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Extension of "A multivariate convergence theorem of the "de Montessus de Ballore" type" to multipoles

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Abstract

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In this journal (1990) we proved a multivariate version of the de Montessus de Ballore theorem stating tł convergence of general order Padé approximants for multivariate meromorphic functions with so-calle "simple" poles. That result is extended here to the case of "multipoles".

Keywords: Padé approximation; meromorphic functions; convergence; multivariate.

The multivariate de Montessus de Ballore theorem

The univariate theorem deals with the case of simple poles as well as with the case c multiple poles. The former means that we have information on the denominator of th meromorphic function while the latter means that we also have information on the derivativ of that denominator. Up to now we only proved a multivariate analogon of the univariate d Montessus de Ballore theorem for the case of "simple" poles. Before stating the more gener theorem we repeat the necessary notations.

Given a Taylor series expansion

$$f(x, y) = \sum_{(i,j)\in\mathbb{N}^2} c_{ij} x^i y^j,$$

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with

$$c_{ij} = \frac{1}{i!}\frac{1}{j!}\frac{\partial^{i+j}f}{\partial x^i\,\partial y^j}(0, 0),$$

we compute a multivariate general order Padé approximant $p(x, y)/q(x, y)$ to $f(x, y)$ from $p(x, y)$ and $q(x, y)$ given by [1]

$$\begin{aligned} p(x, y) &= \sum_{(i,j)\in N} a_{ij}x^iy^j, && N\subset\mathbb{N}^2,\\ q(x, y) &= \sum_{(i,j)\in M} b_{ij}x^iy^j, && M\subset\mathbb{N}^2,\\ (fq-p)(x, y) &= \sum_{(i,j)\in\mathbb{N}^2\setminus E} d_{ij}x^iy^j, && E\subset\mathbb{N}^2, \end{aligned} \tag{1}$$

where the index sets N and M indicate in a way the "degree" of $p(x, y)$ and $q(x, y)$ and the index set E satisfies the inclusion property [1], and contains the numerator set N as a subset. For detailed information we refer to [1]. Let us denote $\#N = n + 1$, $\#M = m + 1$ and this general order multivariate Padé approximant by $[N/M]_E$. We can arrange successive Padé approximants in a table after fixing an enumeration of the degree sets N and M and the equation set E. Numbering the points in $\mathbb{N}^2$, for instance, as (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), (3, 0),... and carrying this enumeration over to the index sets N, M and E which are infinite subsets of $\mathbb{N}^2$ provides us with an enumeration:

$$N = \{(i_0, j_0),\ldots,(i_n, j_n)\}, \tag{2a}$$

$$M = \{(d_0, e_0),\ldots,(d_m, e_m)\}, \tag{2b}$$

$$E = N\cup\{(i_{n+1}, j_{n+1}),\ldots,(i_{n+m}, j_{n+m})\}. \tag{2c}$$

By means of this numbering we can set up descending chains of index sets, defining bivariate polynomials of "lower degree" and bivariate Padé approximation problems of "lower order":

$$N = N_n \supset \cdots \supset N_k = \{(i_0, j_0),\ldots,(i_k, j_k)\} \supset \cdots \supset N_0 = \{(i_0, j_0)\},\quad k = 0,\ldots,n, \tag{3a}$$

$$M = M_m \supset \cdots \supset M_l = \{(d_0, e_0),\ldots,(d_l, e_l)\} \supset \cdots \supset M_0 = \{(d_0, e_0)\},\quad l = 0,\ldots,m, \tag{3b}$$

$$E = E_{n+m} \supset \cdots \supset E_{k+l} = \{(i_0, j_0),\ldots,(i_{k+l}, j_{k+l})\} \supset \cdots \supset E_0 = \{(i_0, j_0)\},\quad k+l = 0,\ldots,n+m. \tag{3c}$$

In order to set up a "table" of multivariate Padé approximants, the enumeration of N and E should be such that all subsets E_{k+l} of E satisfy the inclusion property too. This allows us to compute all the following entries:

$$\begin{matrix} [N_0/M_0]_{E_0} & \cdots & [N_0/M_m]_{E_m}\\ \vdots & & \vdots\\ [N_n/M_0]_{E_n} & \cdots & [N_n/M_m]_{E_{n+m}} \end{matrix} \tag{4}$$

By the set $N * M$ we denote the index set that results from the multiplication of a polynomial indexed by N with a polynomial indexed by M, namely $N * M = \{(i+k, j+l) \mid (i, j) \in N, (k, l) \in M\}$. Since the set E satisfies the inclusion property, we can inscribe isosceles triangles in E, with top in (0, 0) and base along the diagonal. Let τ be the largest of these inscribed triangles. On the other hand, because $N * M$ is a finite subset of $\mathbb{N}^2$, we can circumscribe it with such triangles. Let T be the smallest of these circumscribing triangles. For figures we refer to [2, Figs. 1 and 2]. In both cases we call r_τ and r_T the "range" of the triangles τ and T, respectively, meaning that τ and T, respectively, cut the axes of $\mathbb{N}^2$ in $(r_\tau, 0)$, $(0, r_\tau)$ and $(r_T, 0)$, $(0, r_T)$.

