EXPLORING COVARIANCE, CONSISTENCY AND CONVERGENCE IN PADE APPROXIMATION THEORY

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Introduction.

A given function f can be approximated with a high degree of contact by its Padé approximant. Let us call the operator that associates with f its Padé approximant $r_{n,m}$ of degree n in the numerator and degree m in the denominator, the Padé operator $\mathcal{P}_{n,m}$. The fact that the Padé approximant is a rational function gives rise to a number of interesting questions. Since the concept of Padé approximant is defined both for univariate and multivariate functions, the following topics will each be discussed for both cases.

Suppose that we construct rational expressions $\phi(f) = (af+b)/(cf+d)$ of f. Then one can investigate when the Padé approximant co-varies with f in this case, meaning that $\phi\left[\mathcal{P}_{n,m}(f)\right] = \mathcal{P}_{n_\phi,m_\phi}\left[\phi(f)\right]$.

Let us consider the case that f is itself an irreducible rational function. Then it is reasonable to expect that a suitable choice of the numerator and denominator degree delivers $\mathcal{P}_{n,m}(f) = r_{n,m} = f$.

Taking it one step further, we consider a meromorphic function f, which has a polynomial denominator and a holomorphic numerator. We can prove that both in the univariate and the multivariate case the function f can be rediscovered as the limit of a sequence of Padé approximants with increasing numerator degree and suitably chosen denominator degree. What's more, the polar singularities of f can be computed from the knowledge of its Taylor series expansion as a consequence of this convergence property.

S. P. Singh (ed.), Approximation Theory, Wavelets and Applications, 55–86.

1. Notations and definitions.

1.1. THE UNIVARIATE CASE.

Consider a formal power series expansion

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots \tag{1}$$

in a complex variable x, with $c_0 \neq 0$. In the sequel of the text we shall write ∂p for the exact degree of a polynomial $p(x)$ and $\omega f = \min\{i \mid c_i \neq 0\}$ for the order of a power series $f(x)$. The Padé approximation problem of order (n, m) for f consists in finding polynomials

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

and

$$q(x) = \sum_{i=0}^{m} b_i\, x^i$$

such that in the power series $(fq - p)(x)$ the coefficients of x^i for $i = 0, \dots, n + m$ disappear,

$$\begin{cases} \partial p \leq n \\ \partial q \leq m \\ \omega(fq - p) \geq n + m + 1 \end{cases} \tag{2}$$

Condition (2) is equivalent with the following two linear systems of equations

$$\begin{cases} c_0 b_0 = a_0 \\ c_1 b_0 + c_0 b_1 = a_1 \\ \quad \vdots \\ c_n b_0 + c_{n-1} b_1 + \dots + c_{n-m} b_m = a_n \end{cases} \tag{3a}$$

$$\begin{cases} c_{n+1} b_0 + c_n b_1 + \dots + c_{n-m+1} b_m = 0 \\ \quad \vdots \\ c_{n+m} b_0 + c_{n+m-1} b_1 + \dots + c_n b_m = 0 \end{cases} \tag{3b}$$

with $c_i = 0$ for $i < 0$. For $m = 0$ the system of equations (3b) is empty. In this case $a_i = c_i$ for $i = 0, \dots, n$ and $b_0 = 1$ satisfy (2), in other words the partial sums of (1) solve the Padé approximation problem of order $(n, 0)$. In general, a solution for the coefficients a_i is known after substitution of a solution for the b_i in the left hand side of (3a). So the crucial point is to solve the homogeneous system of m equations (3b) in the $m+1$ unknowns b_i. This system has at least one nontrivial solution because one of the unknowns can be chosen freely. The following relationship can be proved for different solutions of the same Padé approximation problem.

Theorem 1 *If the polynomials p_1, q_1 and p_2, q_2 satisfy (2), then $p_1 q_2 = p_2 q_1$.*

proof The polynomial $p_1 q_2 - p_2 q_1$ can also be written as

$$(f q_2 - p_2) q_1 - (f q_1 - p_1) q_2$$

Since

$$\omega(f q_1 - p_1) \geq n + m + 1$$
$$\omega(f q_2 - p_2) \geq n + m + 1$$

we have

$$\omega(p_1 q_2 - p_2 q_1) \geq n + m + 1$$

But $(p_1 q_2 - p_2 q_1)(x)$ is a polynomial of degree at most $n+m$. Consequently $p_1\ q_2 - p_2\ q_1 = 0$. □

A consequence of this theorem is that the rational functions p_1/q_1 and p_2/q_2 are equivalent. Hence all nontrivial solutions of (2) supply the same irreducible form. If $p(x)$ and $q(x)$ satisfy (2) we shall denote by

$$r_{n,m}(x) = \frac{p_{n,m}}{q_{n,m}}(x)$$

the irreducible form of p/q normalized such that $q_{n,m}(0) = 1$. This rational function $r_{n,m}(x)$ is called the **Padé approximant** of order (n, m) for f. As a conclusion we can formulate the next theorem.

Theorem 2 *For every nonnegative n and m a unique Padé approximant of order (n, m) for (1) exists.*

Although $p_{n,m}$ and $q_{n,m}$ are computed from polynomials p and q that satisfy (2), it is not necessarily so that $p_{n,m}$ and $q_{n,m}$ satisfy (2) themselves. A simple example will illustrate this. Consider $f(x) = 1 + x^2$ and take $n = 1 = m$. Condition (2) is then equivalent with

$$\begin{cases} b_0 & = a_0 \\ b_1 & = a_1 \\ b_0 & = 0 \end{cases}$$

A solution is given by $b_0 = 0 = a_0$ and $b_1 = 1 = a_1$. So $p(x) = x = q(x)$. Consequently $p_{1,1} = 1 = q_{1,1}$ with $\omega(fq_{1,1} - p_{1,1}) = 2 < n + m + 1$ and the corresponding equations (2) do not hold. But it is easy to construct, from the knowledge of $p_{n,m}$ and $q_{n,m}$, a solution of (2).

Theorem 3 *If the Padé approximant of order (n,m) for f is given by*

$$r_{n,m}(x) = \frac{p_{n,m}}{q_{n,m}}(x)$$

with $n' = \partial p_{n,m}$ and $m' = \partial q_{n,m}$, then there exists an integer s with $0 \leq s \leq \min(n - n', m - m')$ such that $p(x) = x^s p_{n,m}(x)$ and $q(x) = x^s q_{n,m}(x)$ satisfy (2).

proof Let p_1, q_1 be a nontrivial solution of (2). Hence

$$\partial p_1 \leq n$$
$$\partial q_1 \leq m$$
$$\omega(fq_1 - p_1) \geq n + m + 1$$

Since the irreducible form of p_1/q_1 is $p_{n,m}/q_{n,m}$ we know that

$$\begin{aligned} p_1(x) &= t(x)p_{n,m}(x) \\ q_1(x) &= t(x)q_{n,m}(x) \end{aligned}$$

with $t(x)$ a polynomial of degree at most $\min(n - n', m - m')$. If $s = \omega t$ is the order of the polynomial $t(x)$, then

$$0 \leq s \leq \min(n - n', m - m')$$

Since

$$\begin{aligned} \omega(fq_1 - p_1) &= \omega\left[t(fq_{n,m} - p_{n,m})\right] \\ &= \omega\left[x^s(fq_{n,m} - p_{n,m})\right] \\ &= \omega\left[f(x^s q_{n,m}) - (x^s p_{n,m})\right] \end{aligned}$$

the proof is completed. □

1.2. THE GENERAL MULTIVARIATE CASE.

We restrict ourselves to the case of two variables because the generalization to functions of more variables is only notationally more difficult. Given a Taylor series expansion

$$f(x,y) = \sum_{(i,j)\in N^2} c_{ij}x^i y^j$$

with

$$c_{ij} = \frac{1}{i!}\frac{1}{j!}\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\Big|_{(0,0)}$$

we introduce a multivariate Padé approximant $(p/q)(x, y)$ to $f(x, y)$ where $p(x, y)$ and $q(x, y)$ are determined by a general accuracy–through–order principle. Let the polynomials $p(x, y)$ and $q(x, y)$ be of the form

$$p(x,y) = \sum_{(i,j)\in N} a_{ij} x^i y^j \tag{4a}$$

$$q(x,y) = \sum_{(i,j)\in D} b_{ij} x^i y^j \tag{4b}$$

where N ("Numerator") and D ("Denominator") are finite subsets of $\mathbb{N}^2$. The sets N and D indicate in a way the degree of the polynomials $p(x, y)$ and $q(x, y)$. Let us denote

$$\partial p = \{(i,j) \mid (i,j) \in N, a_{ij} \neq 0\} \subseteq N \qquad n + 1 = \#N$$

$$\partial q = \{(i,j) \mid (i,j) \in D, b_{ij} \neq 0\} \subseteq D \qquad m + 1 = \#D$$

