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A practical error formula for multivariate rational interpolation and approximation

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Abstract We consider exact and approximate multivariate interpolation of a function $f(x_1, \ldots, x_d)$ by a rational function $p_{n,m}/q_{n,m}(x_1, \ldots, x_d)$ and develop an error formula for the difference $f - p_{n,m}/q_{n,m}$. The similarity with a well-known univariate formula for the error in rational interpolation is striking. Exact interpolation is through point values for f and approximate interpolation is through intervals bounding f. The latter allows for some measurement error on the function values, which is controlled and limited by the nature of the interval data. To achieve this result we make use of an error formula obtained for multivariate polynomial interpolation, which we first present in a more general form. The practical usefulness of the error formula in multivariate rational interpolation is illustrated by means of a 4-dimensional example, which is only one of the several problems we tested it on.

1 The univariate case

Rational interpolation is well-known as a technique to deliver a smoother approximation than polynomial interpolation, which may generate a highly

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oscillating approximant with rapidly changing slopes. The problem statement is the following.

Let $s+1$ values f_i be given at datapoints x_i with $i = 0, \dots, s$. We want to compute an irreducible rational function $p_{n,m}(x)/q_{n,m}(x)$ of degree n in the numerator and degree m in the denominator, with $n + m = s$, that satisfies

$$\frac{p_{n,m}}{q_{n,m}}(x_i) = \frac{\sum_{j=0}^{n} a_j x_i^j}{\sum_{j=0}^{m} b_j x_i^j} = f_i, \qquad i = 0, \dots, s. \tag{1}$$

The problem as posed in (1) is nonlinear in the unknown coefficients a_j and b_j. In addition, it does not always have a solution [4, 8]. Very often, the nonlinear problem (1) is therefore replaced by its linearized version

$$f_i q_{n,m}(x_i) - p_{n,m}(x_i) = 0, \qquad i = 0, \dots, s. \tag{2}$$

While (2) always has a nontrivial solution for the coefficients a_j and b_j, the rational function $p_{n,m}(x)/q_{n,m}(x)$ may be reducible and its irreducible form may not satisfy all interpolation conditions of (1) anymore [8]. However, in the sequel of this paper we content ourselves with the polynomials $p_{n,m}(x)$ and $q_{n,m}(x)$ from (2).

When replacing the function values f_i by intervals $[f_i^{\vee}, f_i^{\wedge}]$ and under the assumption that $q_{n,m}(x_i) > 0$, the linear equations (2) are transformed into the linear inequalities

$$f_i^{\vee} q_{n,m}(x_i) - p_{n,m}(x_i) \leq 0$$
$$f_i^{\wedge} q_{n,m}(x_i) - p_{n,m}(x_i) \geq 0$$

and the polynomials $p_{n,m}(x)$ and $q_{n,m}(x)$ are determined from a quadratic programming problem [9]. In that case we denote by f_i the value in the interval $[f_i^{\vee}, f_i^{\wedge}]$ that $p_{n,m}/q_{n,m}$ takes at the point x_i.

One way to view (2) is that the denominator polynomial $q_{n,m}(x)$ is determined such that it is possible for the numerator polynomial $p_{n,m}(x)$ to interpolate the function $f q_{n,m}$ at $s+1 = n+m+1$ points, although it is only of degree n. Clearly (2) implies that for $n+1 \leq k \leq n+m$ the divided differences $(f q_{n,m} - p_{n,m})[x_0, \dots, x_k] = (f q_{n,m})[x_0, \dots, x_k] = 0$. This allows us to obtain an easy error formula for the rational interpolant from the error formula known for polynomial interpolation [1]:

$$f(x) - \frac{p_{n,m}}{q_{n,m}}(x) = \frac{(f q_{n,m})^{(s+1)}(y_x)}{(s+1)! q_{n,m}(x)} \prod_{i=0}^{s} (x - x_i), \qquad x, x_i \in [a, b]$$