In what follows we discuss functions $f(x, y)$ which are meromorphic in a polydisc $B(0; R_1, R_2) = \{(x, y): |x| < R_1, |y| < R_2\}$, meaning that there exists a polynomial

$$R_M(x, y) = \sum_{(d, e) \in M \subseteq \mathbb{N}^2} r_{de} x^d y^e = \sum_{i=0}^{m} r_{d_i e_i} x^{d_i} y^{e_i},$$

such that $(fR_M)(x, y)$ is analytic in the polydisc above. The denominator polynomial $R_M(x, y)$ can completely be determined by m zeros $(x_h, y_h) \in B(0; R_1, R_2)$ of $R_M(x, y)$:

$$R_M(x_h, y_h) = 0, \quad h = 1, \ldots, m, \tag{5a}$$

or by a combination of zeros of R_M and some of its partial derivatives. For instance in the point (x_h, y_h) the partial derivatives

$$\left. \frac{\partial^{s_h + t_h} R_M}{\partial x^{s_h} \partial y^{t_h}} \right|_{(x_h, y_h)}, \quad (s_h, t_h) \in I^{(h)}, \tag{5b}$$

can be given with $I^{(h)}$ a finite subset of $\mathbb{N}^2$ of cardinality $\mu(h) + 1$ and satisfying the inclusion property. We can again enumerate the indices indicating the known and vanishing partial derivatives as follows:

$$I^{(h)} = \left\{ \left(s_0^{(h)}, t_0^{(h)} \right), \ldots, \left(s_{\mu(h)}^{(h)}, t_{\mu(h)}^{(h)} \right) \right\}, \quad \left(s_0^{(h)}, t_0^{(h)} \right) = (0, 0).$$

In the multivariate de Montessus de Ballore convergence theorem given in [2] all $\mu(h) = 0$ and $I^{(h)} = \{(0, 0)\}$. The following theorem extends this result.

Theorem 1. *Let $f(x, y)$ be a function which is meromorphic in the polydisc $B(0; R_1, R_2) = \{(x, y): |x| < R_1, |y| < R_2\}$, meaning that there exists a polynomial*

$$R_M(x, y) = \sum_{(d, e) \in M \subseteq \mathbb{N}^2} r_{de} x^d y^e = \sum_{i=0}^{m} r_{d_i e_i} x^{d_i} y^{e_i},$$

such that $(fR_M)(x, y)$ is analytic in the polydisc above. Further, we assume that $R_M(0, 0) \neq 0$ so that necessarily $(0, 0) \in M$ in the above expression for R_M. Let there also be given k zeros $(x_h, y_h) \in B(0; R_1, R_2)$ of $R_M(x, y)$ and k sets $I^{(h)} \subset \mathbb{N}^2$ with the inclusion property, satisfying

$$(fR_M)(x_h, y_h) \neq 0, \quad h = 1, \ldots, k, \tag{6a}$$

$$\begin{cases} \left. \dfrac{\partial^{s_h + t_h} R_M}{\partial x^{s_h} \partial y^{t_h}} \right|_{(x_h, y_h)} = 0, \quad (s_h, t_h) \in I^{(h)}, \ h = 1, \ldots, k, \\ \displaystyle\sum_{h=1}^{k} (\mu(h) + 1) = m, \quad \# I^{(h)} = \mu(h) + 1, \end{cases} \tag{6b}$$

and satisfying

$$\begin{vmatrix} x_1^{d_1}y_1^{e_1} & \cdots & x_1^{d_m}y_1^{e_m} \\ \vdots & & \vdots \\ \frac{d_1!}{(d_1-\mu(1))!}\frac{e_1!}{(e_1-\mu(1))!}x_1^{d_1-\mu(1)}y_1^{e_1-\mu(1)} & \cdots & \frac{d_m!}{(d_m-\mu(1))!}\frac{e_m!}{(e_m-\mu(1))!}x_1^{d_m-\mu(1)}y_1^{e_m-\mu(1)} \\ \vdots & & \vdots \\ x_k^{d_1}y_k^{e_1} & \cdots & x_k^{d_m}y_k^{e_m} \\ \vdots & & \vdots \\ \frac{d_1!}{(d_1-\mu(k))!}\frac{e_1!}{(e_1-\mu(k))!}x_k^{d_1-\mu(k)}y_k^{e_1-\mu(k)} & \cdots & \frac{d_m!}{(d_m-\mu(k))!}\frac{e_m!}{(e_m-\mu(k))!}x_k^{d_m-\mu(k)}y_k^{e_m-\mu(k)} \end{vmatrix} \neq 0. \qquad (6c)$$

