix}.$$

A proper normalization of p and q can make $\delta_{d_0e_0}=\delta_{00}=0$ and leaves us with an $m\times m$ homogeneous system with coefficient matrix

$$\begin{pmatrix} x_1^{d_1}y_1^{e_1}(fR_M)(x_1,y_1) & \cdots & x_1^{d_m}y_1^{e_m}(fR_M)(x_1,y_1)\\ \vdots & & \vdots\\ x_m^{d_1}y_m^{e_1}(fR_M)(x_m,y_m) & \cdots & x_m^{d_m}y_m^{e_m}(fR_M)(x_m,y_m)\end{pmatrix},$$

which we know to be regular because of (7a) and (7b). Hence for $n\to\infty$ the solution of the linear system governing the coefficients of $\Delta(x,y)=(R_M-q)(x,y)$ converges to zero, in other words

$$\delta_{kl}\to 0,\quad (k,l)\in M,$$

or the polynomial $q(x,y)$ being the denominator of the Padé approximant, converges to $R_M(x,y)$ because $\Delta(x,y)$ converges to 0. The uniform convergence to $f(x,y)$ is based on the following error formula which is the multivariate counterpart of the univariate error formula in Theorem 1. From (1) and (8) we have

$$(fqR_M-pR_M)(x,y)=\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \sum_{\mathbb{N}^2\setminus E} x^i y^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t,u)}{t^{i+1}u^{j+1}}\mathrm{d}t\,\mathrm{d}u$$

or

$$f(x, y) - [N/M]_E = \left(\frac{1}{2\pi i}\right)^2 \frac{1}{(qR_M)(x, y)} \sum_{\mathbb{N}^2 \setminus E} x^i y^j$$
$$\int_{|t|=R_1} \int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1} u^{j+1}} \mathrm{d}t \, \mathrm{d}u,$$

which converges to zero for $r_\tau \to \infty$ and $(x, y) \in B(0; R_1, R_2)$. □

4. Example and discussion

In the above proof the main difficulty in comparison with the univariate theorem lies in the fact that in formula (9)

$$\Pi_T(x, y) \neq (pR_m)(x, y),$$

due to terms indexed by $T \setminus E$. In the univariate case we would have $T = N * M = E$.

In order to illustrate Theorem 2, we consider the following example. Let

$$f(x, y) = \frac{e^{x+y}}{(x-1)^2 + (y-1)^2 - 1} = \frac{e^{x+y}}{1 - 2(x+y) + x^2 + y^2},$$

which is holomorphic in $B(0; 1 - \frac{1}{2}\sqrt{2}, 1 - \frac{1}{2}\sqrt{2})$ and meromorphic in $B(0; \infty, \infty)$. Using the enumeration of $\mathbb{N}^2$ given before, namely (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), (3, 0),... for the sets N, M and E, and setting up a table of Padé approximants as in (6) where N_k, M_l and E_{k+l} respectively contain the first k, l and $k+l$ points of $\mathbb{N}^2$, we find that the denominator index set for $R_M(x, y)$ is

$$M = M_5 = \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2)\}$$
$$= \{(d_0, e_0), (d_1, e_1), (d_2, e_2), (d_3, e_3), (d_4, e_4), (d_5, e_5)\},$$
$$\#M = m + 1 = 6.$$

For the m points (x_h, y_h) satisfying (7a), (7b) we can take (see Fig. 3)

$$(x_1, y_1) = (1, 0), \quad (x_2, y_2) = (0, 1), \quad (x_3, y_3) = (2, 1),$$
$$(x_4, y_4) = (1 - \tfrac{1}{2}\sqrt{2}, 1 + \tfrac{1}{2}\sqrt{2}), \quad (x_5, y_5) = (1 + \tfrac{1}{2}\sqrt{2}, 1 + \tfrac{1}{2}\sqrt{2}).$$

The respective values of the Padé approximants in the column $[N_n/M_5]$, $n \to \infty$ evaluated at (1, 1) can be found in Table 1. These values are converging to

$$-e^2 = -7.389056098930\ldots.$$

The respective coefficients of the denominators $q(x, y)$ in the column $[N_n/M_5]$, $n \to \infty$ can be found in Table 2.