or

$$\left| f(x) - \frac{p_{n,m}}{q_{n,m}}(x) \right| \leq \frac{\sup_{y \in [a,b]} \left| (f q_{n,m})^{(s+1)}(y) \right|}{(s+1)! \left| q_{n,m}(x) \right|} \prod_{i=0}^{s} |x - x_i|, \qquad x, x_i \in [a, b].$$
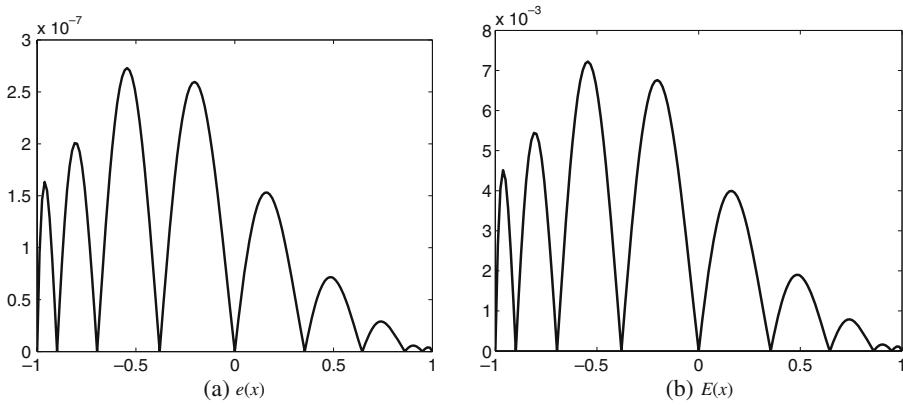
Fig. 1 Case $s + 1 = 10$, $f - p_{5,4}/q_{5,4}$ (point data)

We show that a similar approach in the multivariate case also leads to a useful error formula.

A practical way to make use of this error formula is to regard $\prod_{i=0}^{s}(x - x_i)/q_{n,m}(x)$ as an estimate of the behaviour of the function $(f - p_{n,m}/q_{n,m})(x)$. It cannot be used as an estimate of the magnitude because the factor $(fq_{n,m})^{(s+1)}(y_x)/(s + 1)!$ is neglected. The following example clearly illustrates that the difference between the two expressions can be twofold. First of all, it is easy to see that $\sup_{x \in [a,b]} \left|(f - p_{n,m}/q_{n,m})(x)\right|$ and $\sup_{x \in [a,b]} \left|\prod_{i=0}^{s}(x - x_i)/q_{n,m}(x)\right|$ can be quite different. Second, also pointwise the magnitude of the two expressions can vary because of the neglection of $(fq_{n,m})^{(s+1)}(y_x)$ which depends on x. Take for instance $f(x) = \cos(\pi x/2)$ on the interval $[-1, 1]$ and interpolate it using 10 and 11 points with $n = 5, m = 4$ and $n = 6, m = 4$ respectively. Now let $e(x) = \left|(f - p_{n,m}/q_{n,m})(x)\right|$
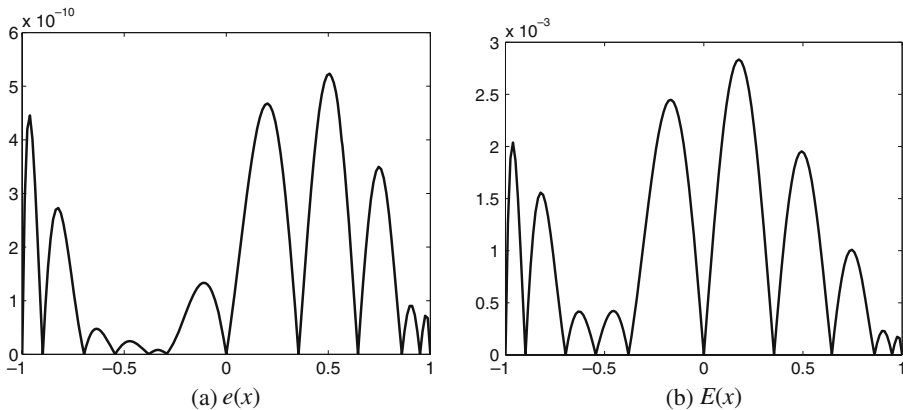

Fig. 2 Case $s + 1 = 11$, $f - p_{6,4}/q_{6,4}$ (point data)

and $E(x) = \left| \prod_{i=0}^{s}(x - x_i)/q_{n,m}(x) \right|$. The interpolation points can be recognized as the zeroes of $e(x)$. Then for $s + 1 = 10$ the function $(fq_{n,m})^{(s+1)}(y_x)$ is almost constant, whereas for $s + 1 = 11$ it is not. But in both cases $E(x)$ contains valuable information (see Figs. 1 and 2).