Then the $[N/M]_E = (p/q)(x, y)$ Padé approximant, with M fixed as given above and N and E growing, converges to $f(x, y)$ uniformly on compact subsets of

$$\{(x, y): |x| < R_1, |y| < R_2, R_M(x, y) \neq 0\},$$

and its denominator

$$q(x, y) = \sum_{i=0}^{m} b_{d_i e_i} x^{d_i} y^{e_i}$$

converges to $R_M(x, y)$ under the following conditions for N and E: the range of the largest inscribed triangle in E and the range of the smallest triangle circumscribing $N * M$ should both tend to infinity as the sets N and E grow along a column in the multivariate Padé table.

Proof. Let the polynomials $p(x, y)$ and $q(x, y)$ respectively of "degree" N and M satisfy the Padé conditions (1). We also assume that the sets N, M and E are enumerated as in (2). Since the function fR_M is an analytic function, we can write, using Cauchy's integral representation [3],

$$(fqR_M)(x, y) = \left(\frac{1}{2\pi i}\right)^2 \sum_{(i, j)\in\mathbb{N}^2} x^i y^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1}u^{j+1}} \mathrm{d}t\,\mathrm{d}u. \qquad (7)$$

From the Padé conditions we know that the series development of fq is indexed by $N \cup \mathbb{N}^2 \backslash E$. Hence the series development of fqR_M is indexed by $N * M \cup \mathbb{N}^2 \backslash E$. The partial sum of this series containing the terms indexed by T circumscribing $N * M$, will be denoted by Π_T and because of the Padé conditions it is indexed by $N * M \cup T \backslash E$ and given by

$$\Pi_T(x, y) = (pR_M)(x, y) + \left(\frac{1}{2\pi i}\right)^2 \sum_{(i,j)\in T\backslash E} x^i y^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1}u^{j+1}} \mathrm{d}t\,\mathrm{d}u.$$

Let us write

$$q(x, y) = R_M(x, y) + \Delta(x, y),$$

with

$$\Delta(x, y) = \sum_{(i,j)\in M} \delta_{ij} x^i y^j,$$

and introduce the notation

$$\partial^{(s_h,t_h)}f = \frac{\partial^{s_h+t_h} f}{\partial x^{s_h} \partial y^{t_h}}(x_h, y_h).$$

We know that the coefficients in $R_M(x, y)$ are determined by k of its zeros (x_h, y_h) satisfying (6). We study $\partial^{(s_h,t_h)}\Pi_T$ and see how this affects $\Delta(x, y)$. On the one hand,

$$\begin{aligned}\partial^{(s_h,t_h)}\Pi_T &= \partial^{(s_h,t_h)}(pR_M)\\ &\quad+\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \partial^{(s_h,t_h)}\left[\sum_{(i,j)\in T\setminus E} x^i y^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1}u^{j+1}} \mathrm{d}t\,\mathrm{d}u\right]\\ &= \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \sum_{(i,j)\in T\setminus E} \frac{i!}{(i-s_h)!}\,\frac{j!}{(j-t_h)!} x_h^{i-s_h} y_h^{j-t_h}\\ &\quad\times \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1}u^{j+1}} \mathrm{d}t\,\mathrm{d}u,\end{aligned}$$

and on the other hand,

$$\begin{aligned}\partial^{(s_h,t_h)}\Pi_T &= \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \sum_{(i,j)\in T} \frac{i!}{(i-s_h)!}\,\frac{j!}{(j-t_h)!} x_h^{i-s_h} y_h^{j-t_h}\\ &\quad\times \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1}u^{j+1}} \mathrm{d}t\,\mathrm{d}u\\ &= \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \sum_{(i,j)\in T} \frac{i!}{(i-s_h)!}\,\frac{j!}{(j-t_h)!} x_h^{i-s_h} y_h^{j-t_h}\\ &\quad\times \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fR_M^2 + f\Delta R_M)(t, u)}{t^{i+1}u^{j+1}} \mathrm{d}t\,\mathrm{d}u.\end{aligned}$$