It is now possible to let $p(x, y)$ and $q(x, y)$ satisfy the following condition for the power series $(fq - p)(x, y)$, namely

$$(fq - p)(x,y) = \sum_{(i,j)\in \mathbb{N}^2 \setminus E} d_{ij} x^i y^j \tag{4c}$$

if, in analogy with the univariate case, the set of indices E ("Equations") is such that

$$N \subseteq E \tag{5a}$$

$$\#(E \setminus N) = m = \#D - 1 \tag{5b}$$

$$E \text{ satisfies the inclusion property} \tag{5c}$$

Here (5c) means that when a point belongs to the index set E, then the rectangular subset of points emanating from the origin with the given point as its furthermost corner, also lies in E. In other words,

$$(i,j) \in E \Longrightarrow \{(k,\ell) \mid k \leq i, \ell \leq j\} \subseteq E$$

Condition (5a) enables us to split the system of equations

$$d_{ij} = 0 \qquad (i,j) \in E$$

in an inhomogeneous part defining the numerator coefficients

$$\sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{\mu\nu}b_{i-\mu,j-\nu} = a_{ij} \qquad (i,j)\in N \tag{6a}$$

and a homogeneous part defining the denominator coefficients

$$\sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{\mu\nu}b_{i-\mu,j-\nu} = 0 \qquad (i,j)\in E\backslash N \tag{6b}$$

By convention $b_{k\ell} = 0$ if $(k,\ell) \notin D$. Condition (5b) guarantees the existence of a nontrivial denominator $q(x,y)$ because the homogeneous system has one equation less than the number of unknowns and so one unknown coefficient can be chosen freely. Condition (5c) finally takes care of the Padé approximation property, namely

$$q(0,0) \neq 0 \Longrightarrow (f-\frac{p}{q})(x,y) = \sum_{(i,j)\in \mathbb{N}^2\backslash E} \tilde{d}_{ij}x^iy^j$$

If E does not satisfy the inclusion property, as in figure 1, then

$$(fq-p)(x,y) = \sum_{(i,j)\in \mathbb{N}^2\backslash E} d_{ij}x^iy^j$$

does not imply

$$\left(f-\frac{p}{q}\right)(x,y) = \sum_{(i,j)\in \mathbb{N}^2\backslash E} \tilde{d}_{ij}x^iy^j$$

since in that case $f-p/q$ also contains terms resulting from a multiplication of the "hole" in E by $(1/q)(x,y)$ as can be seen from figure 1. For more information we refer to [11, 8].

We denote the set of rational functions p/q satisfying (4) by $[N/D]_E$ and we call it the **general multivariate Padé approximant** of order (n,m) for f.

In general, uniqueness of the general order multivariate Padé approximant, in the sense that all rational functions in $[N/D]_E$ reduce to the same irreducible form, is not guaranteed, unless the index set $E\setminus N$ supplies a homogeneous system of linearly independent equations (6b). It is obvious that at least one nontrivial solution of (4) exists, but it is not so (unlike in the univariate case) that different solutions p_1, q_1 and p_2, q_2 of (4) are necessarily equivalent, meaning that $(p_1q_2)(x,y) = (p_2q_1)(x,y)$. Hence p_1/q_1 and p_2/q_2 may be different functions. Consider the following approximant:

$$(p/q)(x,y) = \frac{\alpha+\alpha x+(1-\alpha)y}{1+x+y}$$

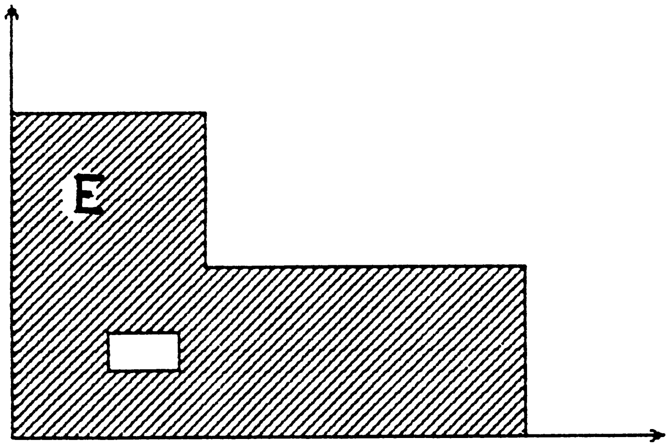

Figure 1.

Depending on α, it has 3 different irreducible forms, namely

$$\begin{aligned} \alpha = 0.0 &: \quad \frac{y}{1+x+y} \\ \alpha = 0.5 &: \quad 0.5 \\ \alpha = 1.0 &: \quad \frac{1+x}{1+x+y} \end{aligned}$$

For particular choices of N, D and E however it can be a general rule that all solutions in $[N/D]_E$ are equivalent and reduce to a single irreducible form, as can be seen in the following section. In general, with a free choice for N, D and E, only subject to (5), it is not true. We shall discuss this in detail further on.

Most of the older definitions for multivariate Padé approximants appear to be special cases of the very general definition introduced here [20, 21, 5, 19, 10]. Hence all the results mentioned in the following sections are applicable to these special cases. In this way the older theories are complemented with a lot of new theorems and algorithms. The general definition also contains the univariate theory as a special case.

Theorem 4 *If the index sets N, D and E are such that*

$$\begin{aligned} N &\supset \{(i,0) \mid 0 \le i \le n\} \\ D &\supset \{(i,0) \mid 0 \le i \le m\} \\ E &\supset \{(i,0) \mid 0 \le i \le n+m\} \end{aligned}$$

then the univariate Padé approximant of order (n,m) to $f(x,0)$ is given by the irreducible form of $[N/D]_E^f(x,0)$

1.3. THE HOMOGENEOUS MULTIVARIATE CASE.

The approach we have taken in the previous section to define and construct multivariate Padé approximants is essentially based on rewriting the double series expansion

$$\sum_{(i,j)\in \mathbb{N}^2} c_{ij}x^iy^j \tag{7}$$

as the single sum

$$\sum_{r_E(i,j)=0}^{\infty} c_{ij}x^iy^j$$

In general, a numbering r_E of $\mathbb{N}^2$ places the points in $\mathbb{N}^2$ one after the other. By doing so, the dimension of the problem description is reduced. When the input is indexed by integer numbers $r_E(i,j) \in \mathbb{N}$ and not by multi-indices $(i,j) \in \mathbb{N}^2$, the explicit determinant representation of the solution as well as the algorithms for its computation depend on a numbering r_E in $\mathbb{N}^2$ and not on the number of variables. Another way to work with the bivariate power series (7) is the following

$$\sum_{(i,j)\in \mathbb{N}^2} c_{ij}x^iy^j = \sum_{\ell=0}^{\infty}\left(\sum_{i+j=\ell} c_{ij}x^iy^j\right)$$

This approach is taken in [10, pp. 59–62] to construct **homogeneous multivariate Padé approximants**. These homogeneous multivariate Padé approximants are a special case of the general definition (4) where for chosen ν and μ in $\mathbb{N}$, which are comparable to the degrees n and m of the univariate Padé approximant, the numerator and denominator degree sets N and D are given by

$$N = \{(i,j) \in \mathbb{N}^2 \mid \nu\mu \le i+j \le \nu\mu+\nu\} \tag{8a}$$

$$D = \{(d,e) \in \mathbb{N}^2 \mid \nu\mu \le d+e \le \nu\mu+\mu\} \tag{8b}$$

while

$$E = E_{(\nu,\mu)} \cup E_\Phi \tag{8c}$$

$$E_{(\nu,\mu)} = \{(i,j) \in \mathbb{N}^2 \mid \nu\mu \le i+j \le \nu\mu+\nu+\mu\}$$

$$E_\Phi = \{(i,j) \in \mathbb{N}^2 \mid 0 \le i+j < \nu\mu\}$$

$$\#E_{(\nu,\mu)} = \#N + \#D - 1$$

The conditions in E_Φ are automatically satisfied by the choice of N and D and hence void.

An advantage of homogeneous Padé approximants is that they preserve the properties and the nature of univariate Padé approximants even better than the general order definition (4). This is for instance reflected in a tremendous simplification of the algorithms for their computation [7, 9]. Let us introduce the notation

$$\begin{aligned} A_\ell(x,y) &= \sum_{i+j=\nu\mu+\ell} a_{ij}x^iy^j & \ell = 0,\ldots,\nu \\ B_\ell(x,y) &= \sum_{i+j=\nu\mu+\ell} b_{ij}x^iy^j & \ell = 0,\ldots,\mu \\ C_\ell(x,y) &= \sum_{i+j=\ell} c_{ij}x^iy^j & \ell = 0,1,2\ldots \end{aligned}$$

and rewrite

$$\begin{aligned} p(x,y) &= \sum_{(i,j)\in N} a_{ij}x^iy^j = \sum_{\ell=0}^{\nu} A_\ell(x,y) \\ q(x,y) &= \sum_{(i,j)\in D} b_{ij}x^iy^j = \sum_{\ell=0}^{\mu} B_\ell(x,y) \end{aligned}$$

Then the conditions

$$(fq-p)(x,y) = \sum_{(i,j)\in N^2\setminus E} d_{ij}x^iy^j = \sum_{i+j\geq\nu\mu+\nu+\mu+1} d_{ij}x^iy^j$$

can be reformulated as

$$\begin{cases} C_0(x,y)B_0(x,y) = A_0(x,y) \\ C_1(x,y)B_0(x,y) + C_0(x,y)B_1(x,y) = A_1(x,y) \\ \quad\vdots \\ C_\nu(x,y)B_0(x,y) + \ldots + C_{\nu-\mu}(x,y)B_\mu(x,y) = A_\nu(x,y) \end{cases} \tag{9a}$$

$$\begin{cases} C_{\nu+1}(x,y)B_0(x,y) + \ldots + C_{\nu+1-\mu}(x,y)B_\mu(x,y) = 0 \\ \quad\vdots \\ C_{\nu+\mu}(x,y)B_0(x,y) + \ldots + C_\nu(x,y)B_\mu(x,y) = 0 \end{cases} \tag{9b}$$

where $C_\ell(x,y) \equiv 0$ if $\ell < 0$. This is exactly the system of defining equations (3) for univariate Padé approximants if the univariate term $c_\ell x^\ell$ is substituted by

$$C_\ell(x,y) = \sum_{i+j=\ell} c_{ij}x^iy^j \qquad \ell = 0,1,2\ldots$$

For the homogeneous Padé approximants we can prove a multivariate analogon of the theorems 1–3. To this end we define the order ωf of a power series $f(x,y)$ as $\omega f = \min\{i+j \mid c_{ij} \neq 0\}$.

