To my father Pierre F.M. Cuyt.

ABSTRACT PADÉ-APPROXIMANTS IN OPERATOR THEORY**

by

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The use of Padé-approximants for the solution of mathematical problems in science has great development. Padé-approximants have proved to be very useful in numerical analysis too : the solution of a nonlinear equation, acceleration of convergence, numerical integration by using nonlinear techniques, the solution of ordinary and partial differential equations. Especially in the presence of singularities the use of Padé-approximants has been very interesting.
Yet we have tried to generalize the concept of Padé-approximant to operator theory, departing from "power-series-expansions" as is done in the classical theory*.
A lot of interesting properties of classical Padé-approximants remain valid and the classical Padé-approximant is now a special case of the theory. The notion of abstract Padé-table is introduced; it also consists of squares of equal elements as in the classical theory.

* Roman figures between brackets refer to a work in the reference-list.

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0. NOTATIONS

R_o^+	{positive real numbers}
X,Y	always normed vectorspaces or Banach-spaces or Banach-algebras with unit
L(X,Y)	{linear bounded operators $L : X \to Y$}
$L(X^k,Y)$	{k-linear bounded operators $L : X \to L(X^{k-1},Y)$}
Λ	field R or C
$\lambda,\mu,\ldots$	elements of Λ
0	unit for addition in a Banach-space, or multilinear operator $L \in L(X^k,Y)$ such that $Lx_1 \ldots x_k = 0 \quad \forall (x_1,\ldots,x_k) \in X^k$
I	unit for multiplication in a Banach-algebra
1	unit for multiplication in Λ
F,G,...	non-linear operators : $X \to Y$
$B(x_o,r)$	open ball with centre $x_o \in X$ and radius $r > o$
$\overline{B}(x_o,r)$	closed ball with centre $x_o \in X$ and radius $r > o$
P,Q,R,S,T,...	non-linear operators : $X \to Y$, usually abstract polynomials
$\partial P, \partial Q, \ldots$	exact degree of the abstract polynomial P,Q,...
$F^{(k)}(x_o)$	k^{th} Fréchet-derivative of the operator $F : X \to Y$ in x_o
D(G)	$\{x \in X \mid G(x)$ is regular in Y$\}$ for the operator $G : X \to Y$ (=Banach-algebra)
A_i, B_j, C_k, D_s	i-linear, j-linear, k-linear, s-linear operators

1. INTRODUCTION

A lot of attempts have been made to generalize in some way classical Padé-approximants. We refer e.g. to quadratic Padé-approximants (X,XV), Chebyshev-Padé or

Legendre-Padé (VII), operator Padé-approximants for formal power series in a parameter with non-commuting elements of a certain algebra as coefficients (VI), N-variable rational approximants (VIII, IX, XI, XII, XIII, XIV).

Another genralisation now is the following one.

Let X and Y be Banach-spaces (same field Λ). We always work in the norm-topology. We define $L(X^k,Y)=\{L|L$ is a k-linear bounded operator, $L:X\to L(X^{k-1},Y)\}$ and $L(X^0,Y)=Y$. So $Lx_1\ldots x_k=(Lx_1)(x_2\ldots x_k)\in Y$ with $x_1,\ldots,x_k\in X$ and $Lx_1\in L(X^{k-1},Y)$ (V pp. 100). $L\in L(X^k,Y)$ is called symmetric if $Lx_1\ldots x_k=Lx_{i_1}\ldots x_{i_k}$, $\forall(x_1,\ldots,x_k)\in X^k$ and $\forall$ permutations $(i_1,\ldots,i_k)$ of $(1,\ldots,k)$ (V pp. 103).

We remark that the operator $\bar{L}\in L(X^k,Y)$ defined by $\bar{L}x_1\ldots x_k=\frac{1}{k!}\sum_{(i_1,\ldots,i_k)} Lx_{i_1}\ldots x_{i_k}$ for a given $L\in L(X^k,Y)$ is symmetric.

Let us identify $y\in Y$ with the constant operator $X\to Y: x\to y$ and call it o-linear.

Definition 1.1. : An abstract polynomial is a non-linear operator $P:X\to Y$ such that

$$P(x)=A_nx^n+\ldots+A_0\in Y \text{ with } \begin{cases} A_i\in L(X^i,Y)\\ A_i \text{ symmetric}\end{cases}$$

The degree of $P(x)$ is n.

The notation for the exact degree of $P(x)$ is ∂P.

Definition 1.2. : Let X be a Banach-space, Y a Banach-algebra; let $F:X\to Y$ and $G:X\to Y$ be operators.

The <u>product F.G</u> is defined by : $(F.G)(x)=F(x).G(x)$ in Y.

Definition 1.3. : Let $X_1,\ldots,X_p$, $Z_1,\ldots,Z_q$ be vector spaces and Y an algebra (same field Λ). Let $F:X_1\times\ldots\times X_p\to Y$ be bounded and p-linear, and $G:Z_1\times\ldots\times Z_q\to Y$ be bounded and q-linear.

The <u>tensorproduct $F\otimes G$</u> : $X_1\times\ldots\times X_p\times Z_1\times\ldots\times Z_q\to Y$ is bounded and (p+q)-linear when defined by $(F\otimes G)x_1\ldots x_pz_1\ldots z_q=Fx_1\ldots x_p.Gz_1\ldots z_q$ (IIpp.318).

One can easily prove that in a Banach-algebra Y :

$$(F.G)'(x_0) = F'(x_0) \otimes G(x_0) + F(x_0) \otimes G'(x_0) \;,$$

where the accent stands for Fréchet-differentiation.

We call $y \in Y$ regular if there exists $y^{-1} \in Y$ such that : $y.y^{-1} = I = y^{-1}.y$;

we call $y \in Y$ singular if it is not regular.

Definition 1.4. : Let $G : X \to Y$ with X a Banach-space and Y a Banach-algebra;
$D(G) = \{x \in X | G(x) \text{ is regular in } Y\}$ is an open set in X (III pp.31).
The operator $\frac{1}{G}$ is defined by $\frac{1}{G} : D(G) \subset X \to Y : x \to [G(x)]^{-1}$.

One can easily prove that in a commutative Banach-algebra Y :

$$\left(\frac{1}{G}\right)'(x_0) = -G'(x_0) \otimes \left(\frac{1}{G}(x_0)\right)^2 \;.$$

Let again X and Y both be Banach-spaces.

We note the fact that $F^{(k)}(x_0)$, the k^{th} derivative of an operator $F : X \to Y$ in x_0, is a symmetric k-linear operator (V pp. 110).

Abstract polynomials are differentiated as in elementary calculus :

if $P(x) = A_n x^n + \dots + A_0$ with $A_i \in L(X^i,Y)$ and A_i symmetric, then

$$P'(x_0) = n.A_n x_0^{n-1} + \dots + A_1 \in L(X,Y)$$

$$P^{(2)}(x_0) = n.(n-1).A_n x_0^{n-2} + \dots + 2A_2 \in L(X^2,Y)$$

$$\vdots$$

$$P^{(n)}(x_0) = n.\; A_n \in L(X^n,Y)$$

We now can easily prove the fact that if for an abstract polynomial $P(x) = \sum_{i=0}^{n} C_i x^i$ with $C_i \in L(X^i,Y)$ and C_i symmetric: $P(x) = 0 \;\; \forall x \in X$, then $C_i \equiv 0$ $\forall i \in \{o,\dots,n\}$.

Let $B(x_0,r) = \{x \in X | \|x_0 - x\| < r\}$ for $r \in R_0^+$ and $x_0 \in X$.