In the last section we demonstrate the use of our new multivariate error formula on a 4-dimensional rational interpolation problem from computational finance. Similar results were found when using it for the 2- and 3-dimensional rational interpolation problems from electromagnetics in [3].

2 Multivariate polynomial error formula

A nice survey of multivariate polynomial interpolation is given in [5]. For our purpose we need some results derived in [11]. Let P_n^d denote the subspace of polynomials in d variables of total degree at most n. The tuple of variables is abbreviated as $v = (x_1, \ldots, x_d)$. Let us denote the length of a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ by $|\alpha| = \alpha_1 + \ldots + \alpha_d$ and the exact total degree of a multivariate polynomial p by ∂p. Using multi-index notation a polynomial in P_n^d is represented as

$$p(v) = p(x_1, \ldots, x_d) = \sum_{|\alpha|=0}^{n} a_\alpha v^\alpha = \sum_{|\alpha|=0}^{n} a_{\alpha_1, \ldots, \alpha_d} x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \qquad \partial p \leq n.$$

We have at our disposal $s + 1$ distinct interpolation points in $\mathbb{R}^d$ denoted by $v_0, \ldots, v_s$ with $\dim P_n^d \leq s + 1 < \dim P_{n+1}^d$. At the point v_i the function value $f_i = f(v_i) = f(x_1^{(i)}, \ldots, x_d^{(i)})$ is given. We decide upon an ordering $\prec$ of the multi-indices (e.g. when $d = 2$ along upward sloping diagonals in $\mathbb{N}^2$: $(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \ldots$) and re-index the interpolation points using multi-indices (e.g. when $s = 5$ and $d = 2$ we re-index $v_0, v_1, \ldots, v_5$ as $v_{0,0}, v_{1,0}, \ldots, v_{0,2}$). Note that every way of indexing puts the interpolation points in a certain order and may influence the formulas to come. Let us fill the sets $V_k := \{v_\alpha : |\alpha| \leq k\}$ and $A_k := \{\alpha : |\alpha| \leq k\}$ till all available interpolation points and their respective multi-indices are used (e.g. for $s = 5$ and $d = 2$ we have $A_2 = \{(0, 0), \ldots, (0, 2)\}$ and $V_2 = \{v_{0,0}, \ldots, v_{0,2}\}$). Here $A_{-1} = \emptyset = V_{-1}$. When $s + 1 = \dim P_n^d$ then V_n is the last set to contain interpolation points and $\#V_n = s + 1 = \#A_n$. When $s + 1 > \dim P_n^d$ then V_{n+1} and A_{n+1} are not filled up to their full capacity. Again $\#V_{n+1} = s + 1 = \#A_{n+1}$. In what follows we sometimes refer to the interpolation points with their multi-index $v_{\alpha_1, \ldots, \alpha_d}$ and sometimes just as v_i or $(x_1^{(i)}, \ldots, x_d^{(i)})$. Similarly the i-th multi-index can be referred to as $\alpha^{(i)}$.

The Newton fundamental polynomials $p_\alpha(v)$ are of the form

$$p_\alpha(v) = \sum_{v \prec \alpha} a_v v^v, \qquad v \in \mathbb{N}^d$$

and satisfy

$$p_\alpha(v_\beta) = \delta_{\alpha,\beta}, \qquad \beta \preceq \alpha.$$

In [10] an ordering of the interpolation points v_i is proposed that improves the robustness when computing the polynomials $p_\alpha(v)$.

We define finite difference functions recursively by

$$\lambda_0[v]f := f(v),$$

$$\lambda_{k+1}[V_k, v]f := \lambda_k[V_{k-1}, v]f - \sum_{|\alpha|=k} \lambda_k[V_{k-1}, v_\alpha]f \cdot p_\alpha(v), \qquad 0 \le k \le n.$$

The following two properties of finite difference functions are easy to prove:

- If $f \in P_n^d$ then

$$\lambda_{k+1}[V_k, v]f = 0, \qquad k \ge n.$$

- Finite difference functions satisfy

$$\lambda_k[V_{k-1}, v](f - g) = \lambda_k[V_{k-1}, v]f - \lambda_k[V_{k-1}, v]g.$$