In order to point out the role of the enumeration of $\mathbb{N}^2$, we remark the following. A small but admissible permutation of the enumeration of $\mathbb{N}^2$, namely (0, 0), (1, 0), (0, 1), (2, 0), (0, 2), (1, 1), (3, 0),... implies that

$$M = M_4 = \{(0, 0), (1, 0), (0, 1), (2, 0), (0, 2)\},$$

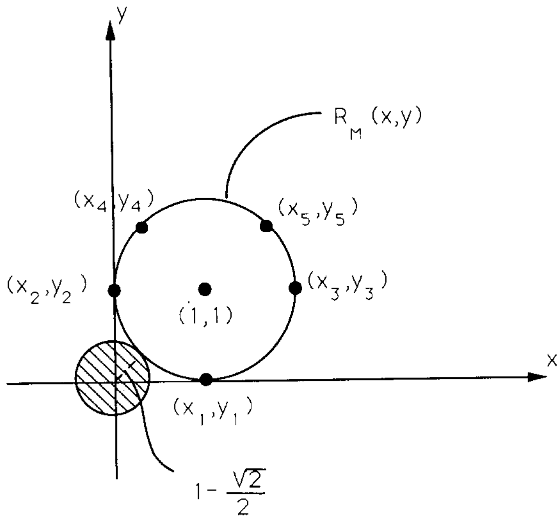

Fig. 3.

and that we have to look in the column $[N_n/M_4]$, $n \to \infty$. The set E_{k+l} still contains the first $k+l$ points of $\mathbb{N}^2$, now according to the new enumeration. For the 4 points (x_h, y_h) satisfying (7a), (7b) we can take for instance

$$(x_1, y_1) = (1, 0), \quad (x_2, y_2) = (0, 1), \quad (x_3, y_3) = (2, 1), \quad (x_4, y_4) = (1, 2).$$

Table 1

n	$[N_n/M_5](1, 1)$
0	$-0.271739 \cdot 10^0$
5	$-0.997561 \cdot 10^1$
10	$-0.625945 \cdot 10^1$
15	$-0.686971 \cdot 10^1$
20	$-0.722663 \cdot 10^1$
25	$-0.730786 \cdot 10^1$
30	$-0.734118 \cdot 10^1$
35	$-0.737707 \cdot 10^1$
40	$-0.738549 \cdot 10^1$
45	$-0.738666 \cdot 10^1$
50	$-0.739130 \cdot 10^1$
55	$-0.738855 \cdot 10^1$
60	$-0.738628 \cdot 10^1$
65	$-0.738860 \cdot 10^1$
70	$-0.738897 \cdot 10^1$
75	$-0.738940 \cdot 10^1$
80	$-0.738904 \cdot 10^1$
85	$-0.738838 \cdot 10^1$
90	$-0.738946 \cdot 10^1$

Table 3

n	$[N_n/M_4](1, 1)$
0	$-0.273684 \cdot 10^0$
5	$-0.432800 \cdot 10^1$
10	$-0.604486 \cdot 10^1$
15	$-0.688903 \cdot 10^1$
20	$-0.724014 \cdot 10^1$
25	$-0.731819 \cdot 10^1$
30	$-0.734785 \cdot 10^1$
35	$-0.738730 \cdot 10^1$
40	$-0.740896 \cdot 10^1$
45	$-0.738716 \cdot 10^1$
50	$-0.740747 \cdot 10^1$
55	$-0.738885 \cdot 10^1$
60	$-0.738946 \cdot 10^1$
65	$-0.739063 \cdot 10^1$
70	$-0.738898 \cdot 10^1$
75	$-0.738988 \cdot 10^1$
80	$-0.738904 \cdot 10^1$
85	$-0.738928 \cdot 10^1$
90	$-0.738814 \cdot 10^1$
95	$-0.738905 \cdot 10^1$