Definition 1.5. : The operator $F : X \to Y$ possesses an abstract Taylor-series in x_0 if

$\exists B(x_0,r)$ with $r > o$:

$$F(x_0+h) = \sum_{k=o}^{\infty} \frac{1}{k!} \cdot F^{(k)}(x_0)\, h^k \text{ for } x_0 + h \in B(x_0,r).$$

We then call F abstract analytic in x_0 (V pp. 113).

2. DEFINITION OF ABSTRACT PADE-APPROXIMANT

To generalize the notion of Padé-approximant we start from analyticity, as in elementary calculus.

Let $F : X \to Y$ be a non-linear operator, X a Banach-space and Y a Banach-algebra. Let F be analytic in $B(x_0,r)$ with $r > o$.

So F has the following abstract Taylor-series :

$$F(x_0+x) = \sum_{k=o}^{\infty} \frac{1}{k!} F^{(k)}(x_0)\, x^k \tag{1}$$

with $\frac{1}{o!} F^{(o)}(x_0)x^0 = F(x_0)$

and $F^{(k)}(x_0) \in L(X^k, Y)$

We give some examples of such series :

a) $C([o,1])$ with the supremum-norm and $(f.g)(x) = f(x).g(x)$ for $f,g \in C([o,1])$, is a commutative Banach-algebra. Consider the Nemyckii-operator $G : C([o,1]) \to C([o,1]) : x \to g(s,x(s))$ with $g \in C^{(\infty)}([o,1] \times C([o,1]))$ (V pp. 95).

Let $I_x : C([o,1]) \to C([o,1]) : x \to x$.

Then clearly $G^{(n)}(x_0) = \frac{\partial^n g}{\partial x^n}(s,x_0(s)).\ \underbrace{I_x \otimes \ldots \otimes I_x}_{n \text{ times}}$, n-linear and bounded.

b) Consider the Urysohn integral operator $U : C([o,1]) \to C([o,1])$:

$x \to \int_0^1 f(s,t,x(t))dt$ with $f \in C^{(\infty)}([o,1] \times [o,1] \times C([o,1]))$ (V pp. 97).

Let [ ] indicate a place-holder for $x(t) \in C([o,1])$ (V pp. 90).

Then we write $U^{(n)}(x_0) = \int_0^1 \frac{\partial^n f}{\partial x^n}(s,t,x_0(t)) \underbrace{[\ ]\ldots[\ ]}_{n \text{ times}}\, dt$

c) Consider the operator $P : C'([o,T]) \to C([o,T]) : y \to \frac{dy}{dt} - f(t,y)$ in the initial value problem $P(y) = 0$ with $y(o) = a \in R$.

Let $f \in C^{(\infty)}([o,T] \times C'([o,T]))$ and $I_y : C'([o,T]) \to C([o,T]) : y \to y$.

We remark that $C^{(i)}([o,T])$ with the supremum-norm is a Banach space.

We see that $P'(y_o) = \frac{d}{dt} - \frac{\partial f(t,y)}{\partial y}(t,y_o).I_y$ and

$$P^{(n)}(y_o) = \frac{-\partial^n f(t,y)}{\partial y^n}(t,y_o) \cdot \underbrace{I_y \otimes \ldots \otimes I_y}_{n \text{ times}} \text{ for } n \geq 2.$$

d) Finally let this nonlinear system of 2 real variables $F\binom{\xi}{\eta} = \binom{\xi + \sin(\xi\eta) + 1}{\xi^2 + \eta^2 - 4\xi\eta}$ be given; let $x_o = \binom{0}{0}$. $\mathbb{R}^2$ with component-wise multiplication is a Banach-algebra with unit $\binom{1}{1}$.

Then $F(x) = \binom{1}{0} + \binom{\xi}{0} + \binom{\xi\eta}{\xi^2 + \eta^2 - 4\xi\eta} + \sum_{k=1}^{\infty} \begin{pmatrix} (-1)^k \cdot \frac{(\xi\eta)^{2k+1}}{(2k+1)!} \\ 0 \end{pmatrix}$

Definition 2.1. : Let $F : X \to Y$ be an operator with X and Y Banach-spaces. We say that <u>$F(x) = 0(x^j)$</u> if $\exists J \in R_0^+$, $\exists B(0,r)$ with $o < r < 1$: $\forall x \in B(0,r) : \|F(x)\| \leq J.\|x\|^j$ $(j \in N)$

Now let $x_o = 0$ without loss of generality, and let Y be a commutative Banach-algebra. In Y we can use the fact that for $y,z \in Y : y.z$ regular $\Rightarrow$ y regular and z regular.

Definition 2.2. : In <u>Padé-approximation</u> we try to find a couple of abstract polynomials $(P(x),Q(x)) = (A_{n.m+n}\, x^{n.m+n} + \ldots + A_{n.m}\, x^{n.m},$ $B_{n.m+m}\, x^{n.m+m} + \ldots + B_{n.m}\, x^{n.m})$ such that the abstract power series

$$F(x).(B_{n.m+m}\, x^{n.m+m} + \ldots + B_{n.m}\, x^{n.m}) - (A_{n.m+n}\, x^{n.m+n} + \ldots + A_{n.m}\, x^{n.m}) = 0(x^{n.m+n+m+1}).$$

(In 5.f) we justify the choice of (P(x), Q(x)) made here).

Write $\frac{1}{k!} \cdot F^{(k)}(0) = C_k \in L(X^k,Y)$.

The condition in definition 2.2 is equivalent with (1a) and (1b) :

$$(1a)\begin{cases} C_o \cdot B_{n.m}\ x^{n.m} = A_{n.m}\ x^{n.m} \quad \forall x \in X \\ C_1 x . B_{n.m}\ x^{n.m} + C_o \cdot B_{n.m+1}\ x^{n.m+1} = A_{n.m+1}\ x^{n.m+1} \quad \forall x \in X \\ \vdots \\ C_n\ x^n . B_{n.m}\ x^{n.m} + C_{n-1}\ x^{n-1} . B_{n.m+1}\ x^{n.m+1} + \dots + C_o \cdot B_{n.m+n}\ x^{n.m+n} = \\ \qquad A_{n.m+n}\ x^{n.m+n} \quad \forall x \in X \end{cases}$$

with $B_j \equiv 0 \in L(X^j,Y)$ if $j > n.m+m$

$$(1b)\begin{cases} C_{n+1}\ x^{n+1} . B_{n.m}\ x^{n.m} + \dots + C_{n+1-m}\ x^{n+1-m} . B_{n.m+m}\ x^{n.m+m} = 0 \quad \forall x \in X \\ \vdots \\ C_{n+m}\ x^{n+m} . B_{n.m}\ x^{n.m} + \dots + C_n\ x^n . B_{n.m+m}\ x^{n.m+m} = 0 \quad \forall x \in X \end{cases}$$

with $C_k \equiv 0 \in L(X^k,Y)$ if $k < o$.

For every solution $\{B_{n.m+j}\ x^{n.m+j} \mid j = o,\dots,m\}$ of (1b), a solution $\{A_{n,m+i}\ x^{n.m+i} \mid i = o,\dots,n\}$ of (1a) can be computed.

3. EXISTENCE OF A SOLUTION

a) case : $m = o$

Choose $B_{n.m} = B_o = I$, unit for the multiplication in Y.

Then $A_i = C_i$ for $i = o,\dots,n$ are a solution of (1a).