If $s + 1 = \dim P_n^d$ and the interpolation points $v_0, \ldots, v_s$ in V_n admit a unique polynomial interpolant $L_n(v)$ of total degree at most n, satisfying $L_n(v_i) = f_i, i = 0, \ldots, s$, then $L_n(v)$ is given by

$$L_n(v) = \sum_{|\alpha|=0}^{n} \lambda_{|\alpha|} \left[V_{|\alpha|-1}, v_\alpha \right] f \cdot p_\alpha(v)$$

with

$$f(v) - L_n(v) = \lambda_{n+1}[V_n, v]f. \tag{3}$$

More generally, when $s + 1 \ge \dim P_n^d$ and the polynomial interpolant $L_{n+1^\circ}(v)$ of the form

$$L_{n+1^\circ}(v) = \sum_{\alpha \in A_{n+1}} a_\alpha v^\alpha,$$

satisfying $L_{n+1^\circ}(v_i) = f_i, i = 0, \ldots, s$ is unique, then $L_{n+1^\circ}(v)$ is given by

$$L_{n+1^\circ}(v) = \sum_{|\alpha|=0}^{n} \lambda_{|\alpha|} \left[V_{|\alpha|-1}, v_\alpha \right] f \cdot p_\alpha(v) + \sum_{\alpha \in A_{n+1} \setminus A_n} \lambda_{n+1}[V_n, v_\alpha]f \cdot p_\alpha(v)$$

with

$$f(v) - L_{n+1^\circ}(v) = \lambda_{n+1}[V_n, v]f - \sum_{\alpha \in A_{n+1} \setminus A_n} \lambda_{n+1}[V_n, v_\alpha]f \cdot p_\alpha(v). \tag{4}$$

The notation $n + 1^\circ$ signifies that the index set A_{n+1} indexing the terms in the polynomial $L_{n+1^\circ}(v)$ is not filled up completely since $s + 1 < \dim P_{n+1}^d$, whereas A_n indexing the terms in $L_n(v)$ is. When $s + 1 = \dim P_n^d$ then $A_{n+1} \setminus A_n = \emptyset$ and $L_{n+1^\circ}(v)$ reduces to and is denoted by $L_n(v)$.

The error formulas (3) and (4) can be bounded above when we rewrite the finite difference function in integral form. We start with (3) and introduce some more terminology. Let $\gamma(i)$ denote a multi-index of length $|\gamma(i)| = i$. Then we can define a path $v(n) := [v_{\gamma(0)}, v_{\gamma(1)}, \ldots, v_{\gamma(n)}]$ of length n among the interpolation points, when considering the interpolation points with their multi-indices (e.g. when $s = 5$, $d = 2$ then $[v_{0,0}, v_{1,0}, v_{2,0}]$ and $[v_{0,0}, v_{0,1}, v_{1,1}]$ are different paths of length $n = 2$). A path in fact arranges interpolation points in a specific way determined by the multi-index ordering. It connects a point in the set V_{k-1} with a point in the set V_k and this for $k = 1, \ldots, n$. Let us denote by Γ_n the set of all different paths of length n.

The following result is proved in [11]. Let f possess continuous partial derivatives of order up to $n + 1$ with respect to all d variables $(x_1, \ldots, x_d)$. To refer to a general partial derivative of order $n + 1$ with respect to a particular combination of the variables we use the multi-index notation

$$\frac{\partial^{n+1} f}{\partial v^\alpha} := \frac{\partial^{n+1} f}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}, \qquad |\alpha| = n + 1.$$

Let $|| \cdot ||_1$ denote the familiar 1-norm in $\mathbb{R}^d$ and $H(V_n, v)$ denote the convex hull of the points $v_0, \ldots, v_s, v$. For a path $v(n)$ of length n we denote

$$\ell_{v(n)}(v) := ||v_{\gamma(1)} - v_{\gamma(0)}||_1 \cdots ||v_{\gamma(n)} - v_{\gamma(n-1)}||_1 \cdot ||v - v_{\gamma(n)}||_1,$$

$$\pi_{v(n)}(v) := p_{\gamma(n)}(v) p_{\gamma(n-1)}(v_{\gamma(n)}) \cdots p_{\gamma(0)}(v_{\gamma(1)}).$$

Then

$$|\lambda_{n+1}[V_n, v]f| \leq \frac{\max\limits_{|\alpha|=n+1} \sup\limits_{\omega \in H(V_n, v)} \left| \dfrac{\partial^{n+1} f(\omega)}{\partial v^\alpha} \right|}{(n+1)!} \sum\limits_{v(n) \in \Gamma_n} \ell_{v(n)}(v) \left| \pi_{v(n)}(v) \right|. \quad (5)$$