Theorem 5 *If p_1, q_1 and p_2, q_2 both satisfy condition (4) written down for the homogeneous multivariate Padé approximation problem defined by (8), then*

$$(p_1 q_2)(x,y) = (p_2 q_1)(x,y)$$

proof We proceed as in the univariate case. Write $p_1 q_2 - p_2 q_1$ as

$$(f q_2 - p_2)\, q_1 - (f q_1 - p_1)\, q_2$$

We know that

$$\begin{aligned} &\omega(f q_1 - p_1) \geq \nu\mu + \nu + \mu + 1 \\ &\omega(f q_2 - p_2) \geq \nu\mu + \nu + \mu + 1 \\ &\omega q_1 \geq \nu\mu \\ &\omega q_2 \geq \nu\mu \end{aligned}$$

and consequently

$$\omega(p_1 q_2 - p_2 q_1) = \omega\left[(f q_2 - p_2) q_1 - (f q_1 - p_1) q_2\right] \geq 2\nu\mu + \nu + \mu + 1$$

Now the polynomial $(p_1 q_2 - p_2 q_1)$ is indexed by a subset of $\{(i,j) \mid 0 \leq i+j \leq 2\nu\mu + \nu + \mu\}$ and hence $p_1 q_2 - p_2 q_1$ must be identically zero. □

The homogeneous multivariate Padé approximant of order (ν,μ) for $f(x,y)$ can now be defined as the unique irreducible form

$$r_{\nu,\mu}(x,y) = \frac{p_{\nu,\mu}(x,y)}{q_{\nu,\mu}(x,y)}$$

of a solution $p(x,y)/q(x,y)$ of (9). For these $r_{\nu,\mu}$ the following properties of the univariate Padé approximant remain valid. Theorem 6 is the multivariate version of theorem 2 and theorem 7 generalizes theorem 3.

Theorem 6 *For every ν and μ a unique homogeneous multivariate Padé approximant of order (ν,μ) for $f(x,y)$ exists.*

Before we proceed let us first take a look at an example to better understand the difference between general and homogeneous multivariate Padé approximants.

Consider

$$\begin{aligned} f(x,y) &= 1 + \frac{x}{0.1 - y} + \sin(xy) \\ &= 1 + \sum_{i=1}^{\infty} 10^i x y^{i-1} + \sum_{i=0}^{\infty} (-1)^i \frac{(xy)^{2i+1}}{(2i+1)!} \\ &= 1 + 10x + 101xy + 1000xy^2 + \ldots \end{aligned}$$

Take $n = 2$ and $m = 1$ in the general order Padé approximation problem with

$$\begin{aligned} N &= \{(0,0),(0,1),(1,0)\} \\ D &= \{(0,0),(0,1)\} \\ E &= \{(0,0),(0,1),(1,0),(1,1)\} \end{aligned}$$

A solution of (4) is given by

$$\begin{aligned} p(x,y) &= 1 + 10x - 10.1y \\ q(x,y) &= 1 - 10.1y \\ [N/D]_E &= \frac{1 + 10x - 10.1y}{1 - 10.1y} \end{aligned}$$

Take $\nu = 1 = \mu$ in the homogeneous Padé approximation problem. Then we have to look for $p(x,y)$ and $q(x,y)$ of the form

$$p(x,y) = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2$$

$$q(x,y) = b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2$$

such that

$$(fq - p)(x,y) = \sum_{i+j\geq 4} d_{ij}x^iy^j$$

A solution of (9) is given by

$$\begin{aligned} p(x,y) &= 10x + 100x^2 - 101xy \\ q(x,y) &= 10x - 101xy \end{aligned}$$

and

$$r_{1,1}(x,y) = \frac{1 + 10x - 10.1y}{1 - 10.1y}$$

Here the shift of the degrees over $\nu\mu = 1$ has disappeared by taking the irreducible form. This is not always the case. Take $\nu = 1$ and $\mu = 2$. Then $p(x,y)$ and $q(x,y)$ satisfying the homogeneous Padé approximation problem are given by

$$\begin{aligned} p(x,y) &= 100x^2 - 101xy + 1000x^3 - 2020x^2y + 1000xy^2 \\ q(x,y) &= 100x^2 - 101xy - 1010x^2y + 1000xy^2 + 201x^2y^2 \end{aligned}$$

with

$$r_{1,2}(x,y) = \frac{x - 1.01y + 10x^2 - 20.2xy + 10y^2}{x - 1.01y - 10.1xy + 10y^2 + 2.01xy^2}$$

In general the following results can be proved about the order and degree of numerator and denominator of $r_{\nu,\mu}(x,y)$. In this context we mean by order and degree the total homogeneous order and degree. If $r_{\nu,\mu} = p_{\nu,\mu}/q_{\nu,\mu}$ is the irreducible form of a solution p/q of the homogeneous Padé approximation problem (9), we can write

$$p(x,y) = p_{\nu,\mu}(x,y)T(x,y)$$

$$q(x,y) = q_{\nu,\mu}(x,y)T(x,y)$$

with $\omega q_{\nu,\mu} \le \omega p$ and $\omega q_{\nu,\mu} \le \omega q$ So we can define $\nu' = \partial p_{\nu,\mu} - \omega q_{\nu,\mu}$ and $\mu' = \partial q_{\nu,\mu} - \omega q_{\nu,\mu}$. Obviously

$$\begin{aligned} \nu' &= \partial p_{\nu,\mu} - \omega q_{\nu,\mu} = (\partial p - \partial T) - \omega q_{\nu,\mu} \\ &\le \partial p - \omega T - \omega q_{\nu,\mu} = \partial p - \omega q \le \nu\mu + \nu - \nu\mu = \nu \end{aligned}$$

$$\begin{aligned} \mu' &= \partial q_{\nu,\mu} - \omega q_{\nu,\mu} = (\partial q - \partial T) - \omega q_{\nu,\mu} \\ &\le \partial q - \omega T - \omega q_{\nu,\mu} = \partial q - \omega q \le \nu\mu + \mu - \nu\mu = \mu \end{aligned}$$

This definition of ν' and μ' is an extension of the univariate definition, because in the univariate case $\omega q_{\nu,\mu} = 0$.

Theorem 7 *If the homogeneous Padé approximant of order (ν,μ) for $f(x,y)$ is given by $r_{\nu,\mu}(x,y)$ with ν' and μ' defined as above, then an integer s with $0 \le s \le \min(\nu - \nu', \mu - \mu')$ and a homogeneous bivariate polynomial*

$$S(x,y) = \sum_{i+j=\nu\mu-\omega q_{\nu,\mu}+s} s_{ij} x^i y^j$$

exist such that $p(x,y) = S(x,y)p_{\nu,\mu}(x,y)$ and $q(x,y) = S(x,y)q_{\nu,\mu}(x,y)$ satisfy (4).

proof Since $p_{\nu,\mu}/q_{\nu,\mu}(x,y)$ is computed from a solution of (4), we may consider nontrivial polynomials $p_1(x,y)$ and $q_1(x,y)$ and write

$$p_1(x,y) = T(x,y)p_{\nu,\mu}(x,y)$$

$$q_1(x,y) = T(x,y)q_{\nu,\mu}(x,y)$$

with p_1 and q_1 satisfying (4) and with

$$T(x,y) = \sum_{i+j=\omega T}^{\partial T} t_{ij} x^i y^j$$

Clearly from (8)

$$\omega q_1 = \omega T + \omega q_{\nu,\mu} \ge \nu\mu$$

and hence

$$\omega T = \nu\mu - \omega q_{\nu,\mu} + s$$

with $s \geq 0$. Also $\omega T \leq \partial T$ with

$$\begin{aligned} \partial T &= \partial p_1 - \partial p_{\nu,\mu} \leq \nu\mu + \nu - (\nu' + \omega q_{\nu,\mu}) = \nu\mu - \omega q_{\nu,\mu} + \nu - \nu' \\ \partial T &= \partial q_1 - \partial q_{\nu,\mu} \leq \nu\mu + \mu - (\mu' + \omega q_{\nu,\mu}) = \nu\mu - \omega q_{\nu,\mu} + \mu - \mu' \end{aligned}$$

Hence

$$\partial T \leq \nu\mu - \omega q_{\nu,\mu} + \min(\nu - \nu', \mu - \mu')$$

which implies

$$0 \leq s \leq \min(\nu - \nu', \mu - \mu')$$

Now consider the homogeneous polynomial consisting of the lowest order terms in $T(x, y)$, namely

$$S(x,y) = \sum_{i+j=\nu\mu-\omega q_{\nu,\mu}+s} t_{ij} x^i y^j$$

Because

$$\begin{aligned} \nu\mu + \nu + \mu + 1 \leq \omega(f q_1 - p_1) &= \omega\left[(f q_{\nu,\mu} - p_{\nu,\mu})T\right] \\ &= \omega\left[(f q_{\nu,\mu} - p_{\nu,\mu})S\right] \end{aligned}$$

the proof is completed. □

2. Covariance.

2.1. THE UNIVARIATE CASE.

In this section we are looking for operators Φ working on the series development f that commute more or less with the Padé operator $\mathcal{P}_{n,m}$ which associates with f its (n,m) Padé approximant:

$$\Phi\left[\mathcal{P}_{n,m}(f)\right] = \mathcal{P}_{n_\Phi, m_\Phi}\left[\Phi(f)\right]$$

with n_Φ and m_Φ depending on the considered Φ. It's easy to see that the operators Φ have to be rational.