The partial sums of (1) are the sought abstract polynomials.

b) case : $m \neq o$

Compute $D_{n.m} = \sum_{i_1=1}^{m} \dots \sum_{i_m=1}^{m} [\varepsilon_{i_1 \dots i_m} \bigotimes_{j=1}^{m} C_{n-(j-1)+(i_j-1)}]$

with $i_1,\dots,i_m \in \{1,\dots,m\}$, and $\varepsilon_{i_1\dots i_m} = +1$ when $i_1\dots i_m$ is an even permutation of $1\dots m$, and $\varepsilon_{i_1\dots i_m} = -1$ when $i_1\dots i_m$ is an odd permutation of $1\dots m$, and $\varepsilon_{i_1\dots i_m} = 0$ elsewhere.

Compute for $h = 1,\dots,m$: $D_{n.m+h}$ by replacing in $D_{n.m}$ the operator $C_{n-(h-1)+(i_h-1)}$ by the operator $-C_{n+1+(i_h-1)}$.

Clearly $D_{n.m+h} \in L(X^{n.m+h},Y)$ for $h = 0,\dots,m$.

Now $D_{n.m+h}\, x^{n.m+h}$ is a solution of system (1b); and $D_{n.m+h}\, x^{n.m+h} = \overline{D}_{n.m+h}\, x^{n.m+h}$.

We thus can consider a symmetric solution, also for (1a).

This is a correct procedure to calculate a solution. But in some cases it can be more practical to solve the system otherwise, e.g. to get the most general form of the solution.

4. UNICITY OF A SOLUTION

From now on $F : X \to Y$ is a nonlinear operator with X a Banach-space and Y a commutative Banach-algebra such that for each polynomial $T : X \to Y$ with $D(T) \neq \phi$, the set $D(T)$ is dense in X (or any other equivalent condition).

This is the case e.g. for $F : R^p \to R^q$; if $T(x) = \left(\sum_{j_1+\dots+j_p=0}^{m} \alpha_{ij_1\dots j_p} x_1^{j_1}\dots x_p^{j_p},\ i = 1,\dots,q\right)$, $D(T) \neq \phi$, the set $X \setminus \bigcup_{i=1}^{q} \{(x_1,\dots,x_p) \in R^p \mid \sum_{j_1+\dots+j_p=0}^{m} \alpha_{ij_1\dots j_p} x_1^{j_1}\dots x_p^{j_p} = 0\}$ is dense in X with the norm-topology. We then have the following important lemma.

Lemma 4.1. :

Let U,T be abstract polynomials : $X \to Y$ $\left.\begin{array}{l} U(x).T(x) = 0 \quad \forall x \in X \\ \{x \in X \mid T(x) \text{ regular}\} \text{ is dense in } X \end{array}\right\} \Rightarrow U \equiv 0$

After calculating the solution of (1a) and (1b) we are going to look for an irreducible rational approximant.

Definition 4.1. : Let P and Q be 2 abstract polynomials. We call $\frac{1}{Q}.P$ <u>reducible</u> if there exist abstract polynomials T,R,S such that $P = T.R = R.T$ and $Q = T.S = S.T$ and $\partial T \geq 1$, $\partial R \geq o$, $\partial S \geq o$.

For reducible $\frac{1}{Q}.P$ we know that $\forall x \in D(Q) : (\frac{1}{Q}.P)(x) = (\frac{1}{S}.R)(x)$.
It is possible that $\frac{1}{S}$ is defined on a greater domain than $\frac{1}{Q}$.

Lemma 4.2. :

> Let P,Q,R be abstract polynomials : $X \to Y$
> For $R = P.Q$: $\begin{cases} D(R) = D(P) \cap D(Q) \\ D(R) = \phi \Leftrightarrow D(P) = \phi \text{ or } D(Q) = \phi \end{cases}$

Proof : R(x) regular $\Leftrightarrow$ P(x) regular and Q(x) regular

so $D(R) = D(P) \cap D(Q)$

We know that D(P) is open (and so is D(Q))

D(Q) is dense in X if $D(Q) \neq \phi$ (and so is D(P))

If $D(P) = \phi$ or $D(Q) = \phi$ then evidently $D(R) = \phi$.

The second implication is proved by contraposition.

If $D(R) = \phi$ and $\exists x \in D(P)$ then $\exists r_0 > o : B(x\ ,r_0) \subset D(P)$.

Now $\forall x \in X$, $\forall\ r > o : B(x,r) \cap D(Q) \neq \phi$.

And so $\phi \neq B(x,r_0) \cap D(Q) \subseteq D(P) \cap D(Q)$.

This implies a contradiction.

Definition 4.2. : Let (P,Q) be a couple of abstract polynomials satisfying definition 2.2 and suppose $D(Q) \neq \phi$ or $D(P) \neq \phi$. Possibly $\frac{1}{Q}.P$ is reducible. Let $\frac{1}{Q_\star}.P_\star$ be the irreducible form of $\frac{1}{Q}.P$ such that $0 \in D(Q_\star)$ and and $Q_\star(0) = I$, if it exists. We then call $\frac{1}{Q_\star}.P_\star$ an <u>abstract Padé-approximant</u> of order (n,m) for F.

That irreducible form $\frac{1}{Q_\star}.P_\star$ with $Q_\star(0) = I$ is unique because if $P = P_{\star 1}.T_1 = P_{\star 2}.T_2$

and $Q = Q_{\star 1}.T_1 = Q_{\star 2}.T_2$ with $\frac{1}{Q_{\star 1}}.P_{\star 1}$ and $\frac{1}{Q_{\star 2}}.P_{\star 2}$ irreducible, $Q_{\star 1}(0) = I = Q_{\star 2}(0)$, $D(T_1) \neq \phi$ and $D(T_2) \neq \phi$, then $P_{\star 1}.Q_{\star 2} = P_{\star 2}.Q_{\star 1}$ because of lemma 4.1 and so we can prove that $\exists$ polynomial R $\ni$ $\begin{cases} P_{\star 1} = R.P_{\star 2}, \\ Q_{\star 1} = R.Q_{\star 2} \\ R(0) = I \end{cases}$ what contradicts the irreducible character

of $\frac{1}{Q_{\star_1}}.P_{\star_1}$ unless $\partial R = o$.

Call n' the exact degree of $P_\star$ and m' the exact degree of $Q_\star$.
When $(P(x) = P_\star(x).T(x),\ Q(x) = Q_\star(x).T(x))$ is a solution of (1a) and (1b) and $\frac{1}{Q_\star}.P_\star$ is an abstract Padé-approximant of order (n,m) for F, then $\partial T \geq n.m$ and $n' \leq n$ and $m' \leq m$.

We have the following theorem concerning the solutions of (1a) and (1b).

Theorem 4.1 : If the couples (P,Q) and (R,S) of abstract polynomials both satisfy (1a) and (1b), then $P.S = R.Q$; in other words : $\forall x \in X : P(x).S(x) = R(x).Q(x)$.

Proof : Regard $P(x).S(x)-R(x).Q(x) =$

$$[F(x).S(x)-R(x)].Q(x)-[F(x).Q(x)-P(x)].S(x)$$

Now $(F.Q-P)(x) = O(x^{n.m+n+m+1})$

$(F.S-R)(x) = O(x^{n.m+n+m+1})$

But $(P.S-R.Q)(x)$ is an abstract polynomial of degree at most $2n.m+n+m$, while $[(F.S-R).Q-(F.Q-P).S](x) = O(x^{2n.m+n+m+1})$

So $(P.S-R.Q)(x) = 0 \quad \forall x \in X$.