Let us denote $M_n(v) = \max_{\alpha \in A_{n+1} \setminus A_n}(1, |p_\alpha(v)|)$. So when $V_{n+1} \setminus V_n = \emptyset = A_{n+1} \setminus A_n$ we have $M_n(v) = 1$. From (4) we then obtain

$$|f(v) - L_{n+1 \circ}(v)| \leq \sum\limits_{w \in \{v\} \cup V_{n+1} \setminus V_n} |\lambda_{n+1}[V_n, w]f| \, M_n(v)$$

and subsequently

$$|f(v) - L_{n+1 \circ}(v)| \leq \frac{\max\limits_{|\alpha|=n+1} \sup\limits_{\omega \in H(V_{n+1}, v)} \left| \dfrac{\partial^{n+1} f(\omega)}{\partial v^\alpha} \right|}{(n+1)!}$$
$$\times \sum\limits_{w \in \{v\} \cup V_{n+1} \setminus V_n} \sum\limits_{v(n) \in \Gamma_n} \ell_{v(n)}(w) \left| \pi_{v(n)}(w) \right| M_n(v) \quad (6)$$

which coincides with (5) when $V_{n+1} \setminus V_n = \emptyset$.

To summarize, in multivariate polynomial interpolation, we distinguish between two cases: either $s + 1 = \dim P_n^d$ for some n and then the error formula is given by (5), or $\dim P_n^d < s + 1 < \dim P_{n+1}^d$ in which case the more general expression (6) applies. In multivariate rational interpolation we have

to distinguish between three cases, as can be seen from Theorem 1 in the next section.

3 Multivariate rational error formula

We have again $s + 1$ distinct interpolation points $v_0, \ldots, v_s$ at our disposal, without any connection between s and the dimension of some P_n^d. In addition we assume that the ordering $\alpha \prec \beta$ of the multi-indices in $\mathbb{N}^d$ is such that it respects

$$\forall \alpha \in [0, \beta_1] \times \ldots \times [0, \beta_d] \setminus \{\beta\} : \alpha \prec \beta.$$

Again we denote the i-th point in this ordering by $\alpha^{(i)}$ (e.g. when $d = 2$ then $\alpha^{(0)} = (0, 0), \alpha^{(1)} = (1, 0), \alpha^{(2)} = (0, 1), \alpha^{(3)} = (2, 0), \ldots$).

We look for a rational function with numerator $p_{n,m}(v)$ and denominator $q_{n,m}(v)$ of the form

$$p_{n,m}(v) = \sum_{i=0}^{n} a_i v^{\alpha^{(i)}},$$

$$q_{n,m}(v) = \sum_{i=0}^{m} b_i v^{\alpha^{(i)}}$$

with $n + m = s$ and satisfying

$$f_i q_{n,m}(v_i) - p_{n,m}(v_i) = 0, \qquad i = 0, \ldots, s.$$

For more information on multivariate rational interpolation we refer to [2]. Multivariate polynomial interpolation, as treated in the previous section, appears as a special case when $m = 0$. Here we are interested in an expression for

$$(f q_{n,m} - p_{n,m})(v). \tag{7}$$

Using the results from the previous section, the following formula can be proved. The very last case can only occur when $m \neq 0$ and therefore the third error formula is particular for rational interpolation.

Theorem 1 *For $p_{n,m}(v)$ and $q_{n,m}(v)$ satisfying* (7) *we find:*

- *If $s + 1 = \dim P_v^d$ for some v, then with $V_v = \{v_0, \ldots, v_s\}$ we have*

$$(f q_{n,m} - p_{n,m})(v) = \lambda_{v+1}[V_v, v] \left(f q_{n,m} \right).$$

- *If $P_{v-1}^d < s + 1 < \dim P_v^d$ for some v then with $\partial p_{n,m} = v$ we have*

$$(f q_{n,m} - p_{n,m})(v) = \lambda_v [V_{v-1}, v] \left(f q_{n,m} - p_{n,m} \right),$$

and with $\partial p_{n,m} \leq v - 1$ we have

$$(f q_{n,m} - p_{n,m})(v) = \lambda_v [V_{v-1}, v] \left(f q_{n,m} \right).$$

Proof We prove the first case, since the proof is very similar in the other cases. To proceed we arrange the interpolation points in a chain of sets $V_k, k = 0, \ldots, \nu$ (or $\nu - 1$ for the other cases) as in the previous section: $V_0 = \{v_0\} \subset \ldots \subset V_\nu = \{v_0, \ldots, v_s\}$. Because $(fq_{n,m} - p_{n,m})(v_i) = 0$ for $i = 0, \ldots, s$, we find that the finite differences

$$\lambda_{|\alpha|}[V_{|\alpha|-1}, v_\alpha](fq_{n,m} - p_{n,m}) = 0, \qquad |\alpha| = 0, \ldots, \nu.$$