A first property we are going to prove is called the reciprocal covariance property.

Theorem 8 *If $r_{n,m} = p_{n,m}/q_{n,m}$ is the (n,m) Padé approximant to the series development (1) with $c_0 \neq 0$, then*

$$r_{m,n} = \frac{q_{n,m}/c_0}{p_{n,m}/c_0}$$

is the (m,n) Padé approximant to $1/f$.

proof Since $p_{n,m}/q_{n,m}$ is the (n, m) Padé approximant to f, a polynomial $t(x)$ exists such that $p = tp_{n,m}$ and $q = tq_{n,m}$ satisfy (2). Since $c_0 \neq 0$,

$$\omega(fq - p) \geq n + m + 1 \Longrightarrow \omega\left[\frac{1}{f}(fq - p)\right] = \omega\left(\frac{1}{f}p - q\right) \geq n + m + 1$$

from which we can conclude that q and p satisfy (2) for $1/f$. Since $p_{n,m}(0) = c_0 \neq 0$, $q_{n,m}/p_{n,m}$ can also be normalized. □

A second property is called the homographic covariance property.

Theorem 9 *Let a, b, c and d be complex numbers with $cc_0 + d \neq 0$. If $r_{n,n} = p_{n,n}/q_{n,n}$ is the (n, n) Padé approximant to f, then*

$$\frac{(ap_{n,n} + bq_{n,n})/(c\,c_0 + d)}{(cp_{n,n} + dq_{n,n})/(c\,c_0 + d)}$$

is the (n, n) Padé approximant to $(af + b)/(cf + d)$.

proof We know that $r_{n,n}$ is computed from a solution $p = tp_{n,n}$ and $q = tq_{n,n}$ of (2). Now

$$\partial(ap + bq) \leq n$$
$$\partial(cp + dq) \leq n$$
$$\omega(fq - p) \geq n + m + 1 \Longrightarrow \omega\left(\frac{1}{cf + d}(fq - p)(ad - bc)\right) \geq n + m + 1$$

Since

$$\frac{af + b}{cf + d}(cp + dq) - (ap + bq) = \frac{1}{cf + d}(fq - p)(ad - bc)$$

and $cp(0) + dq(0) = c\,c_0 + d \neq 0$ the proof is completed. □

In general the theorem is not valid for the (n, m) Padé approximant with $n \neq m$ because then

$$\partial(ap + bq) \leq \max(n, m)$$
$$\partial(cp + dq) \leq \max(n, m)$$

instead of n and m respectively.

2.2. THE GENERAL MULTIVARIATE CASE.

In this section we study some covariance properties of the general order multivariate Padé approximant. For the sake of simplicity we denote a particular element of the set of solutions $[N/D]_E$ for the general Padé approximation problem of $f(x, y)$ by $[N/D]_E^f$.

Let the formal Taylor series development of $f(x,y)$ be such that $c_{00} \neq 0$. Then the formal Taylor series development of $g(x,y) = (1/f)(x,y)$ is defined by

$$g(x,y) = \sum_{(i,j)\in \mathbb{N}^2} e_{ij}x^i y^j$$

with

$$f(x,y)g(x,y) = 1$$

If

$$(fq-p)(x,y) = \sum_{(i,j)\in \mathbb{N}^2\setminus E} d_{ij}x^i y^j$$

then after multiplication by $-g(x,y)$, we get

$$(gp-q)(x,y) = \sum_{(i,j)\in \mathbb{N}^2\setminus E} \tilde{e}_{ij}x^i y^j$$

From this we can conclude the following theorem.

Theorem 10 *Let $[N/D]_E^f$ be a general order multivariate Padé approximant to $f(x,y)$ as defined above and let $g(x,y) = (1/f)(x,y)$. Then*

$$[D/N]_E^g = 1/[N/D]_E^f$$

If we study the homographic function covariance of the general order multivariate Padé approximant, we cannot consider denominator index sets D different from the numerator index set N. Indeed, when transforming the function f into the function $\tilde{f} = (af+b)/(cf+d)$, a general order Padé approximant p/q for f transforms into

$$\frac{ap+bq}{cp+dq}(x,y) = \frac{\sum_{(i,j)\in\tilde{N}} \tilde{a}_{ij}x^i y^j}{\sum_{(i,j)\in\tilde{D}} \tilde{b}_{ij}x^i y^j}$$

which can not necessarily be written in the form $\tilde{p}/\tilde{q} = [N/D]_E^{\tilde{f}}$.

Theorem 11 *Let $[N/N]_E^f = p/q$ be a general order multivariate Padé approximant to $f(x,y)$ and let $\tilde{f} = (af+b)/(cf+d)$, then*

$$[N/N]_E^{\tilde{f}} = \tilde{p}/\tilde{q}$$

with

$$\begin{aligned} \tilde{p}(x,y) &= ap(x,y) + bq(x,y) \\ \tilde{q}(x,y) &= cp(x,y) + dq(x,y) \end{aligned}$$

2.3. THE HOMOGENEOUS MULTIVARIATE CASE.

Since the homogeneous multivariate Padé approximant can be considered as a special case of the general multivariate Padé approximant, the theorems 10 and 11 remain valid. The condition $N = D$ in theorem 11, is replaced by the equivalent condition $\nu = \mu$.

3. Consistency.

3.1. THE UNIVARIATE CASE.

Last but not least the consistency property of the Padé approximant. If we are given an irreducible rational function $f(x)$ right from the start, but know only its Taylor series, do we come across it when calculating the appropriate Padé approximant? This consistency property is in fact quite logic and hence very desirable. In the next section we consider the more general problem of approximating functions with polar singularities, in other words Taylor series coming from functions with a polynomial denominator but not necessarily a polynomial numerator.

Theorem 12 *If* $f(x) = g(x)/h(x)$ *with* $h(0) = 1$ *and*

$$\begin{aligned} g(x) &= \sum_{i=0}^{n} g_i x^i \\ h(x) &= \sum_{i=0}^{m} h_i x^i \end{aligned}$$

then for $f(x)$ *irreducible we find* $r_{n,m} = f$.

proof For $f(x)$ we can write

$$\omega(fh - g) \geq n + m + 1$$

Since $\partial g \leq n$ and $\partial h \leq m$ we see that g and h satisfy (2) for f. Hence $r_{n,m}$ is the irreducible form of g/h or f itself. □

3.2. THE GENERAL MULTIVARIATE CASE.

Let's investigate the same question. If we are given an irreducible rational function $f(x, y)$ right from the start, do we come across it when calculating the appropriate general order Padé approximant? By this we mean that for an irreducible function

$$f(x,y) = \frac{g(x,y)}{h(x,y)} = \frac{\sum_{(i,j)\in N} g_{ij} x^i y^j}{\sum_{(i,j)\in D} h_{ij} x^i y^j}$$

and for a solution $p(x,y)/q(x,y)$ to the Padé approximation problem of $f(x,y)$, we want to find that p/q and g/h are equivalent. In other words, that

$$(ph - gq)(x,y) = 0$$

It is clear that this is the case if the general order multivariate Padé approximation problem to f has a unique solution, because then both p/q and g/h satisfy the approximation conditions (4c). If the solution is non-unique we can get in trouble because of the non-unicity of the irreducible form of the Padé approximant as pointed out in the previous section. A solution of the form

$$\frac{\alpha + \alpha x + (1-\alpha)y}{1 + x + y}$$

has 3 different irreducible forms. These irreducible forms cannot all together coincide with g/h. In general we can only say that

$$(ph - gq)(x,y) = \sum_{(i,j)\in N*D\setminus E} \tilde{e}_{ij} x^i y^j$$

where

$$N * D = \{(i+d, j+e) \mid (i,j) \in N, (d,e) \in D\}$$

3.3. THE HOMOGENEOUS MULTIVARIATE CASE.

However for the homogeneous Padé approximants, the consistency property holds.