This theorem implies that $(\frac{1}{Q}.P)(x) = (\frac{1}{S}.R)(x) \quad \forall x \in D(Q) \cap D(S)$.
If $D(Q.S) \neq \phi$ then $D(Q.S)$ is dense in X.
Possibly $\frac{1}{Q}.P$ and $\frac{1}{S}.R$ are reducible. If $P = P_\star.T$, $Q=Q_\star.T$, $R=R_\star.U$, $S=S_\star.U$ with

$D(T) \neq \phi$ and $D(U) \neq \phi$, then :

$$P.S = R.Q \Rightarrow P_\star.S_\star = R_\star.Q_\star \quad \text{because of lemma 4.1.}$$

We then know that $(\frac{1}{Q_\star} . P_\star)(x) = (\frac{1}{S_\star} . R_\star)(x) \quad \forall x \in D(Q_\star) \cap D(S_\star)$; if $D(Q_\star.S_\star) \neq \phi$ then $D(Q_\star.S_\star)$ is dense in X.

We can define an equivalence relation ... ~ ... in

$A = \{(P,Q) | (P,Q)$ satisfies definition 2.2 and $(D(P) \neq \phi$ or $D(Q) \neq \phi)\} \cup$

$\{(P_\star,Q_\star) | (P = P_\star.T,\ Q = Q_\star.T)$ satisfies definition 2.2 and $(D(P) \neq \phi$ or $D(Q) \neq \phi)$ and $\frac{1}{Q_\star} . P_\star$ is irreducible$\}$ where $P_\star, Q_\star, T, P, Q$ are abstract polynomials, by

$(P,Q) \sim (R,S) \Leftrightarrow P(x).S(x) = R(x).Q(x)\ \forall x \in X.$

If there exists a solution $(P,Q) \in A$ such that $Q_\star(0) = I$, then for all equivalent solutions $(R,S) \in A : 0 \in D(S_\star)$ because $P_\star S_\star = R_\star Q_\star$ implies : $\exists$ polynomial $V \ni \begin{cases} R_\star = VP_\star \\ S_\star = VQ_\star \\ V(0) = S(0) \end{cases}$,

what contradicts the irreducible character of $\frac{1}{S_\star} . R_\star$ unless $\partial V = o$ and so $\begin{cases} R_\star = S(0).P_\star ; \\ S_\star = S(0).Q_\star \end{cases}$

if now S(0) were not regular then (R,S) were no element of A.

If $S_\star(0) = I = Q_\star(0)$ then $P_\star.S_\star = R_\star.Q_\star$ implies that $\exists$ polynomial $V \ni \begin{cases} P_\star = V.R_\star \\ Q_\star = V.S_\star \\ V(0) = I \end{cases}$

In other words : for $\frac{1}{S_\star} . R_\star$ irreducible we have $\partial V = o$ and so $\frac{1}{Q} . P$ and $\frac{1}{S} . R$ supply the same abstract Padé-approximant of order (n,m) for F when (P,Q) and (R,S) both satisfy (1a) and (1b).

We call $\frac{1}{Q_\star} . P_\star$ satisfying definition 4.2 <u>the abstract Padé-approximant (APA) of order (n,m) for F</u>.

Definition 4.3. : If for all the solutions (P,Q) of (1a) and (1b) with $D(P) \neq \phi$ or $D(Q) \neq \phi$ the irreducible form $\frac{1}{Q_\star} . P_\star$ (representant of the equivalence relation-class) is such that $D(Q_\star) \not\ni 0$, then we call $\frac{1}{Q_\star} . P_\star$ the abstract rational approximant (ARA) of order (n,m) for F.

(We do come back on abstract rational approximants in 5.f).
We remark that, although $F(0) = C_0$ is defined, $(\frac{1}{Q} . P)(0) = \frac{0}{0}$ is always undefined for (P,Q) satisfying definition 2.2 with $n > o$ and $m > o$, since 0 is always singular in Y.
If for all the solutions (P,Q) of (1a) and (1b) : $0 \notin D(Q_\star)$ or $D(Q) = \phi = D(P)$, we shall call the abstract Padé-approximant undefined.
If for the ARA $D(Q_\star) = \phi$ then for all solutions (R,S) of (1a) and (1b) : $D(S_\star) = \phi$ because $D(P_\star) \cap D(S_\star) = D(R_\star) \cap D(Q_\star) = \phi$ and $D(P) \neq \phi$; the ARA is in fact useless then.
An example will prove that it is very well possible that for an operator $F : X \to Y$, the (n,m) Padé-approximant is defined, while the (l,k) Padé-approximant is undefined for $l \neq n$ or $k \neq m$.
Consider the operator $F\binom{\xi}{\eta} = \binom{\xi + \sin(\xi n) + 1}{\xi^2 + \eta^2 - 4\xi\eta} = \binom{1}{0} + \binom{\xi}{0} + \binom{\xi\eta}{\xi^2 + \eta^2 - 4\xi\eta} + \ldots$

Then : (1,1)-APA is $\begin{pmatrix} \frac{1+\xi-\eta}{1-\eta} \\ 0 \end{pmatrix}$, $P_\star(x) = P_\star\binom{\xi}{\eta} = \binom{1}{0} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\binom{\xi}{\eta}$

$$Q_\star(x) = Q_\star\binom{\xi}{\eta} = \binom{1}{1} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\binom{\xi}{\eta}$$

$$D(Q_\star) = R^2 \setminus \{(\xi, 1) \mid \xi \in R\}$$

(2,1)-APA is $\begin{pmatrix} 1+\xi+\xi\eta \\ \xi^2 + \eta^2 - 4\xi\eta \end{pmatrix}$ $P_\star(x) = C_0 + C_1 x + C_2 x^2$

$$Q_\star(x) = I$$

$$D(Q_\star) = R^2$$

(1,2)-APA is undefined.

The next theorem is a summary of the previous results.

Theorem 4.2. :

> For every non-negative value of n and m, the systems (1a) and (1b) are solvable; if the abstract Padé-approximant of order (n,m) for $F : X \to Y$ is defined, it is unique.
> For the (n,m)-APA $\frac{1}{Q_\star} . P_\star$ we know that $P_\star$ and $Q_\star$ are abstract polynomials, respectively of degree at most n and at most m.

Proof : Evident.

From now on, when mentioning abstract Padé-approximants, we consider only the abstract Padé-approximants that are not undefined. Let (P,Q) be a solution of (1a) and (1b). Because of definition 4.2 it is very well possible that $(P_\star,Q_\star)$ itself does not satisfy definition 2.2.

Theorem 4.3. :

> Let $\frac{1}{Q_\star} . P_\star$ be the abstract Padé-approximant of order (n,m) for F.
> Then $\exists s : o \leq s \leq \min(n-n',m-m')$, $\exists$ an abstract polynomial
> $T(x) = \sum_{k=n.m}^{n.m+s} T_k x^k$, $T_{n.m+s} \neq 0$, $D(T) \neq \phi \supset (P_\star.T, Q_\star.T)$ satisfies definition 2.2 ; $\partial(P_\star.T) = n.m+n'+s$ and $\partial(Q_\star.T) = n.m+m'+s$.
> If then $T(x) = T_{n.m+r} x^{n.m+r} + T_{n.m+r+1} x^{n.m+r+1} + \ldots + T_{n.m+s} x^{n.m+s}$ with $D(T_{n.m+r}) \neq \phi$, also $(P_\star.T_{n.m+r}, Q_\star.T_{n.m+r})$ satisfies definition 2.2 and $o \leq r \leq s \leq \min(n-n',m-m')$.