Hence

$$(fq_{n,m} - p_{n,m})(v) = \sum_{|\alpha|=0}^{\nu} \lambda_{|\alpha|}[V_{|\alpha|-1}, v_\alpha](fq_{n,m} - p_{n,m})p_\alpha(v)$$

$$+\lambda_{\nu+1}[V_\nu, v](fq_{n,m} - p_{n,m})$$

$$= \lambda_{\nu+1}[V_\nu, v](fq_{n,m} - p_{n,m}).$$

Since $n + 1 \leq s + 1 = \dim P_\nu^d$ we have $\partial p_{n,m} \leq \nu$. Using the additivity of the finite difference functions and the fact that

$$\lambda_{\nu+1}[V_\nu, v]p_{n,m} = 0$$

we obtain the error formula given in the theorem.   □

The first case is very similar to the polynomial case (3) and reduces to it when $m = 0$.

The second case with $\partial p_{n,m} = \nu$ occurs only when the denominator contains a small number of terms, in other words when m is small. When $m = 0$ it coincides with the polynomial case (4).

The last case mentioned in the theorem, where $s + 1 \neq \dim P_\nu^d$ for some ν and $\partial p_{n,m}$ is not maximal, is the most frequent. Note that in that situation the interpolation points in $\{v_0, \ldots, v_s\} \setminus V_{\nu-1}$ do not directly appear in the expressions $\ell_{\nu(\nu-1)}(v)$ and $\pi_{\nu(\nu-1)}(v)$. They only indirectly appear in the finite difference function $\lambda_\nu[V_{\nu-1}, v]$ through the presence of $q_{n,m}(v)$ of which the coefficients are determined by all points.

When programming the error bounds (5) and (6) we have taken the interpolation points in the order that resulted from the pivoting strategy discussed in [10]. On all the examples that we worked with, much to our surprise, the order of the interpolation points did not have a major impact on the behaviour of the error formula. In several extensive tests, we even compared the order from [10] with all possible permutations of the data points and did not notice any significant differences.

We remark that when the values f_i at the interpolation points v_i are again replaced by intervals $[f_i^\vee, f_i^\wedge]$ as in [9], then the error formula remains valid. It indeed suffices to evaluate $p_{n,m}(v)/q_{n,m}(v)$ satisfying, with $q_{n,m}(v_i) > 0$,

$$f_i^\vee q_{n,m}(v_i) - p_{n,m}(v_i) \leq 0$$

$$f_i^\wedge q_{n,m}(v_i) - p_{n,m}(v_i) \geq 0$$

at the interpolation points v_i to find out through which value $f_i \in [f_i^{\vee}, f_i^{\wedge}]$ the rational function actually passes, before computing the finite difference functions appearing in the error formula. Working with intervals is often more realistic in practical situations because it allows for possible measurement errors. In addition, it allows for $s > n + m$ instead of the classical $s = n + m$.

Each of the error formulas given in the theorem can be bounded accordingly by a bound of the form given in (5) or (6): to the first and third case bound (5) applies and to the second case bound (6).

We now describe an application in which the magnitude of the error bound is irrelevant, but its behaviour as a function of v is. A classical way to estimate an error in interpolation is to compare interpolants of higher and lower degree and look at their differences. In several applications we have found it useful to compute (5) or (6) while skipping the computation of the factor in front of the summations. We illustrate both the situation of exact interpolation through point values and approximate interpolation through intervals.