Theorem 13 *For an irreducible rational function*

$$f(x,y) = \frac{g(x,y)}{h(x,y)} = \frac{\sum_{i+j=0}^{\nu} g_{ij} x^i y^j}{\sum_{i+j=0}^{\mu} h_{ij} x^i y^j}$$

the homogeneous Padé approximant to f of order (ν,μ) is given by $r_{\nu,\mu} = g/h$.

proof For $f = g/h$ we can write

$$(fh - g)(x,y) = \sum_{i+j\geq \nu\mu+\nu+\mu+1} d'_{ij} x^i y^j$$

For $r_{\nu,\mu} = p_{\nu,\mu}/q_{\nu,\mu}$ we know that there exists a polynomial $T(x,y)$ such that

$$(fq_{\nu,\mu}T - p_{\nu,\mu}T)(x,y) = \sum_{i+j\geq \nu\mu+\nu+\mu+1} d_{ij} x^i y^j$$

Because of the equivalence of different solutions for the homogeneous Padé approximation problem we can write

$$g(x,y)q_{\nu,\mu}(x,y)T(x,y) = p_{\nu,\mu}(x,y)T(x,y)h(x,y)$$

and consequently

$$\begin{aligned} g(x,y) &= p_{\nu,\mu}(x,y) \\ h(x,y) &= q_{\nu,\mu}(x,y) \end{aligned}$$

□

This consistency property is an important advantage of the homogeneous multivariate Padé approximants over the general order multivariate Padé approximants: in the general case the consistency property is only satisfied if the linear system of defining equations (6b) has maximal rank.

4. Convergence.

4.1. THE UNIVARIATE CASE.

Let us consider a sequence $S = \{r_0, r_1, r_2, \ldots\}$ of Padé approximants of different order for a given function $f(x)$. We want to investigate the existence of a function $F(x)$ with

$$\lim_{i\to\infty} r_i(x) = F(x)$$

and the properties of that function $F(x)$. In general the convergence of S will depend on the properties of f. A lot of information on the convergence of Padé approximants can also be found in [3].

We are interested in the convergence of columns in the Padé table. First we take $r_i(x) = r_{i,0}(x)$, the partial sums of the Taylor series expansion for $f(x)$. The following result is obvious.

Theorem 14 *If f is analytic in $B(0,r)$ with $r > 0$, then $S = \{r_{i,0}\}_{i\in N}$ converges uniformly to f in $B(0,r)$.*

Next take $r_i(x) = r_{i,1}(x)$, the Padé approximants of order $(i,1)$ for f. It is possible to construct functions f that are analytic in the whole complex plane but for which the poles of the $r_{i,1}$ are a dense subset of $\mathbb{C}$ [23, p. 158]. So in general S will not converge. But the following theorem can be proved [4].

Theorem 15 *If f is analytic in $B(0,r)$ with $r > 0$, then an infinite subsequence of $\{r_{i,1}\}_{i\in N}$ exists which converges uniformly to f in $B(0,r)$.*

In [2] a similar result was proved for $S = \{r_{i,2}(x)\}_{i\in N}$. However, the most interesting result was obtained by de Montessus de Ballore for Padé approximants of meromorphic functions. In that case it is possible to prove the uniform convergence of a particular column in the Padé table [15]. Since the column number in the Padé table is given by the degree of the Padé denominator, this number is determined by the number of poles of the meromorphic function in the considered disk. Aiming at larger regions of convergence, implies considering larger disks and hence dealing with more poles of the function at the same time and increasing the column number to be inspected.

Theorem 16 *If f is analytic in $B(0,r)$ except in the k distinct poles $w_1,\ldots,w_k$ of f with total multiplicity m and with*

$$0 < |w_1| \leq |w_2| \leq \ldots \leq |w_k| < R$$

then $\{r_{i,m}\}_{i\in N}$ converges uniformly to f in $B(0,r)\setminus\{w_1,\ldots,w_k\}$.

Several proofs exist of which the most elegant one is due to Saff [3, pp. 252–254]. In some cases another kind of convergence can be proved for the diagonal approximants. It is called convergence in measure [22].

Theorem 17 *Let f be meromorphic and let G be a closed and bounded subset of $\mathbb{C}$. For every $\epsilon > 0$ and $\delta > 0$ there exists an integer k such that for $i > k$ we have*

$$|r_{i,i}(x) - f(x)| < \epsilon \qquad x \in G_i$$

where G_i is a subset of G such that the measure of $G \setminus G_i$ is less than δ.

The proof of this theorem and more results on convergence in measure of Padé approximants can be found in [3, pp. 263–283].

For meromorphic functions $f(x)$ information on the poles can also be obtained from the columns in the qd-table which we introduce now. In the series development of f we set $c_i = 0$ for $i < 0$. For arbitrary integers n and for integers $m \geq 0$ we define determinants

$$H_m^{(n)} = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+m-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+m} \\ \vdots & & & \vdots \\ c_{n+m-1} & c_{n+m} & \cdots & c_{n+2m-2} \end{vmatrix}$$

with $H_0^{(n)} = 1$. The series (1) is termed k-normal if $H_m^{(n)} \neq 0$ for $m = 0,1,\ldots,k$ and $n \geq 0$. It is called ultimately k-normal if for every $0 \leq m \leq k$ there exists an $n(m)$ such that $H_m^{(n)} \neq 0$ for $n > n(m)$. With (1) we define the qd-scheme where subscripts denote columns and superscripts downward sloping diagonals [18, p. 609]:

(a) the start columns are given by

$$
\begin{aligned}
e_0^{(n)} &= 0 \qquad n = 1,2,\ldots \\
q_1^{(n)} &= \tfrac{c_{n+1}}{c_n} \qquad n = 0,1,\ldots
\end{aligned}
\tag{10}
$$

(b) and the rhombus rules for continuation of the scheme by

$$
\begin{aligned}
e_m^{(n)} &= q_m^{(n+1)} - q_m^{(n)} + e_{m-1}^{(n+1)} \qquad m = 1,2\ldots \quad n = 0,1\ldots \\
q_{m+1}^{(n)} &= \frac{e_m^{(n+1)}}{e_m^{(n)}} q_m^{(n+1)} \qquad m = 1,2\ldots \quad n = 0,1,\ldots
\end{aligned}
\tag{11}
$$

Theorem 18 *If f is analytic in $B(0,r)$ except in k distinct poles with total multiplicity m and with*

$$|w_0| = 0 < |w_1| \le |w_2| \le \ldots \le |w_m| < R \qquad |w_{m+1}| = \infty$$

where each pole occurs as many times in the sequence as its order, and if f is ultimately m-normal, then the qd-scheme associated with f has the following properties:

(a) for each ℓ with $0 < \ell \le m$ and $|w_{\ell-1}| < |w_\ell| < |w_{\ell+1}|$,

$$\lim_{n\to\infty} q_\ell^{(n+1)} = 1/w_\ell$$

(b) for each ℓ with $0 < \ell \le m$ and $|w_\ell| < |w_{\ell+1}|$,

$$\lim_{n\to\infty} e_\ell^{(n+1)} = 0$$

The index ℓ for which $|w_\ell| < |w_{\ell+1}|$, is called a critical index because it indicates in which columns of the qd-table we have to take a look. It is clear that the critical indices of a function do not depend on the order in which the poles of equal modulus are numbered. When ℓ is a critical index, the ℓ^{th} e-column tends to zero and the ℓ^{th} q-column just preceding it in the qd-table contains information on the poles with distinct moduli. Thus the qd-table of a meromorphic function is divided into subtables by those e-columns tending to zero. Any q-column corresponding to a simple pole of isolated modulus is flanked by such e-columns and converges to the reciprocal of the corresponding pole. If a subtable contains more columns of q-values, the presence of poles of equal modulus is indicated. In [18, p. 642] it is also explained how to determine these poles.

Theorem 19 *Let ℓ and $\ell+k$ with $k > 1$ be two consecutive critical indices. Let the polynomials $p_i^{(s)}$ be defined by*

$$
\begin{aligned}
p_0^{(s)}(x) &= 1 \\
p_{i+1}^{(s)}(x) &= x p_i^{(s+1)}(x) - q_{\ell+i+1}^{(s)} p_i^{(s)}(x) \qquad s \ge 0 \qquad i = 0,1,\ldots,k-1
\end{aligned}
$$

Then there exists a subsequence $\{s(n)\}_{n\in N}$ *such that*

$$\lim_{n\to\infty} p_k^{(s(n))}(x) = (x - w_{\ell+1}^{-1})\dots(x - w_{\ell+k}^{-1})$$

From the above theorem the qd-scheme seems to be an ingenious tool for determining the poles of a meromorphic function f directly from its Taylor series at the origin. If f is rational, the last e-column is even theoretically equal to zero, as can be seen from the next theorem. The proof hereof is based on the next lemma [18, pp. 610–613].