Proof : Because of theorem 4.2 we may consider abstract polynomials P and Q that satisfy (1a) and (1b) and supply $P_\star$ and $Q_\star$. Because of definition 4.2, there exists an abstract polynomial T such that : $P = P_\star.T$ and $Q = Q_\star.T$ and $\partial T \geq n.m$. Because of lemma 4.2 $D(T) \neq \phi$ (otherwise $D(P) = \phi = D(Q)$).
Let $n' = \partial P_\star, m' = \partial Q_\star$, $P = \sum_{i=n.m}^{n.m+n} A_i x^i$, $Q = \sum_{j=n.m}^{n.m+m} B_j x^j$.

Consequently $T(x)=\sum_{k=n.m}^{n.m+s} T_k x^k$ with $\begin{cases} \partial T = n.m+s \\ n.m+n'+s \leq n.m+n \\ n.m+m'+s \leq n.m+m \\ s \geq o \end{cases}$

and so $o \leq s \leq \min(n-n', m-m')$.

$F(x).Q(x)-P(x) = T(x).[F(x).Q_\star(x)-P_\star(x)] = O(x^{n.m+n+m+1})$

Because $T(x) = T_{n.m+r}\, x^{n.m+r} + \ldots$ with $T_{n.m+r} \in L(X^{n.m+r}, Y)$, $D(T_{n.m+r}) \neq \phi$,

we have that $T_{n.m+r}\, x^{n.m+r}.[F(x).Q_\star(x)-P_\star(x)] = O(x^{n.m+n+m+1})$.

5. REMARKS AND SPECIAL CASES

a) When $X = R = Y$ $(\Lambda = R)$, then the definition of abstract Padé-approximant is precisely the classical definition. F is now a real-valued function f of 1 real variable, with a Taylor-series development $\sum_{k=o}^{\infty} c_k . x^k$ with $c_k = \frac{1}{k!} f^{(k)}(o)$.

The k-linear operators $C_k \in L(X^k, Y)$ are :

$$C_k x^k = c_k \underbrace{.x\ldots x}_{k} \in R \text{ with } c_k \in R \quad .$$

The j-linear functions $B_j x^j = b_j \underbrace{.x\ldots x}_{j} \in R$, $b_j \in R$, $j = n,m,\ldots,n.m+m$ and such that :

$$\begin{cases} c_{n+1}.b_{n.m} + \ldots + c_{n+1-m}.b_{n.m+m} = o \\ \vdots \\ c_{n+m}.b_{n.m} + \ldots + c_n.b_{n.m+m} = o \end{cases}$$

are a solution of (1b).

The i-linear functions $A_i x^i = a_i \underbrace{.x\ldots x}_{i} \in R$, $a_i \in R$, $i = n.m,\ldots,n.m+n$ such that :

$$\begin{cases} c_o \cdot b_{n.m} = a_{n.m} \\ c_1 \cdot b_{n.m} + c_o \cdot b_{n.m+1} = a_{n.m+1} \\ \vdots \\ c_n \cdot b_{n.m} + \dots + c_o \cdot b_{n.m+n} = a_{n.m+n} \end{cases}$$

are a solution of (1a).

The irreducible form $\frac{1}{Q_\star} \cdot P_\star$ of $(\frac{1}{Q} \cdot P)(x) = \frac{1}{(\sum_{j=n.m}^{n.m+m} b_j x^j)} \cdot (\sum_{i=n.m}^{n.m+n} a_i x^i)$, such that $Q_\star(o) = 1$, is the irreducible form $\frac{1}{Q_\star} \cdot P_\star$ of $(\sum_{i=o}^{n} a_{i+n.m} \, x^i)/(\sum_{j=o}^{m} b_{j+n.m} \, x^j)$, such that $Q_\star(o) = 1$.

b) When we calculate the abstract Padé-approximant of order (n,o) we find the n^{th} partial sum of the abstract Taylor series.
For if $B_{n.m} = I$ then $A_i x^i = C_i x^i$, $i = o,\dots,n$ is a solution of system (1a).
This result has also been found in the classical theory.

c) To find equivalent formulations of the problem of Padé-approximating, we consider a couple of abstract polynomials (P,Q) satisfying definition 2.2. We then know that $(F.Q-P)(x) = O(x^{n.m+n+m+1})$.
The systems (1a) and (1b) are completely equivalent with :

$$(F.Q-P)^{(i)}(0)\ x^i = 0 \quad \forall x \in X \quad \text{and } i = o,\dots,n.m+n+m,$$

because clearly $(F.Q-P)^{(i)}(0) \equiv 0 \in L(X^i,Y)$ for $i = o,\dots,n.m-1$ and $(F.Q-P)^{(i)}(0)\ x^i = 0$ $\forall x \in X$, $i = n.m,\dots,n.m+n$ is system (1a) and $(F.Q-P)^{(i)}(0)\ x^i = 0$ $\forall x \in X$, $i = n.m+n+1,\dots,n.m+n+m$ is precisely system (1b).

d) If $X = R^p$ and $Y = R$ $(\Lambda = R)$, then F is a real-valued function of p real variables. Now $L(X^i,Y)$ is isomorphic with R^{p^i}. Consequently for $(P(x),Q(x))$ satisfying definition 2.2 the operator $(\frac{1}{Q}.P)(x)$ has the following form :

$$\frac{\sum_{j_1+\dots+j_p=n.m}^{n.m+n} \alpha_{j_1\dots j_p} x_1^{j_1}\dots x_p^{j_p}}{\sum_{j_1+\dots+j_p=n.m}^{n.m+m} \beta_{j_1\dots j_p} x_1^{j_1}\dots x_p^{j_p}}$$

This form agrees with the form proposed by J. Karlsson and H. Wallin :

$$\frac{\sum_{j_1+\dots+j_p=o}^{n} \alpha_{j_1\dots j_p} x_1^{j_1}\dots x_p^{j_p}}{\sum_{j_1+\dots+j_p=o}^{m} \beta_{j_1\dots j_p} x_1^{j_1}\dots x_p^{j_p}}$$

if $n=o$ or $m=o$ (III).

Let $p=2$.

To calculate the abstract Padé-approximant we have to calculate the $(n.m+1)+\dots+(n.m+n+1)+(n.m+1)+\dots+(n.m+m+1)$ real coefficients $\alpha_{j_1\dots j_p}$ and $\beta_{j_1\dots j_p}$.

Now $(n.m+1)+\dots+(n.m+n+1)+(n.m+1)+\dots+(n.m+m+1) = n.m.(n+m+2)+\frac{(n+1)(n+2)}{2}+$ $+\frac{(m+1)(m+2)}{2}$.

The formulation in c) supplies us an amount of conditions on the derivatives of $(F.Q-P)$:

in all $\sum_{i=n.m}^{n.m+n+m} \binom{p+i-1}{i}$ conditions.

For $p=2$ these are $(n.m+1)+\dots+(n.m+n+m+1)$ conditions.

If we use the extra condition of definition 4.2, we have in all $n.m.(n+m+1)+$ $+\frac{(n+m+1)(n+m+2)}{2}+1$ conditions, just enough to calculate the $\alpha_{j_1j_2}$ and $\beta_{j_1j_2}$.

The extra condition is : o-linear term in $Q_\star(x)$ is I.

e) If $X = R^p$ and $Y = R^q$ $(\Lambda = R)$, then F is a system of q real-valued functions in p real variables.