4 A practical application of the error formula

To illustrate the usefulness of the multivariate rational error formula, we take a look at some 4-dimensional rational interpolation problem. The interpolation points v_i belong to $[0, \ln 200] \times [0, \ln 100] \times [0, 0.6] \times [0, \sqrt{3}]$ and the function values f_i come from the pricing of a European call option following the Black-Scholes model [6]. The details with respect to the data generation are irrelevant. When S, K, ρ, t and σ respectively denote the asset price, strike price, annual interest rate, time to expiry and market volatility, then the variables are $x_1 = \ln S, x_2 = \ln K, x_3 = \rho t$ and $x_4 = \sigma \sqrt{t}$. For the graphical illustrations we substitute the variables $x_1, \ldots, x_4$ by

$$z_1 = (x_1 - x_2 + x_3)/x_4$$
$$z_2 = x_4$$

as in [7], because $f(x_1, \ldots, x_4)$ can easily be rewritten as $g(z_1, z_2)$, and rescale the interval for z_1 and z_2 to $[-1, 1]$. We illustrate the three different
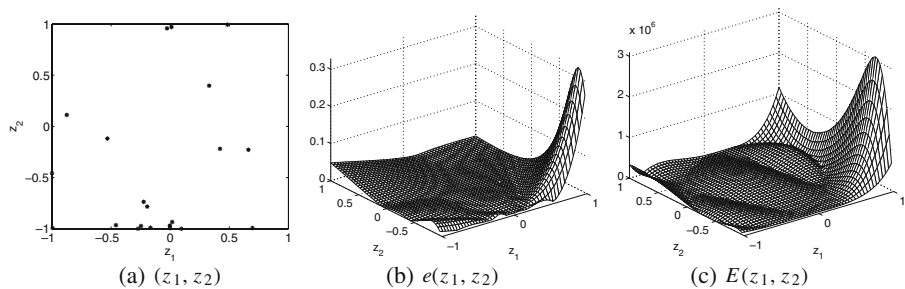

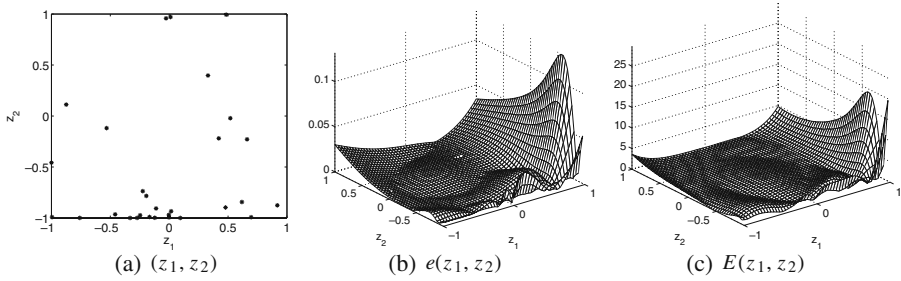**Fig. 3** Case $s + 1 = \dim P_v^d$ with $s = 20, v = 5 : f - p_{7,8}/q_{7,8}$ (interval data)

Fig. 4 Case $P_{\nu-1}^d < s+1 < \dim P_\nu^d$ with $s=23, \nu=6: fq_{11,12} - p_{11,12}$ (point data) with $\partial p_{n,m} = 4$

cases appearing in our theorem. For each case we graph the location of the interpolation points in the (z_1, z_2)-plane, re-express the rational interpolant $p_{n,m}/q_{n,m}$ in the new variables, and then graph the true error $e(z_1, z_2) = \left| g(z_1, z_2) - (p_{n,m}/q_{n,m})(z_1, z_2) \right|$ and the error bound applicable in that case and denoted by $E(z_1, z_2)$. To obtain $E(z_1, z_2)$ we make use of the formula given in Theorem 1 for $(fq_{n,m} - p_{n,m})(z_1, z_2)$, subsequently bound its absolute value by either (5) or (6), disregarding the factor in front of the summation sign, and divide this bound by $|q_{n,m}(z_1, z_2)|$. For our purpose the magnitude of the error is unimportant. It is the behaviour of the error function in the domain of interest that matters. The reason for this is the following. When updating the rational interpolant by imposing more interpolation conditions, additional interpolation points can best be placed where the error is still unsatisfactory. Such information can be obtained from (5) or (6) and in that respect the magnifying factor in front is immaterial, since it is the same allover the domain of interest.