Lemma 1 *Let f be given by its formal Taylor series expansion (1). If there exists a positive integer k such that f is k-normal, then the values $q_m^{(n)}$ and $e_m^{(n)}$ exist for $m = 1, \dots, k$ and $n \geq 0$ and they are given by*

$$q_m^{(n)} = \frac{H_m^{(n+1)} H_{m-1}^{(n)}}{H_m^{(n)} H_{m-1}^{(n+1)}}$$

$$e_m^{(n)} = \frac{H_{m+1}^{(n)} H_{m-1}^{(n+1)}}{H_m^{(n)} H_m^{(n+1)}}$$

Theorem 20 *Let (1) be the Taylor series at $x = 0$ of a rational function of degree n in the numerator and $m \leq n$ in the denominator. Then if the series f is m-normal,*

$$e_m^{(n-m+h)} = 0 \qquad h > 0$$

4.2. THE GENERAL MULTIVARIATE CASE.

The univariate theorem of de Montessus de Ballore deals with the case of simple poles as well as with the case of multiple poles. The former means that we have information on the denominator of the meromorphic function while the latter means that we also have information on the derivatives of that denominator. In this section we give a similar convergence theorem for the general order multivariate Padé approximants.

Let us first introduce the notations

$$\begin{aligned} \#N &= n+1 \\ \mathcal{N}_x(n) &= \max\{i \mid (i,j) \in N\} \\ \mathcal{N}_y(n) &= \max\{j \mid (i,j) \in N\} \end{aligned}$$

In what follows we discuss functions $f(x, y)$ which are meromorphic in a polydisc $B(0; R_1, R_2) = \{(x, y) : |x| < R_1, |y| < R_2\}$, meaning that there exists a polynomial

$$R_m(x,y) = \sum_{(d,e)\in D\subseteq \mathbb{N}^2} r_{de}x^d y^e = \sum_{i=0}^{m} r_{d_i e_i} x^{d_i} y^{e_i}$$

such that $(fR_m)(x,y)$ is analytic in the polydisc above. The denominator polynomial $R_m(x,y)$ can up to a multiplicative factor be determined by m zeros (x_h, y_h) of $R_m(x,y)$ in $B(0; R_1, R_2)$,

$$R_m(x_h, y_h) = 0 \qquad h = 1, \dots, m \tag{12a}$$

or by a combination of zeros of R_m and some of its partial derivatives. For instance in the point (x_h, y_h) the partial derivatives

$$\frac{\partial^{i_h+j_h} R_m}{\partial x^{i_h} \partial y^{j_h}}\bigg|_{(x_h,y_h)} = 0 \qquad (i_h, j_h) \in I_h \tag{12b}$$

can be given with I_h a finite subset of $\mathbb{N}^2$ of cardinality $\mu(h)+1$ and satisfying the inclusion property. We can again enumerate the indices indicating these vanishing partial derivatives as

$$I_h = \{(i_0^{(h)}, j_0^{(h)}), \dots, (i_{\mu(h)}^{(h)}, j_{\mu(h)}^{(h)})\} \qquad (i_0^{(h)}, j_0^{(h)}) = (0,0)$$

For the pole (x_h, y_h) the set I_h substitutes the univariate notion of multiplicity.

Theorem 21 *Let $f(x,y)$ be a function which is meromorphic in the polydisc $B(0; R_1, R_2) = \{(x,y) : |x| < R_1, |y| < R_2\}$, meaning that there exists a polynomial*

$$R_m(x,y) = \sum_{(d,e)\in D\subseteq \mathbb{N}^2} r_{de}x^d y^e = \sum_{i=0}^{m} r_{d_i e_i} x^{d_i} y^{e_i}$$

such that $(fR_m)(x,y)$ is analytic in the polydisc above. Further, we assume that $R_m(0,0) \neq 0$ so that necessarily $(0,0) \in D$. Let there also be given k zeros (x_h, y_h) of $R_m(x,y)$ in $B(0; R_1, R_2)$ and k sets $I_h \subset \mathbb{N}^2$ with inclusion property, satisfying

$$(fR_m)(x_h, y_h) \neq 0 \qquad h = 1, \dots, k \tag{13a}$$

$$\begin{cases} \dfrac{\partial^{i_h+j_h} R_m}{\partial x^{i_h} \partial y^{j_h}}\bigg|_{(x_h,y_h)} = 0 & (i_h, j_h) \in I_h \qquad h = 1, \dots, k \\ \sum_{h=1}^{k} (\mu(h)+1) = m & \# I_h = \mu(h)+1 \end{cases} \tag{13b}$$

and producing the nonzero determinant

$$\begin{vmatrix} x_1^{d_1} y_1^{e_1} & \cdots & x_1^{d_m} y_1^{e_m} \\ \vdots & & \vdots \\ \frac{d_1!}{(d_1-\mu(1))!} \frac{e_1!}{(e_1-\mu(1))!} x_1^{d_1-\mu(1)} y_1^{e_1-\mu(1)} & \cdots & \frac{d_m!}{(d_m-\mu(1))!} \frac{e_m!}{(e_m-\mu(1))!} x_1^{d_m-\mu(1)} y_1^{e_m-\mu(1)} \\ \vdots & & \vdots \\ x_k^{d_1} y_k^{e_1} & \cdots & x_k^{d_m} y_k^{e_m} \\ \vdots & & \vdots \\ \frac{d_1!}{(d_1-\mu(k))!} \frac{e_1!}{(e_1-\mu(k))!} x_k^{d_1-\mu(k)} y_k^{e_1-\mu(k)} & \cdots & \frac{d_m!}{(d_m-\mu(k))!} \frac{e_m!}{(e_m-\mu(k))!} x_k^{d_m-\mu(k)} y_k^{e_m-\mu(k)} \end{vmatrix} \quad (13c)$$

Then the sequence of general order multivariate Padé approximants $[N/D]_E = (p/q)(x,y)$ with D determined and fixed by the index set of $R_m(x,y)$ and $N \subset E$ growing such that

$$\lim_{n\to\infty} \mathcal{N}_x(n) = \infty$$
$$\lim_{n\to\infty} \mathcal{N}_y(n) = \infty$$

converges to $f(x,y)$ uniformly on compact subsets of

$$\{(x,y) : |x| < R_1, |y| < R_2, R_m(x,y) \neq 0\}$$

and its denominator

$$q(x,y) = \sum_{i=0}^{m} b_{d_i e_i} x^{d_i} y^{e_i}$$

converges to $R_m(x,y)$.

The main difference in comparison with the univariate theorem lies in the fact that in the univariate case $N * D = \{(i,0) \mid 0 \leq i \leq n\} * \{(j,0) \mid 0 \leq j \leq m\} = E = \{(i,0) \mid 0 \leq i \leq n+m\}$ which is not necessarily true in the multivariate case. A numerical example illustrating theorem 21 can be found in [12].

In the literature one can find similar attempts to generalize the theorem of "de Montessus de Ballore" to the multivariate case [6, 17, 19]. However, the problem is nowhere treated in such a general way as is done here. We have complete freedom of choice for the numerator (by setting N) and the equations defining the Padé approximation order (by setting E). Also we can deal with any polynomial denominator since the index set D can be any subset of $\mathbb{N}^2$. In [13] the interested reader can find a typical situation where this freedom is necessary for the construction of numerically useful multivariate Padé approximants.

In analogy with the univariate case we now discuss a first multivariate version of the qd-algorithm. Given a formal series expansion of a function $f(x,y)$, an enumeration $r_{\mathbb{N}^2}$ of the points (i,j) in $\mathbb{N}^2$ specifies in which order we are going to deal with the Taylor coefficients c_{ij}. The multi-indices (i,j) of c_{ij} can for instance be counted down along upward sloping diagonals in the order (0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (3,0), ... or in any other order as long as the inclusion property remains satisfied during the enumeration. Now we introduce two enumerations of multi-indices which will play a special role in the super- and subscripts of the general order multivariate qd-algorithm, namely r_N satisfying the inclusion property and enumerating the c_{ij} to be fed as input to the algorithm, and r_D for the time being arbitrary:

$$N = \{(i_0,j_0),\ldots,(i_n,j_n),\ldots\} \qquad (14a)$$
$$D = \{(d_0,e_0),\ldots,(d_m,e_m),\ldots\} \qquad (14b)$$

Here indexing a point (i,j) or (d,e) with ℓ and referring to it as (i_ℓ,j_ℓ) or (d_ℓ,e_ℓ), means that it is the next point in line in N or D. The univariate case appears as a special case if we enumerate only the first axis. A typical multivariate choice would be $N = \mathbb{N}^2$ and D general.