Now L(X,Y) is isomorphic with R^{qxp} and $L(X^k,Y)$ isomorphic with R^{qxp^k} while an element of R^{qxp^k} is represented by a row of p^{k-1} matrices (blocks), each containing q rows and p columns;

$L = (c_{i_1 \ldots i_{k+1}}) \in L(X^k,Y) \Rightarrow$ i_1 is the row-index in the block

$i_2 \ldots i_k$ is the number of the block (the most right index grows the fastest)

i_{k+1} is the column-index in the block.

So $L = (c_{i_1 1 \ldots 11 i_{k+1}} | c_{i_1 1 \ldots 12 i_{k+1}} | \ldots | c_{i_1 1 \ldots 1p i_{k+1}} | c_{i_1 1 \ldots 121 i_{k+1}} | \ldots | c_{i_1 p \ldots p i_{k+1}})$

The abstract polynomials (P(x),Q(x)) satisfying definition 2.2 now have for each of the q components the form of the abstract polynomials of p real variables mentioned in d).

f) When we would try, in order to calculate the (n,m)-APA, to find a couple of abstract polynomials $(A_n x^n + \ldots + A_o, B_m x^m + \ldots + B_o)$ such that :

$$F(x).(B_m x^m + \ldots + B_o) - (A_n x^n + \ldots + A_o) = O(x^{n+m+1}) \qquad (2)$$

instead of $(A_{n.m+n} x^{n.m+n} + \ldots + A_{n.m} x^{n.m},\ B_{n.m+m} x^{n.m+m} + \ldots + B_{n.m} x^{n.m})$ such that:

$$F(x).(B_{n.m+m} x^{n.m+m} + \ldots + B_{n.m} x^{n.m}) - (A_{n.m+n} x^{n.m+n} + \ldots + A_{n.m} x^{n.m}) = O(x^{n.m+n+m+1}) \qquad (3)$$

we would remark that this problem is not always solvable (except with $Q \equiv 0 \equiv P$).

Consider again the example $F\binom{\xi}{\eta} = \binom{\xi + \sin(\xi\eta) + 1}{\xi^2 + \eta^2 - 4\xi\eta} = \binom{1}{0} + \binom{\xi}{0} + \binom{\xi\eta}{\xi^2 + \eta^2 - 4\xi\eta} + \ldots$

and take n=1 and m=2.

The system
$$\begin{cases} C_0.B_0 = A_0 \\ C_1x.B_0+C_0.B_1x = A_1x \\ C_2x^2.B_0+C_1x.B_1x + C_0.B_2x^2=0 \\ C_3x^3.B_0+C_2x^2.B_1x + C_1x_0B_2x^2 = 0 \end{cases}$$
$\forall x \in X$, has only the solution $Q \equiv 0 \equiv P$, and thus is not solvable such that $\frac{1}{Q}.P$ is somewhere defined (for n = 1, m=3 this is the case for the first component of the solution),

while (3) is very well solvable, but the solution (P,Q) is such that the irreducible form of $(\frac{1}{Q}.P)(x)$ is undefined in $\binom{0}{0}$.

So via (3) we find an abstract rational operator $(\frac{1}{Q}.P)(x) = \begin{pmatrix} \frac{\xi-\eta+\xi^2-2\xi\eta}{\xi-\eta-\xi\eta+\xi\eta^2} \\ 0 \end{pmatrix}$ that is useful in points in the vicinity of $\binom{0}{0}$.

In other words : (2) does not provide us any solution at all (except $Q \equiv 0 \equiv P$)
(3) does provide an ARA but no APA .

What's more : the situation cannot occur where (2) supplies us the (n,m)-APA while (3) does not, because for every solution (P,Q) of the systems resulting from (2) such that $Q_\star(0)=I$ and for every $L \in L(X^{n.m},Y)$:

$$\begin{cases} (L.P,L.Q) \text{ is a solution of (1a) and (1b)} \\ \frac{1}{Q_\star}.P_\star \text{ is the (n,m)-APA} \end{cases}$$

And we have to look for an irreducible form anyhow.

6. COVARIANCE-PROPERTIES OF ABSTRACT PADE-APPROXIMANTS

The first property we are going to prove is the reciprocal covariance of abstract Padé-approximants.

Theorem 6.1. :

Suppose F(0) is regular in Y and F is continuous in 0 and $\frac{1}{Q}.P$ is the abstract Padé-approximant of order (n,m) for F, then $\frac{1}{P}.Q$ is the abstract Padé-approximant of order (m,n) for $\frac{1}{F}$.

Proof : Since $\{y\in Y \mid y \text{ is regular}\}$ is an open set in Y, there exists $B(F(0),r_2)$ with $r_2>o$ such that $\forall y\in B(F(0),r_2)$: y is regular. Since F is continuous in 0, there exists $B(0,r_1)$ with $r_1>o$ such that $\forall x\in B(0,r_1)$: F(x) is regular. So $\frac{1}{F}$ is defined in $B(0,r_1)$. We speak about $\frac{1}{Q}.P$ and $\frac{1}{P}.Q$ too only on the set of points on which those operators are defined.

$P(0)=C_0=F(0)$ is regular $\Rightarrow \exists B(0,r) : \forall x\in B(0,r) : P(x)$ is regular.

So $\frac{1}{P}$ exists in B(0,r).

Let $n'=\partial P$ and $m'=\partial Q$.

$\exists s\in N$, $o\leq s\leq \min(n-n',m-m')$, $\exists$polynomial $T(x)=\sum_{k=n.m}^{n.m+s} T_k x^k$, $D(T)\neq\phi \ni$

$(P_1(x)=P(x).T(x),\ Q_1(x)=Q(x).T(x))$ satisfies definition 2.2 for F.

$[(F.Q-P).T](x)=(F.Q_1-P_1)(x)=0(x^{n.m+n+m+1})$

$\Rightarrow (\frac{1}{F}.P_1-Q_1)(x)=0(x^{n.m+n+m+1})$ since $\frac{1}{F}(0)=C_0^{-1}\neq 0$ in the abstract Taylor series for $\frac{1}{F}$.

So $\exists s\in N$, $o\leq s\leq \min(n-n',m-m')$, $\exists$ polynomial $T(x)=\sum_{k=n.m}^{n.m+s} T_k x^k$, $D(T)\neq\phi \ni$

$(Q_1(x)=Q(x).T(x),\ P_1(x)=P(x).T(x))$ satisfies definition 2.2 for $\frac{1}{F}$.

The irreducible form of $\frac{1}{P_1}.Q_1$ is $\frac{1}{P}.Q$ $(D(P_1)\neq\phi$ or $D(Q_1)\neq\phi)$.

If we want the o-linear term in the denominator to be I, then

$\frac{1}{(P(x).C_0^{-1})}.(Q(x).C_0^{-1})$ is the abstract Padé-approximant of order (m,n) for $\frac{1}{F}$.

Theorem 6.2. : Suppose $a,b,c,d \in Y$, $c.F(0)+d$ is regular in Y, $a.d-b.c$ is regular in Y, $\frac{1}{Q}.P$ is the (n,n)-APA for F and $D(c.P+d.Q) \neq \phi$ or $D(a.P+b.Q) \neq \phi$, then $\frac{1}{(c.\frac{1}{Q}.P+d)}.(a.\frac{1}{Q}.P+b)$ is the (n,n)-APA for $\frac{1}{(c.F+d)}.(a.F+b)$.

Proof : $c.F(0)+d$ is regular $\Rightarrow c.(\frac{1}{Q}.P)(0)+d$ is regular since $F(0) = (\frac{1}{Q}.P)(0)$.