From the above formulas we conclude for $\Delta(x, y)$ with $(x, y)\in B(0; R_1, R_2)$,

$$\begin{aligned}&\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \partial^{(s_h,t_h)}\left[\sum_{(k,l)\in M} \int_{|t|=R_1}\int_{|u|=R_2} \sum_{(i,j)\in T} \frac{x^i y^j (fR_M)(t, u)\delta_{kl}}{t^{i+1-k}u^{j+1-l}} \mathrm{d}t\,\mathrm{d}u\right]\\ &\quad= \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \partial^{(s_h,t_h)}\left[\sum_{(i,j)\in T\setminus E} x^i y^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fqR_M)(t, u)}{t^{i+1}u^{j+1}} \mathrm{d}t\,\mathrm{d}u\right]\\ &\quad\quad- \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \partial^{(s_h,t_h)}\left[\sum_{(i,j)\in T} x^i y^j \int_{|t|=R_1}\int_{|u|=R_2} \frac{(fR_M^2)(t, u)}{t^{i+1}u^{j+1}} \mathrm{d}t\,\mathrm{d}u\right],\end{aligned}$$

$(s_h, t_h)\in I^{(h)}, \quad h = 1,\ldots,k.$

The above equations are a linear system of m equations in the $m+1$ unknown coefficients δ_{kl} with $(k, l) \in M$. Knowing that the set T satisfies the inclusion property and is triangular in structure, let us say of range r, we can write for the coefficient of δ_{kl} in the above equations

$$\left(\frac{1}{2\pi i}\right)^2 \partial^{(s_h,t_h)}\left[\int_{|t|=R_1}\int_{|u|=R_2}\sum_{(i,j)\in T}\frac{x^iy^j(fR_M)(t, u)}{t^{i+1-k}u^{j+1-l}}\mathrm{d}t\,\mathrm{d}u\right]$$

$$=\left(\frac{1}{2\pi i}\right)^2 \partial^{(s_h,t_h)}\left[\int_{|t|=R_1}\int_{|u|=R_2} t^ku^l\sum_{i=0}^{r}\sum_{j=0}^{r-i}\frac{x^i}{t^{i+1}}\frac{y^j}{u^{j+1}}(fR_M)(t, u)\,\mathrm{d}t\,\mathrm{d}u\right]$$

$$=\left(\frac{1}{2\pi i}\right)^2 \partial^{(s_h,t_h)}\left[\int_{|t|=R_1}\int_{|u|=R_2} t^ku^l\left(\frac{1-(x/t)^{r+1}}{(t-x)(u-y)}-\left(\frac{y}{tu}\right)\frac{\Sigma_{i=0}^r(y/u)^i(x/t)^{r-i}}{(u-y)}\right)\right.$$

$$\left.\times(fR_M)(t, u)\,\mathrm{d}t\,\mathrm{d}u\right],$$

which for $r\to\infty$ and for (x_h, y_h) satisfying (6a), converges to

$$\left(\frac{1}{2\pi i}\right)^2 \partial^{(s_h,t_h)}\left[\int_{|t|=R_1}\int_{|u|=R_2}\frac{t^ku^l(fR_M)(t, u)}{(t-x)(u-y)}\mathrm{d}t\,\mathrm{d}u\right]$$

$$=\partial^{(s_h,t_h)}\left[x^ky^l(fR_M)(x, y)\right]$$

$$=\sum_{\kappa=0}^{s_h}\sum_{\lambda=0}^{t_h}\frac{k!}{(k-\kappa)!}\frac{l!}{(l-\lambda)!}x_h^{k-\kappa}y_h^{l-\lambda}\,\partial^{(s_h-\kappa,t_h-\lambda)}(fR_M).$$

We shall denote this last expression by $S(k, l, s_h, t_h)$. Similar computations can be made for the right-hand side of the linear system of equations for the δ_{kl}. We have

$$\left(\frac{1}{2\pi i}\right)^2 \partial^{(s_h,t_h)}\left[\sum_{(i,j)\in T\setminus E}x^iy^j\int_{|t|=R_1}\int_{|u|=R_2}\frac{(fqR_M)(t, u)}{t^{i+1}u^{j+1}}\mathrm{d}t\,\mathrm{d}u\right]$$

$$-\left(\frac{1}{2\pi i}\right)^2 \partial^{(s_h,t_h)}\left[\sum_{(i,j)\in T}x^iy^j\int_{|t|=R_1}\int_{|u|=R_2}\frac{(fR_M^2)(t, u)}{t^{i+1}u^{j+1}}\mathrm{d}t\,\mathrm{d}u\right]$$