Let us introduce help entries $g_{0,m}^{(n)}$ by:

$$g_{0,m}^{(n)} = \sum_{k=0}^{n} c_{i_k-d_m,j_k-e_m} x^{i_k-d_m} y^{j_k-e_m} - \sum_{k=0}^{n} c_{i_k-d_{m-1},j_k-e_{m-1}} x^{i_k-d_{m-1}} y^{j_k-e_{m-1}} \qquad (15a)$$

$$g_{m,r}^{(n)} = \frac{g_{m-1,r}^{(n)} g_{m-1,m}^{(n+1)} - g_{m-1,r}^{(n+1)} g_{m-1,m}^{(n)}}{g_{m-1,m}^{(n+1)} - g_{m-1,m}^{(n)}} \qquad r = m+1, m+2, \ldots \qquad (15b)$$

keeping in mind that $c_{ij} = 0$ if $i < 0$ or $j < 0$. The values $g_{m,r}^{(n)}$ are stored as in table 1. The general order multivariate qd-algorithm is then defined by:

$$Q_1^{(n)}(x,y) = \frac{c_{i_{n+1}-d_0,j_{n+1}-e_0} x^{i_{n+1}-d_0} y^{j_{n+1}-e_0}}{c_{i_n-d_0,j_n-e_0} x^{i_n-d_0} y^{j_n-e_0}} \quad \frac{g_{0,1}^{(n+1)}}{g_{0,1}^{(n+1)} - g_{0,1}^{(n+2)}} \qquad (16a)$$

$$Q_m^{(n+1)}(x,y) = \frac{E_{m-1}^{(n+2)}(x,y) Q_{m-1}^{(n+2)}(x,y)}{E_{m-1}^{(n+1)}(x,y)} \quad \frac{g_{m-2,m-1}^{(n+m-1)} - g_{m-2,m-1}^{(n+m)}}{g_{m-2,m-1}^{(n+m-1)}} \quad \frac{g_{m-1,m}^{(n+m)}}{g_{m-1,m}^{(n+m)} - g_{m-1,m}^{(n+m+1)}} \qquad m \geq 2 \qquad (16b)$$

$$E_m^{(n+1)}(x,y)+1=\frac{g_{m-1,m}^{(n+m)}-g_{m-1,m}^{(n+m+1)}}{g_{m-1,m}^{(n+m)}}\left(Q_m^{(n+2)}(x,y)+1\right)\qquad m\geq 1 \tag{16c}$$

If we arrange the values $Q_m^{(n)}(x,y)$ and $E_m^{(n)}(x,y)$ as in the univariate case, where subscripts indicate columns and superscripts indicate downward sloping diagonals, then (16b) links the elements in the rhombus

$$\begin{array}{ccc} & E_{m-1}^{(n+1)}(x,y) & \\ Q_{m-1}^{(n+2)}(x,y) & & Q_m^{(n+1)}(x,y) \\ & E_{m-1}^{(n+2)}(x,y) & \end{array}$$

and (16c) links two elements on an upward sloping diagonal

$$\begin{array}{cc} & E_m^{(n+1)}(x,y) \\ Q_m^{(n+2)}(x,y) & \end{array}$$

In analogy with the univariate Padé approximation case [18, p. 610] it is also possible to give explicit determinant formulas for the general multivariate Q-and E-values. Let us introduce the notations

$$\begin{aligned} C_{m,n}(x,y) &= c_{i_n-d_m,j_n-e_m}x^{i_n-d_m}y^{j_n-e_m} \qquad i_n\geq d_m \quad j_n\geq e_m \\ \pi_x(n,m) &= i_{n+1}+\ldots+i_{n+m}-d_0-\ldots-d_m \\ \pi_y(n,m) &= j_{n+1}+\ldots+j_{n+m}-e_0-\ldots-e_m \end{aligned}$$

where $C_{m,n}(x,y)$ is not to be confused with the homogeneous expression $C_m(x,y)$, and let us introduce the determinants

$$H_{0,m}^{(n)}=\begin{vmatrix} C_{0,n+1}(x,y) & \ldots & C_{m-1,n+1}(x,y) \\ \vdots & & \vdots \\ C_{0,n+m}(x,y) & \ldots & C_{m-1,n+m}(x,y) \end{vmatrix}\qquad H_{0,0}^{(n)}=0$$

$$H_{1,m}^{(n)}=\begin{vmatrix} 1 & \ldots & 1 \\ C_{0,n+1}(x,y) & \ldots & C_{m,n+1}(x,y) \\ \vdots & & \vdots \\ C_{0,n+m}(x,y) & \ldots & C_{m,n+m}(x,y) \end{vmatrix}\qquad H_{1,-1}^{(n)}=0$$

$$H_{2,m}^{(n)} = \begin{vmatrix} \sum_{k=0}^{n} C_{0,k}(x,y) & \dots & \sum_{k=0}^{n} C_{m,k}(x,y) \\ C_{0,n+1}(x,y) & \dots & C_{m,n+1}(x,y) \\ \vdots & & \vdots \\ C_{0,n+m}(x,y) & \dots & C_{m,n+m}(x,y) \end{vmatrix} \qquad H_{2,-1}^{(n)} = 0$$

$$H_{3,m}^{(n)} = \begin{vmatrix} 1 & \dots & 1 \\ \sum_{k=0}^{n} C_{0,k}(x,y) & \dots & \sum_{k=0}^{n} C_{m,k}(x,y) \\ C_{0,n+1}(x,y) & \dots & C_{m,n+1}(x,y) \\ \vdots & & \vdots \\ C_{0,n+m-1}(x,y) & \dots & C_{m,n+m-1}(x,y) \end{vmatrix} \qquad H_{3,-1}^{(n)} = 0$$

By means of the determinant identities of Schweins and Sylvester we can prove the following lemma.

Lemma 2 *For well-defined $Q_m^{(n+1)}(x,y)$ and $E_m^{(n+1)}(x,y)$ the following determinant formulas hold:*

$$Q_m^{(n+1)}(x,y) = -\frac{H_{0,m}^{(n+m)} H_{1,m-1}^{(n+m-1)} H_{3,m}^{(n+m)}}{H_{0,m}^{(n+m-1)} H_{1,m}^{(n+m)} H_{3,m-1}^{(n+m)}}$$

$$E_m^{(n+1)}(x,y) = -\frac{H_{0,m+1}^{(n+m)} H_{1,m-1}^{(n+m)} H_{3,m}^{(n+m+1)}}{H_{0,m}^{(n+m)} H_{1,m}^{(n+m+1)} H_{3,m}^{(n+m)}}$$

Moreover, $H_{1,m}^{(n+m)} / \left[x^{\pi_x(n,m)} y^{\pi_y(n,m)}\right]$ is a determinant representation for the denominator $q(x,y)$ satisfying the Padé approximation conditions. From lemma 2 we then see that if $f(x,y)$ is a meromorphic function, the denominators of $Q_\ell^{(n+1)}(x,y)$ contain information on the poles of f because in that case some determinants $H_{1,\ell}^{(n+\ell)} / \left[x^{\pi_x(n,\ell)} y^{\pi_y(n,\ell)}\right]$ converge to the poles of the meromorphic f as explained in the previous theorem. We reformulate this in terms of the general multivariate qd-algorithm.

Theorem 22 *Let $f(x,y)$ be a function which is meromorphic in the polydisc $B(0; R_1, R_2) = \{(x,y) : |x| < R_1, |y| < R_2\}$, meaning that there exists a polynomial $R_m(x,y)$ such that $(fR_m)(x,y)$ is analytic in the polydisc above. Let the polynomial $R_m(x,y)$ be factored into*

$$R_m(x,y) = \sum_{(d,e)\in D} r_{de} x^d y^e = \sum_{i=0}^{m} r_{d_i e_i} x^{d_i} y^{e_i}$$

$$= \prod_{\kappa=1}^{K} R_\kappa(x,y) = \prod_{\kappa=1}^{K} \left(\sum_{(d,e)\in D_{\ell_\kappa}} r_{de} x^d y^e \right)$$

with $D_{\ell_1} * D_{\ell_2} * \ldots * D_{\ell_K} = D$*. Further, we assume that* $R_m(0,0) \neq 0$ *so that necessarily* $(0,0) \in D$*. Let the conditions (13) be satisfied and let the enumeration (14a) be such that*

$$\begin{aligned} \lim_{n\to\infty} \mathcal{N}_x(n) &= \infty \\ \lim_{n\to\infty} \mathcal{N}_y(n) &= \infty \end{aligned}$$

Then for each $1 \leq \ell \leq m$ *with* $D_\ell = D_{\ell_1} * \ldots * D_{\ell_\kappa}$*:*

$$\lim_{n\to\infty} E_\ell^{(n+1)}(x,y) = 0$$

uniformly in a neighbourhood of the origin excluding a set of Lebesgue measure zero, and

$$\lim_{n\to\infty} H_{1,\ell}^{(n+\ell)} / \left[x^{\pi_x(n,\ell)} y^{\pi_y(n,\ell)}\right] = R_1(x,y) \times \ldots \times R_\kappa(x,y)$$

The column number ℓ satisfying $D_\ell = D_{\ell_1} * \ldots * D_{\ell_\kappa}$ is called a critical column number and it has the same meaning as the critical column number in the univariate qd-scheme. It indicates which column of Q-values has to be inspected because it contains information on the poles of $f(x,y)$. When computing $Q_\ell^{(n+1)}$ algebraically, the factor $H_{1,\ell}^{(n+\ell)} / \left[x^{\pi_x(n,\ell)} y^{\pi_y(n,\ell)}\right]$ is easy to isolate in the denominator of $Q_\ell^{(n+1)}$ because it is the only one that evaluates different from zero at the point around which the given function $f(x,y)$ is developed, here the origin.

4.3. THE HOMOGENEOUS MULTIVARIATE CASE.

Let us first take a look at the sequence of homogeneous multivariate Padé approximants $r_{\nu,\mu}(x,y)$ with $\nu = 0,1,2,\ldots$ and μ fixed. From (8a–b) it is clear that if we increase the numerator degree ν of the homogeneous Padé approximant, also the denominator set D is influenced. Hence a sequence of homogeneous Padé approximants with fixed μ and increasing ν does not correspond to any sequence of general order multivariate Padé approximants with fixed denominator index set D. This immediately implies that theorem 21 cannot be applied to any sequence of homogeneous multivariate Padé approximants.