So $\exists B(0,r)$: $\frac{1}{Q}$ is defined in $B(0,r)$

$\frac{1}{(c.\frac{1}{Q}.P+d)}.(a.\frac{1}{Q}.P+b)$ is defined in $B(0,r)$

$\frac{1}{(c.F+d)}.(a.F+b)$ is defined in $B(0,r)$.

Let $n' = \partial P$ and $n'' = \partial Q$.

$\exists s \in N : o \leqslant s \leqslant \min(n-n',n-n'')$, $\exists$ polynomial $T(x) = \sum_{k=n^2}^{n^2+s} T_k x^k$, $D(T) \neq \phi \ni$

$(P_1(x) = P(x).T(x),\ Q_1(x) = Q(x).T(x))$ satisfies definition 2.2 for F.

In other words : $[(F.Q-P).T](x) = (F.Q_1-P_1)(x) = 0(x^{n^2+2n+1})$.

Now where $\frac{1}{(c.\frac{1}{Q}.P+d)}.(a.\frac{1}{Q}.P+b)$ is defined :

$$\frac{1}{(c.\frac{1}{Q}.P+d)}.(a.\frac{1}{Q}.P+b) = \frac{1}{\frac{1}{Q}.(c.P+d.Q)}.(a.P+b.Q).\frac{1}{Q} = \frac{1}{c.P+d.Q}.(a.P+b.Q).$$

Also $(c.P+d.Q)(0) = c.F(0)+d$ is regular in $B(0,r)$.

$$\begin{cases} \partial(a.P+b.Q) \leqslant \max(\partial P,\partial Q) \text{ and } \partial[(a.P+b.Q).T] \leqslant n^2+n \\ \partial(c.P+d.Q) \leqslant \max(\partial P,\partial Q) \text{ and } \partial[(c.P+d.Q).T] \leqslant n^2+n \end{cases}$$

Since $(F.Q_1-P_1)(x) = 0(x^{n^2+2n+1})$ and $c.F(0)+d$ is regular,

$$[(a.d-b.c).\frac{1}{c.F+d}.(F.Q_1-P_1)](x) = 0(x^{n^2+2n+1}).$$

Now $\frac{1}{(c.F+d)} \cdot (a.F+b).(c.P+d.Q).T - (a.P+b.Q).T =$

$\frac{1}{(c.F+d)} \cdot T.(F.Q-P).(a.d-b.c) = (a.d-b.c).\frac{1}{c.F+d} \cdot (F.Q_1-P_1)$

and $[(a.d-b.c)\frac{1}{c.F+d} \cdot (F.Q_1-P_1)](x) = O(x^{n^2+2n+1})$.

We now search the irreducible form of $\frac{1}{(c.P+d.Q).T} \cdot (a.P+b.Q).T$.

It is $\frac{1}{c.P+d.Q} \cdot (a.P+b.Q)$, for if $\frac{1}{c.P+d.Q} \cdot (a.P+b.Q)$ were reducible :

$$\begin{cases} a.P+b.Q = U.V & \text{with } U,V,W \text{ abstract polynomials} \\ c.P+d.Q = U.W & \text{and } \partial U \geq 1 \end{cases}$$

then : $\begin{cases} (a.d-b.c).P = d.U.V-b.U.W \\ (a.d-b.c).Q = a.U.W-c.U.V \end{cases}$

and so $\frac{1}{Q} \cdot P$ were reducible.

If we want the o-linear term in the denominator to be I,

$\frac{1}{(c.P+d.Q).e} \cdot (a.P+b.Q).e$, with $e = (c.P(0)+d.Q(0))^{-1} = (c.C_0+d)^{-1}$, is the (n,n)-APA for $\frac{1}{(c.F+d)} \cdot (a.F+b)$.

We have to remark that if $\frac{1}{Q} \cdot P$ were the (n,m)-APA for F with $n > m$ for instance, then a.P+b.Q was indeed an abstract polynomial of degree n but c.P+d.Q not necessarily an abstract polynomial of degree m. This clarifies the condition in theorem 6.2 that $\frac{1}{Q} \cdot P$ is the (n,n)-APA for F.

Another property we can prove is the scale-covariance of abstract Padé-approximants.

Theorem 6.3. :

> Let $\lambda \in \Lambda$, $\lambda \neq o$, $y = \lambda x$ and $\frac{1}{Q} . P$ be the (n,m)-APA for F.
> If $S(x) := Q(\lambda x)$, $R(x) := P(\lambda x)$, $G(x) := F(\lambda x)$, then
> $\frac{1}{S} . R$ is the (n,m)-APA for G.

Proof : We remark that if $L \in L(X^i, Y)$, then $\forall \mu \in \Lambda : \mu L \in L(X^i, Y)$.

Because $\frac{1}{Q} . P$ is the (n,m)-APA for F, $\exists s$, $o \leq s \leq \min(n-n', m-m')$,

$\exists$ polynomial $T(x) = \sum_{k=n.m}^{n.m+s} T_k x^k$, $D(T) \neq \phi \ni [(F.Q-P).T](x) = O(x^{n.m+n+m+1})$.

Thus $[(F.Q-P).T]\ (\lambda x) = O(x^{n.m+n+m+1})$.

Now $[(F.Q-P).T](\lambda x) = (G(x).S(x)-R(x)).U(x)$ with $U(x) := T(\lambda x)$ and so

$[(G.S-R).U](x) = O(x^{n.m+n+m+1})$.

We can prove that
$$\begin{cases} D(P) = \lambda . D(R) = \{\lambda x \mid R(x) \text{ regular in } Y\} \\ D(Q) = \lambda . D(S) \\ D(T) = \lambda . D(U) \end{cases}$$

So $D(S.U) \neq \phi$ or $D(R.U) \neq \phi$.

The irreducible form of $\frac{1}{S.U} . (R.U)$ is $\frac{1}{S} . R$ and $S(0) = Q(0) = I$, what finally proves the theorem.

7. THE ABSTRACT PADE-TABLE

Let $R_{n,m}$ denote the (n,m)-APA for F if it is not undefined. The $R_{n,m}$ can be ordered for different values of n and m in a table :

$R_{o,o}$	$R_{o,1}$	$R_{o,2}$	...
$R_{1,o}$	$R_{1,1}$	$R_{1,2}$	...
$R_{2,o}$	$R_{2,1}$	$R_{2,2}$	...
$R_{3,o}$	$\vdots$		
$\vdots$			

Gaps can occur in this Padé-table because of undefined elements. An important property of the table is the next one : the abstract Padé-table consists of squares of equal elements (if one element of the square is defined, all the elements are).

We explicitly restrict ourselves now to spaces $X \underset{\neq}{\supset} \{o\}$ (and $Y \supseteq \{0,I\}$ of course).

Thus $\exists x \in X : x \neq 0$ and $\forall \lambda \in \Lambda : \lambda . I \in Y$.

Lemma 7.1 :

$$\forall n \in \mathbb{N}, \quad \exists D_n \in L(X^n,Y), \quad \exists (x_1,\ldots,x_n) \in X^n : \quad D_n x_1\, x_2 \ldots x_n = I$$

Proof : The reader must be familiar with the well-known functional analysis theorem of Hahn-Banach (Rudin W., Functional Analysis, Mc Graw-Hill, New York, 1973, pp. 57).

Let $n = 1$.

Take $x_0 \in X$, $x_0 \neq 0$ and define the linear functional (V pp.34)

$f : M = \{\lambda\ x_0 | \lambda \in \Lambda\} \rightarrow \Lambda : \lambda x_0 \rightarrow \lambda$.

Now $|f(\lambda\ x_0)| = |\lambda| = \dfrac{\|\lambda\ x_0\|}{\|x_0\|}$.