$$=\left(\frac{1}{2\pi i}\right)^2 \partial^{(s_h,t_h)}\left[\sum_{(i,j)\in T\setminus E}x^iy^j\int_{|t|=R_1}\int_{|u|=R_2}\frac{(fqR_M)(t, u)}{t^{i+1}u^{j+1}}\mathrm{d}t\,\mathrm{d}u\right]$$

$$-\left(\frac{1}{2\pi i}\right)^2 \partial^{(s_h,t_h)}\left[\int_{|t|=R_1}\int_{|u|=R_2}\left(\frac{1-(x/t)^{r+1}}{(t-x)(u-y)}-\left(\frac{y}{tu}\right)\frac{\Sigma_{i=0}^r(y/u)^i(x/t)^{r-i}}{(u-y)}\right)\right.$$

$$\left.\times(fR_M^2)(t, u)\,\mathrm{d}t\,\mathrm{d}u\right],$$

which, for $r \to \infty$ and the range of τ inscribed in E tending to infinity, converges to

$$-\left(\frac{1}{2\pi i}\right)^2 \partial^{(s_h,t_h)}\left[\int_{|t|=R_1}\int_{|u|=R_2} \frac{fR_M^2(t, u)}{(t-x)(u-y)}\,\mathrm{d}t\,\mathrm{d}u\right]$$

$$= -\partial^{(s_h,t_h)}\left(fR_M^2\right)$$

$$= -\sum_{\kappa=0}^{s_h}\sum_{\lambda=0}^{t_h} \partial^{(\kappa,\lambda)}(R_M)\partial^{(s_h-\kappa,t_h-\lambda)}(fR_M) = 0,$$

because of (6b) since $\partial^{(\kappa,\lambda)}(R_M)$ evaluates at (x_h, y_h) with $(\kappa, \lambda) \in I^{(h)}$. Hence for $n \to \infty$ we have in the limit a homogeneous system of m equations in the unknowns $\delta_{d_0e_0}, \ldots, \delta_{d_me_m}$, with coefficient matrix

$$\begin{pmatrix} S(d_0, e_0, s_0^{(1)}, t_0^{(1)}) & \cdots & S(d_m, e_m, s_0^{(1)}, t_0^{(1)}) \\ \vdots & & \vdots \\ S(d_0, e_0, s_{\mu(1)}^{(1)}, t_{\mu(1)}^{(1)}) & \cdots & S(d_m, e_m, s_{\mu(1)}^{(1)}, t_{\mu(1)}^{(1)}) \\ \vdots & & \vdots \\ S(d_0, e_0, s_0^{(k)}, t_0^{(k)}) & \cdots & S(d_m, e_m, s_0^{(k)}, t_0^{(k)}) \\ \vdots & & \vdots \\ S(d_0, e_0, s_{\mu(k)}^{(k)}, t_{\mu(k)}^{(k)}) & \cdots & S(d_m, e_m, s_{\mu(k)}^{(k)}, t_{\mu(k)}^{(k)}) \end{pmatrix}.$$

The fact that $I^{(h)}$ satisfies the inclusion property allows us to make the appropriate linear combinations within each block of $\mu(h)+1$ rows related to $I^{(h)}$, and this for $h = 1, \ldots, k$, such that the preceding coefficient matrix is reduced to the matrix of (6c) after dividing each equation of the hth block by $(fR_M)(x_h, y_h)$ which is nonzero because of (6a) and after normalizing p and q such that $\delta_{d_0e_0} = \delta_{00} = 0$. We know that this new $m \times m$ coefficient matrix is regular because of (6c), and hence for $n \to \infty$ the solution of the linear system governing the coefficients of $\Delta(x, y) = (R_M - q)(x, y)$ converges to zero, in other words,

$$\delta_{kl} \to 0, \quad (k, l) \in M,$$

or the polynomial $q(x, y)$, being the denominator of the Padé approximant, converges to $R_M(x, y)$ because $\Delta(x, y)$ converges to 0. The uniform convergence to $f(x, y)$ is based on the same error formula as for the case $I^{(h)} = \{(0, 0)\}$ treated in [2]. □

It is clear that the choice of the equation set E plays an important role, as already pointed out in [2], and that Padé approximants, for instance computed from

$$M = \{(0, 0), (1, 0), (0, 1)\}, \quad m = 2,$$

$$N = \{(i, 0) \mid 0 \leqslant i \leqslant n\},$$

$$E = N \cup \{(0, 1), (1, 1)\}, \quad \#E = \#N + m,$$

generate error expressions $O(y^2 + xy^2 + x^{2+k}y)$ which do not necessarily converge to zero.

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