Table 2

n	Denominator of $[N_n/M_5](1, 1)$										
0	1.00000	+	$-3.00000\,x$	+	$-3.00000\,y$	+	$3.50000\,x^2$	+	$5.00000\,xy$	+	$3.50000\,y^2$
5	1.00000	+	$-2.13768\,x$	+	$-2.13768\,y$	+	$1.19686\,x^2$	+	$0.48551\,xy$	+	$1.19686\,y^2$
10	1.00000	+	$-2.01685\,x$	+	$-2.00950\,y$	+	$1.02142\,x^2$	+	$0.03888\,xy$	+	$1.00653\,y^2$
15	1.00000	+	$-1.99918\,x$	+	$-1.99918\,y$	+	$0.99789\,x^2$	+	$-0.00456\,xy$	+	$0.99789\,y^2$
20	1.00000	+	$-1.99980\,x$	+	$-1.99976\,y$	+	$0.99963\,x^2$	+	$-0.00096\,xy$	+	$0.99955\,y^2$
25	1.00000	+	$-1.99997\,x$	+	$-1.99995\,y$	+	$0.99995\,x^2$	+	$-0.00014\,xy$	+	$0.99981\,y^2$
30	1.00000	+	$-1.99994\,x$	+	$-1.99996\,y$	+	$0.99990\,x^2$	+	$-0.00019\,xy$	+	$0.99994\,y^2$
35	1.00000	+	$-1.99999\,x$	+	$-1.99999\,y$	+	$0.99999\,x^2$	+	$-0.00003\,xy$	+	$0.99998\,y^2$
40	1.00000	+	$-1.99999\,x$	+	$-2.00000\,y$	+	$0.99999\,x^2$	+	$-0.00002\,xy$	+	$0.99999\,y^2$
45	1.00000	+	$-2.00000\,x$	+	$-2.00000\,y$	+	$1.00000\,x^2$	+	$0.00000\,xy$	+	$1.00000\,y^2$
50	1.00000	+	$-2.00000\,x$	+	$-2.00000\,y$	+	$1.00000\,x^2$	+	$0.00000\,xy$	+	$1.00000\,y^2$

The results for $[N_n/M_4](1, 1)$, $n \to \infty$, and $q(x, y)$ in the column $[N_n/M_4]$ can be found in Tables 3 and 4, respectively.

We can even use an enumeration for the set M different from the one used for N and E and find similar convergence results. One can play around with these things as long as the conditions (7) are not violated. Especially for (7b) we refer the interested reader to [4] where the regularity of such generalized Vandermonde determinant is discussed.

In the literature one can find similar attempts to generalize the theorem of "de Montessus de Ballore" to the multivariate case. Chisholm and Graves-Morris [3,8] give a highly technical multivariate convergence theorem for the Canterbury approximants [2,11]. They do not yet treat the material in such a general way as is done here. We have complete freedom of choice for the numerator (by setting N) and the equations defining the Padé approximation order (by setting E). Karlsson and Wallin [12] provide some counterexamples for a "de Montessus de Ballore" convergence theorem. These counterexamples serve to confirm their point of view that the several variable case is much more complicated than the one variable case. The reader can verify that the counterexamples in question deal with situations in which the conditions of Theorem 2 are not satisfied. It has also been shown by Graves-Morris and Roberts [9] that good convergence results