A generalization of theorem 17 on the convergence in measure of diagonal homogeneous Padé approximants is currently under investigation. That there is evidence for such convergence in measure is illustrated numerically in [13]. We remark that because ν and μ play the same role for multivariate homogeneous Padé approximants as n and m for univariate Padé approximants, the notion of diagonal approximant is very natural for multivariate homogeneous Padé approximants. It suffices to let $\nu = \mu$. For general order multivariate Padé approximants the notion of diagonal approximant is not so clear because of the possibility to choose the enumeration for the numerator different from the one for the denominator.

Let us now take a look at a homogeneous multivariate qd-algorithm. The series expansion of $f(x,y)$ is rewritten as a single sum by grouping terms into homogeneous expressions:

$$f(x,y) = \sum_{\ell \in \mathbb{N}} \left(\sum_{i+j=\ell} c_{ij} x^i y^j \right)$$

The homogeneous multivariate qd-algorithm is then defined by:

$$E_0^{(n)}(x,y) = 0 \qquad n = 1,2,\ldots \tag{17a}$$

$$Q_1^{(n)}(x,y) = \frac{\sum\limits_{i+j=n+1} c_{ij} x^i y^j}{\sum\limits_{i+j=n} c_{ij} x^i y^j} \tag{17b}$$

$$E_m^{(n)}(x,y) = Q_m^{(n+1)}(x,y) - Q_m^{(n)}(x,y) + E_{m-1}^{(n+1)}(x,y) \qquad m = 1,2\ldots \quad n = 0,1\ldots$$

$$Q_{m+1}^{(n)}(x,y) = \frac{E_m^{(n+1)}(x,y) Q_m^{(n+1)}(x,y)}{E_m^{(n)}(x,y)} \qquad m = 1,2,\ldots \quad n = 0,1,\ldots \tag{17c}$$

If we arrange the values $Q_m^{(n)}(x,y)$ and $E_m^{(n)}(x,y)$ as in the univariate case, where subscripts indicate columns and superscripts indicate downward sloping diagonals, then the entire construction is very similar to the univariate scheme (10–11). It can also be proved that for $y = \lambda x$ and for $n, m \geq 1$

$$Q_m^{(n)}(x, \lambda x) = \tilde{q}_m^{(n)} \cdot x$$
$$E_m^{(n)}(x, \lambda x) = \tilde{e}_m^{(n)} \cdot x$$

where $\tilde{q}_m^{(n)}$ and $\tilde{e}_m^{(n)}$ come from the univariate qd-scheme computed for the function $f(x, \lambda x)$. In other words, the homogeneous multivariate qd-scheme and the univariate qd-scheme coincide when the multivariate function is projected on rays $y = \lambda x$. When we then want to use the homogeneous qd-algorithm to detect the polar singularities of $f(x, y)$, we proceed as follows. Theorem 23 generalizes the results of theorem 18. A generalization of theorem 19 can be formulated in the same way.

Theorem 23 *Let the Taylor series at the origin be given of a function* $f(x,y)$ *meromorphic in the polydisc* $B(0,R) = \{(x,y) : |x| < R, |y| < R\}$, *meaning that there exists a polynomial* $q(x,y)$ *such that* $(fq)(x,y)$ *is holomorphic in* $B(0,R)$. *Let for* $\lambda \in \mathbb{R}$ *the function* $f_\lambda(x)$ *be defined by*

$$f_\lambda(x) = f(x, \lambda x)$$

and let the poles w_i *of* f_λ *in* $B(0,R)$ *be numbered such that*

$$w_0 = 0 < |w_1| \leq |w_2| \leq \ldots < R$$

each pole occuring as many times in the sequence $\{w_i\}_{i \in N}$ *as indicated by its order. If* f_λ *is ultimately* m*-normal for some integer* $m > 0$, *then the homogeneous qd-scheme associated with* f *has the following properties (put* $w_{m+1} = \infty$ *if* f_λ *has only* m *poles):*

(a) for each ℓ *with* $0 < \ell \leq m$ *and* $|w_{\ell-1}| < |w_\ell| < |w_{\ell+1}|$,

$$\lim_{n \to \infty} Q_\ell^{(n)}(x, \lambda x) = w_\ell^{-1} \cdot x$$

(b) for each ℓ *with* $0 < \ell \leq m$ *and* $|w_\ell| < |w_{\ell+1}|$,

$$\lim_{n \to \infty} E_\ell^{(n)}(x, \lambda x) = 0$$

How the parameter λ affects the order in which the poles of $f(x, y)$ are detected pointwise as $(w_\ell, \lambda w_\ell)$ with $w_\ell = x / \lim_{n \to \infty} Q_\ell^{(n)}(x, \lambda x)$ and not curvewise as in theorem 22, can be learned from a numerical example given in [14]. If we compare this convergence result to the one for the general order multivariate qd-algorithm given in the previous section, we see that there the algorithm discovers and identifies the polar factors as separate objects. The price one has to pay for this elegance is that the general multivariate qd-algorithm must be programmed in order to deal with algebraic expressions instead of with numeric data. The homogeneous qd-algorithm delivers the poles point by point (numeric output) while the general order qd-algorithm delivers the poles as algebraic curves (formula output). This implies that the general qd-algorithm is considerably slower than the homogeneous qd-algorithm when used for pole detection. However its reply is considerably more accurate.

In analogy with the univariate Padé approximation case [18, p. 610] it is also possible to give explicit determinant formulas for the homogeneous multivariate Q- and E-values. Let us re-introduce the notation

$$C_\ell(x,y) = \sum_{i+j=\ell} c_{ij} x^i y^j \qquad \ell = 0, 1, \ldots$$

and define the determinants

$$H_m^{(n)}(x,y) = \begin{vmatrix} C_n(x,y) & C_{n+1}(x,y) & \ldots & C_{n+m-1}(x,y) \\ C_{n+1}(x,y) & C_{n+2}(x,y) & \ldots & C_{n+m}(x,y) \\ \vdots & & & \vdots \\ C_{n+m-1}(x,y) & C_{n+m}(x,y) & \ldots & C_{n+2m-2}(x,y) \end{vmatrix}$$

The series development of $f(x,y)$ is termed k-normal if $H_m^{(n)}(x,y) \not\equiv 0$ for $m = 0, 1, \ldots, k$ and $n \geq 0$. It is called ultimately k-normal if for every $0 \leq m \leq k$ there exists an $n(m)$ such that $H_m^{(n)}(x,y) \not\equiv 0$ for $n > n(m)$. By means of the determinant identities of Sylvester and Schweins we can prove the following lemma for k-normal multivariate series [9].

Lemma 3 *Let $f(x,y)$ be given by its formal Taylor series expansion. If there exists a positive integer k such that $f(x,y)$ is k-normal then the functions $Q_m^{(n)}(x,y)$ and $E_m^{(n)}(x,y)$ exist for $m = 1, \ldots, k$ and $n \geq 0$ and they are given by*

$$Q_m^{(n)}(x,y) = \frac{H_m^{(n+1)} H_{m-1}^{(n)}}{H_m^{(n)} H_{m-1}^{(n+1)}}(x,y)$$

$$E_m^{(n)}(x,y) = \frac{H_{m+1}^{(n)} H_{m-1}^{(n+1)}}{H_m^{(n)} H_m^{(n+1)}}(x,y)$$

We can now complete the list of results with the following multivariate analogue of theorem 20 of which the proof can be found in [14].

Theorem 24 *Let the Taylor series expansion at the origin be given of a multivariate rational function of homogeneous degree n in the numerator and $m \leq n$ in the denominator. Then if the series $f(x,y)$ is m-normal,*

$$E_m^{(n-m+h)}(x,y) \equiv 0 \qquad h > 0$$

Table 1

$g_{0,1}^{(0)}$	$g_{0,2}^{(0)}$			$g_{0,r}^{(0)}$					$g_{0,m}^{(0)}$		
		$g_{1,2}^{(0)}$			$g_{1,r}^{(0)}$					$\ddots$	
$g_{0,1}^{(1)}$	$g_{0,2}^{(1)}$			$g_{0,r}^{(1)}$		$\ddots$			$g_{0,m}^{(1)}$		
		$g_{1,2}^{(1)}$			$g_{1,r}^{(1)}$		$g_{r-1,r}^{(0)}$		$\vdots$		$g_{m-1,m}^{(0)}$
$g_{0,1}^{(2)}$	$g_{0,2}^{(2)}$	$\vdots$	$\cdots$	$g_{0,r}^{(2)}$	$\vdots$		$\vdots$	$\cdots$			$\vdots$
$\vdots$	$\vdots$			$\vdots$			$g_{r-1,r}^{(n+m-r+1)}$				$g_{m-1,m}^{(n+1)}$
						$\iddots$					
		$g_{1,2}^{(n+m-1)}$			$g_{1,r}^{(n+m-1)}$					$\iddots$	
$g_{0,1}^{(n+m)}$	$g_{0,2}^{(n+m)}$			$g_{0,r}^{(n+m)}$					$g_{0,m}^{(n+m)}$		

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