We remark that $s = n + m$ only in the case of point data, meaning exact interpolation. From the graphical illustrations (see Figs. 3, 4, 5 and 6) it is clear that the error bound $E(z_1, z_2)$ can indeed provide a valuable idea of the behaviour of the true error $e(z_1, z_2)$.
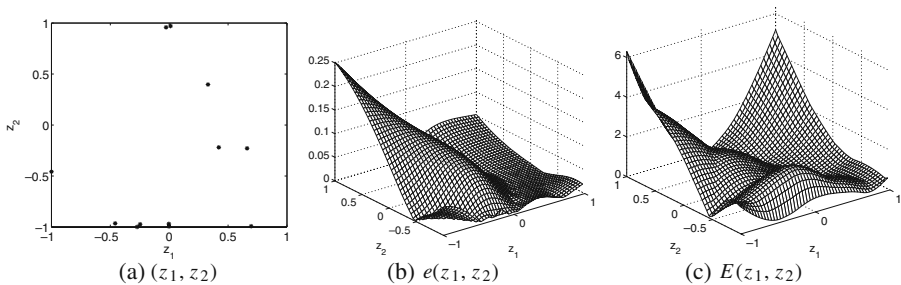

Fig. 5 Case $P_{\nu-1}^d < s+1 < \dim P_\nu^d$ with $s=11, \nu=4: f - p_{6,3}/q_{6,3}$ (interval data) with $\partial p_{n,m} = 3$
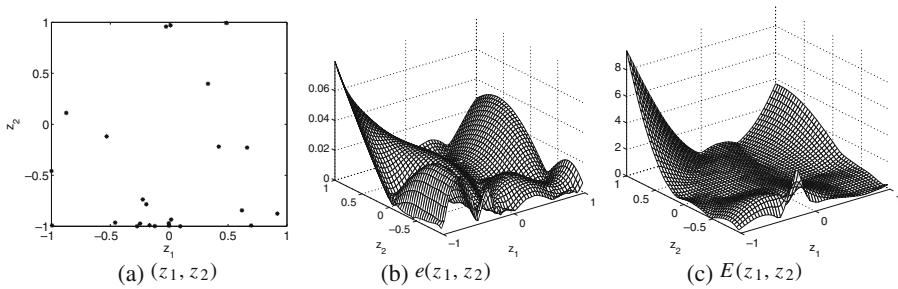
Fig. 6 Case $P_{\nu-1}^d < s+1 < \dim P_\nu^d$ with $s = 19, \nu = 5 : f - p_{15,4}/q_{15,4}$ (point data) with $\partial p_{n,m} = 5$

References

1. Baker, G.A., Jr., Graves-Morris, P.: Padé Approximants, 2nd edn. Cambridge University Press (1996)
2. Becuwe, S., Cuyt, A., Verdonk, B.: Multivariate rational interpolation of scattered data. In: Lirkov, I., Margenov, S., Waśniewski, J., Yalamov, P. (eds.) LNCS, vol. 2907, pp. 204–213 (2004)
3. Cuyt, A., Lenin, R.B., Becuwe, S., Verdonk, B.: Adaptive multivariate rational data fitting with applications in electromagnetics. IEEE Trans. Microwave Theor. Tech. **54**, 2265–2274 (2006)
4. Cuyt, A., Wuytack, L.: Nonlinear Methods in Numerical Analysis. North-Holland, Amsterdam (1987)
5. Gasca, M., Sauer, T.: Polynomial interpolation in several variables. Adv. Comput. Math. **12**(4), 377–410 (2000)
6. Higham, D.J.: An Introduction to Financial Option Valuation: Mathematics, Stochastics, and Computation. Cambridge University Press (2004)
7. Li, M.: Approximate inversion of the Black-Scholes formula using rational functions. Eur. J. Oper. Res. **185**(2) 743–759 (2008)
8. Salazar Celis, O., Cuyt, A., Van Deun, J.: Symbolic and interval rational interpolation: the problem of unattainable data. In: Simos, T.E., Psihoyios, G., Tsitouras, C. (eds.) International Conference on Numerical Analysis and Applied Mathematics. AIP Conference Proceedings, vol. 1048, pp. 466–469 (2008)
9. Salazar Celis, O., Cuyt, A., Verdonk, B.: Rational approximation of vertical segments. Numer. Algorithms **45**, 375–388 (2007)
10. Sauer, T.: Computational aspects of multivariate polynomial interpolation. Adv. Comput. Math. **3**(3), 219–237 (1995)
11. Sauer, T., Xu, Y.: On multivariate Lagrange interpolation. Math. Comput. **64**(211), 1147–1170 (1995)