Table 4

n	Denominator of $[N_n/M_4](1, 1)$								
0	1.00000	+	$-3.00000\,x$	+	$-3.00000\,y$	+	$3.50000\,x^2$	+	$3.50000\,y^2$
5	1.00000	+	$-2.02564\,x$	+	$-2.02564\,y$	+	$0.99145\,x^2$	+	$0.99145\,y^2$
10	1.00000	+	$-2.00462\,x$	+	$-2.00200\,y$	+	$1.00071\,x^2$	+	$0.99540\,y^2$
15	1.00000	+	$-2.00033\,x$	+	$-2.00033\,y$	+	$0.99959\,x^2$	+	$0.99959\,y^2$
20	1.00000	+	$-1.99991\,x$	+	$-2.00019\,y$	+	$0.99977\,x^2$	+	$1.00039\,y^2$
25	1.00000	+	$-2.00000\,x$	+	$-1.99998\,y$	+	$0.99998\,x^2$	+	$0.99985\,y^2$
30	1.00000	+	$-2.00001\,x$	+	$-1.99999\,y$	+	$1.00001\,x^2$	+	$0.99998\,y^2$
35	1.00000	+	$-2.00000\,x$	+	$-2.00001\,y$	+	$0.99999\,x^2$	+	$1.00002\,y^2$
40	1.00000	+	$-2.00001\,x$	+	$-2.00000\,y$	+	$1.00002\,x^2$	+	$0.99999\,y^2$
45	1.00000	+	$-2.00000\,x$	+	$-2.00000\,y$	+	$1.00000\,x^2$	+	$1.00000\,y^2$
50	1.00000	+	$-2.00000\,x$	+	$-2.00000\,y$	+	$1.00000\,x^2$	+	$1.00000\,y^2$

are in a way insensitive to the choice of E_{k+l}. This fact is clearly confirmed here and re-established in a very general setting.

We believe that among the papers dealing with this type of convergence problem, the result obtained in the previous section is the most general and flexible to be found. It is our aim to further develop Theorem 2 as to include the multivariate analogon of "multiple" poles and the case of multivariate Newton–Padé approximation [5,6].

References

[1] G.A. Baker and P.R. Graves-Morris, *Padé Approximants Part I: Basic Theory*, Encyclopedia of Mathematics **13** (Addison-Wesley, Reading, MA, 1981).

[2] J. Chisholm, Rational approximants defined from double power series, *Math. Comp.* **27** (1973) 124–141.

[3] J. Chisholm and P.R. Graves-Morris, Generalizations of the theorem of de Montessus to two-variable approximants, *Proc. Roy. Soc. London Ser. A* **342** (1975) 341–372.

[4] C.K. Chui and H.C. Lai, Vandermonde determinant and Lagrange interpolation in $\mathbb{R}^s$, in: *Nonlinear and Convex Analysis* (Marcel Dekker, New York, 1987) 23–35.

[5] G. Claessens, Convergence of multipoint Padé approximants, Preprint 77-26, Univ. Antwerp, 1976.

[6] A. Cuyt, General order multivariate rational Hermite interpolants, Monograph, Univ. Antwerp, 1986.

[7] A. Cuyt and B. Verdonk, General order Newton–Padé approximants for multivariate functions, *Numer. Math.* **43** (1984) 293–307.

[8] P.R. Graves-Morris, Generalisations of the theorem of de Montessus using Canterbury approximants, in: E. Saff and R. Varga, Eds., *Padé and Rational Approximation: Theory and Applications* (Academic Press, New York, 1977) 73–82.

[9] P.R. Graves-Morris and R.E. Roberts, Calculation of Canterbury approximants, *Comput. Phys. Comm.* **10** (1975) 234–244.

[10] R. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables* (Prentice-Hall, Englewood Cliffs, NJ, 1965).

[11] R. Hughes Jones, General rational approximants in n variables, *J. Approx. Theory* **16** (1976) 201–233.

[12] J. Karlsson and H. Wallin, Rational approximation by an interpolation procedure in several variables, in: E. Saff and R. Varga, Eds., *Padé and Rational Approximation: Theory and Applications* (Academic Press, New York, 1977) 83–100.

[13] E. Saff and R. Varga, Eds., *Padé and Rational Approximation: Theory and Applications* (Academic Press, New York, 1977).