Define the norm $p(x) = \dfrac{\|x\|}{\|x_0\|}$ on X. Thus $|f(x)| \leq p(x)\ \forall x \in M$.

This linear functional f can be extended to a linear functional $\tilde{f} : X \rightarrow \Lambda$ such that $\tilde{f}(x) = f(x)\ \forall x \in M$ and $|\tilde{f}(x)| \leq p(x) \forall x \in X$.

We now define $D_1 : X \rightarrow Y : x \rightarrow \tilde{f}(x).I$.

Clearly $D_1 \in L(X,Y)$ and $D_1\ x_0 = I$ since $\tilde{f}(x_0) = f(x_0) = 1$.

If $D_{n-1} \in L(X^{n-1},Y),(x_1,\ldots,x_{n-1}) \in X^{n-1} \ni D_{n-1}\ x_1 \ldots x_{n-1} = I$,

then we can define for $x \in X : D_n x = \tilde{f}(x).D_{n-1} \in L(X^{n-1},Y)$.

Then $D_n \in L(X^n,Y)$ and $D_n\ x_0 x_1 \ldots x_{n-1} = \tilde{f}(x_0).D_{n-1}\ x_1 \ldots x_{n-1} = I$.

This lemma implies that $\forall n \in \mathbb{N}, \exists D_n \in L(X^n,Y) : D(D_n) \neq \phi$. We shall use this result in the proofs of the following theorems.

Theorem 7.1. : Let $\frac{1}{Q}.P = R_{n,m}$ be the abstract Padé-approximant of order (n,m) for F.

Then : a) $(F.Q-P)(x) = \sum_{i=o}^{\infty} J_i x^{n'+m'+t+i+1}$

with $J_i \in L(X^{n'+m'+t+i+1}, Y)$,

$t \geqslant o$ and $J_o \neq 0$

b) $\begin{cases} n \leqslant n'+t \\ m \leqslant m'+t \end{cases}$

c) $\forall k,l \succ \begin{cases} n' \leqslant k \leqslant n'+t : R_{k,l} = R_{n,m} \\ m' \leqslant l \leqslant m'+t \end{cases}$

Proof : a) Suppose $(F.Q-P)(x) = O(x^j)$ with $j < n'+m'+1$.

Then $\forall r \succ o \leqslant r \leqslant \min(n-n',m-m') : j+r+n.m < n.m+n+m+1$

This is in contradiction with theorem 4.3.

b) Suppose $n > n'+t$ or $m > m'+t$.

Then $\forall r \in \mathbb{N}$, $o \leqslant r \leqslant \min(n-n',m-m')$, $\forall T_{n.m+r} \in L(X^{n.m+r},Y)$, $D(T_{n.m+r}) \neq \phi$,

we know that $(F.Q.T_{n.m+r} - P.T_{n.m+r})(x)$ is not $O(x^{n.m+n+m+1})$ since

$(F.Q-P)(x) = \sum_{i=o}^{\infty} J_i x^{n'+m'+t+i+1}$ with $J_o \neq 0$ and $n.m+n'+m'+t+r+1 < n.m+n+m+1$.

This is in contradiction with theorem 4.3.

c) Let $\begin{cases} s = \min(k-n', l-m'), \ D_s \in L(X^{k.l+s},Y), \ D(D_s) \neq \phi \\ P_1 = P.D_s \qquad \partial P_1 \leqslant k.l+k \\ Q_1 = Q.D_s \qquad \partial Q_1 \leqslant k.l+l \end{cases}$

$(F.Q_1-P_1)(x) = O(x^{n'+m'+1+t+s+k.l})$ because of a).

Now for $k \leqslant n'+t$ and $l \leqslant m'+t : k.l+k+l+1 \leqslant k.l+n'+m'+t+s+1$.

So $(F.Q_1-P_1)(x) = O(x^{k.l+k+l+1})$ and $D(P_1) \neq \phi$ or $D(Q_1) \neq \phi$.

Definition 7.1. : The (n,m)-APA for F is called normal if it occurs only once in the abstract Padé-table.

The abstract Padé-table is called normal if each of its elements is normal.

Theorem 7.2. :

> The (n,m)-APA $R_{n,m} = \frac{1}{Q} . P$ for F is normal if and only if :
>
> a) $\partial P = n$ and $\partial Q = m$
>
> and
>
> b) $(F.Q-P)(x) = \sum_{i=o}^{\infty} J_i x^{n+m+1+i}$
>
> with $J_i \in L(X^{n+m+1+i},Y)$ and $J_o \neq 0$

Proof : $\Rightarrow$

We proof it by contraposition.

Let $n' = \partial P < n$ or $m' = \partial Q < m$.

According to theorem 7.1 a): $(F.Q-P)(x) = 0(x^{n'+m'+1})$ at least.

Then $R_{n',m'} = \frac{1}{Q} . P$ (irreducible and satisfying $Q(0)=I$) since for $D \in L(X^{n'.m'},Y)$, $D(D) \neq \phi$:

$[(F.Q-P).D](x) = 0(x^{n'.m'+n'+m'+1})$ and $\partial(P.D) = n'.m'+n'$ and $\partial(Q.D) = n'.m'+m'$.

$R_{n',m'} = R_{n,m}$ contradicts the normality of $R_{n,m}$.

If b) is not valid, then according to theorem 7.1 a) :

$(F.Q-P)(x) = 0(x^{n+m+1+t})$ with $t > o$ (for $t = o$ b) would be valid).

This implies that $\forall k,l : n \leq k \leq n+t$ and $m \leq l \leq m+t$:

$R_{k,l} = R_{n,m}$ and thus contradicts the normality of $R_{n,m}$.

$\Leftarrow$

The proof goes again by contraposition.

Suppose $R_{k,l} = R_{n,m}$ for k,l such that $k > n$ or $l > m$.

Now b) implies : $(F.Q-P)(x) = O(x^{n+m+1})$.

If $s \in N$, $D_s \in L(X^{k.l+s}, Y)$, $D(D_s) \neq \phi \Rightarrow (F.Q.D_s - P.D_s)(x) = O(x^{k.l+k+l+1})$,

then $k.l+k+l+1 \leq n+m+1+k.l+s$ and thus $s > k-n$ or $s > l-m$.

This is a contradiction with theorem 4.3.

8. INTERPOLATING OPERATORS

Theorem 7.1 a) and 7.2 b) allow us to write down the following conclusions.

If $\frac{1}{Q}.P$ is the (n,m)-APA for F then $(F.Q-P)(x) = O(x^{n'+m'+1+t})$ with $t \geq o$.

This implies $(F - \frac{1}{Q}.P)(x) = O(x^{n'+m'+1+t})$ with $t \geq o$, since $Q(0) = I$ is regular.

In other words : $(F - \frac{1}{Q}.P)^{(i)}(0) \equiv 0 \in L(X^i, Y)$ for $i = o, \ldots, n'+m'+t$.

Thus : for $R_{n,m} = \frac{1}{Q}.P$: $F^{(i)}(0) = (\frac{1}{Q}.P)^{(i)}(0)$ $i = o, \ldots, n'+m'+t$ with $t \geq o$.

What is more, if $R_{n,m}$ is normal then $n' = n$, $m' = m$ and $(F.Q-P)(x) = O(x^{n+m+1})$.

Thus : for $R_{n,m} = \frac{1}{Q}.P$ normal : $F^{(i)}(0) = (\frac{1}{Q}.P)^{(i)}(0)$ $i = o, \ldots, n+m$.

This also agrees with the classical theory of Padé-approximants.

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