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On the continuity of the multivariate Padé operator

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Abstract: Continuity of the univariate Padé operator was proved in [5,6]. We discuss the limitations of a multivariate generalization and prove a multivariate analogon of the continuity property.

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1. The multivariate Padé operator

Let $f(x_1,\dots,x_p)$ be a multivariate function given by its Taylor series expansion

$$f(x_1,\dots,x_p)=\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_p=k} c_{k_1\cdots k_p} x_1^{k_1}\cdots x_p^{k_p}.$$

Choose n and m in $\mathbb{N}$ and define multivariate polynomials

$$p(x_1,\dots,x_p)=\sum_{i=nm}^{nm+n}\sum_{i_1+\cdots+i_p=i} a_{i_1\cdots i_p} x_1^{i_1}\cdots x_p^{i_p}$$

and

$$q(x_1,\dots,x_p)=\sum_{j=nm}^{nm+m}\sum_{j_1+\cdots+j_p=j} b_{j_1\cdots j_p} x_1^{j_1}\cdots x_p^{j_p}.$$

In the multivariate Padé approximation problem, defined in [2], we calculate the coefficients $a_{i_1\cdots i_p}$ and $b_{j_1\cdots j_p}$ such that

$$(f\cdot q-p)(x_1,\dots,x_p)=\sum_{k=nm+n+m+1}^{\infty}\sum_{k_1+\cdots+k_p=k} d_{k_1\cdots k_p} x_1^{k_1}\cdots x_p^{k_p}. \tag{1}$$

It is always possible to compute a nontrivial solution of (1) and different solutions supply equivalent rational functions p/q. For more details we refer to [1].

The irreducible form $(p_\star/q_\star)(x_1,\ldots,x_p)$ of $(p/q)(x_1,\ldots,x_p)$ is called the (n, m) multivariate Padé approximant for f and it is unique up to a multiplicative constant.

In [1] we proved the following results for $p_\star/q_\star$. Let $\partial_0 q_\star$ denote the order of the multivariate polynomial $q_\star$, in other words the degree of the first nonzero term in $q_\star(x_1,\ldots,x_p)$; let $\partial p_\star$ and $\partial q_\star$ respectively denote the exact degree of the multivariate polynomials $p_\star$ and $q_\star$. Then it is easy to see that $\partial_0 p_\star \geqslant \partial_0 q_\star$ and that for

$$n' = \partial p_\star - \partial_0 q_\star, \qquad m' = \partial q_\star - \partial_0 q_\star,$$

we have

(a) $n' \leqslant n$,

(b) $m' \leqslant m$,

(c) there exists an s in $\mathbb{N}$, $0 \leqslant s \leqslant \min(n-n', m-m')$, such that for $\bar{s} = nm - \partial_0 q_\star + s$, we can find a nontrivial

$$r(x_1,\ldots,x_p) = \sum_{k_1+\cdots+k_p=\bar{s}} e_{k_1\cdots k_p} x_1^{k_1}\cdots x_p^{k_p} \tag{2}$$

with $\partial_0[(f\cdot q_\star - p_\star)\cdot r] \geqslant nm+n+m+1$,

(d) $\partial_0(f\cdot q_\star - p_\star) = \partial_0 q_\star + n' + m' + t + 1$ with $t \geqslant \max(n-n', m-m')$.

In other words, the polynomials $p_\star$ and $q_\star$ themselves do not necessarily satisfy (1) anymore, but we can multiply them by an appropriate homogenous polynomial r to obtain a solution of (1).

Let us denote $\min(n-n', m-m')$ by $d_{n,m}$ and call it the defect of $(p_\star/q_\star)(x_1,\ldots,x_p)$. This terminology is chosen for $d_{n,m}$ because one can see from (2c) that

$$\partial_0(f\cdot q_\star - p_\star) \geqslant \partial_0 q_\star + n + m + 1 - d_{n,m}.$$

Let $T_{n,m}$ be the operator which associates with $f(x_1,\ldots,x_p)$ the (n, m) multivariate Padé approximant; then $T_{n,m}$ is called the multivariate Padé operator.

2. Continuity

In the Taylor series expansion of $f(x_1,\ldots,x_p)$, the contribution

$$\sum_{k_1+\cdots+k_p=k} c_{k_1\cdots k_p} x_1^{k_1}\cdots x_p^{k_p}$$

is the result of a k-linear operator on $\mathbb{R}^p$ [4, pp. 100–107]. If we call that k-linear operator C_k and write $x = (x_1,\ldots,x_p)$, then we have

$$f(x_1,\ldots,x_p) = \sum_{k=0}^{\infty} C_k x^k, \quad \text{where } C_k x^k = C_k \underbrace{x\cdots x}_{k \text{ times}}.$$

Analogously we can write

$$p(x) = \sum_{i=nm}^{nm+n} A_i x^i, \qquad q(x) = \sum_{j=nm}^{nm+m} B_j x^j.$$

We can now introduce seminorms for the power series f as follows:

$$\|f(x_1,\dots,x_p)\|_l = \max_{0\leqslant k\leqslant l} \|C_k\|,$$

where $\|C_k\| = \max_{\|x\|=1} |C_k x^k|$.

Let $x_0 = (x_{01},\dots,x_{0p})$ be such that $q(x_0) \neq 0$. We do not necessarily have $x_0 = 0$ because $\partial_0 q$ may be strictly positive. Then, because of the continuity of q, there is a finite poly-interval $I_1 \times \cdots \times I_p$ around x_0, where q is nonzero. We call this poly-interval I. Multivariate functions g, continuous on I, are normed by the Chebyshev norm

$$\|g\|_\infty = \max_{x\in I} |g(x_1,\dots,x_p)|.$$

Continuity of the univariate Padé operator was extensively discussed in [6,5]. We shall now attempt to prove a multivariate analogon of those conclusions. But first of all we want to show why we cannot expect a continuity property of this multivariate Padé operator $T_{n,m}$ to hold in the origin $x_0 = 0$.

Consider

$$f(x_1, x_2) = \frac{1}{1-x_1} = 1 + x_1 + x_1^2 + x_1^3 + \cdots$$

and

$$\bar{f}(x_1, x_2) = f(x_1, x_2) + \alpha x_2^2$$
$$= 1 + x_1 + x_1^2 + \alpha x_2^2 + x_1^3 + \cdots.$$

Then

$$\lim_{\alpha\to 0} \|\bar{f} - f\|_{n+m} = 0.$$

In other words, $\bar{f}$ can be chosen as close to f as we want.

Take $n = 1 = m$ and calculate the (n, m) Padé approximants for f and $\bar{f}$. We get

$$\frac{p_\star}{q_\star}(x_1, x_2) = \frac{1}{1-x_1} \quad \text{for } f \quad \text{and} \quad \frac{\bar{p}_\star}{\bar{q}_\star}(x_1, x_2) = \frac{x_1 - \alpha x_2^2}{x_1 - x_1^2 - \alpha x_2^2} \quad \text{for } \bar{f}.$$

In both cases the denominator polynomials $q(x_1, x_2)$ and $\bar{q}(x_1, x_2)$ of all solutions of (1) are zero in the origin because $nm > 0$.

The order $\partial_0 q$ or $\partial_0 \bar{q}$ and what is left of it in $q_\star(x_1, x_2)$ or $\bar{q}_\star(x_1, x_2)$ are responsible for the singularity in the origin and hence for

$$\|T_{n,m} f - T_{n,m}\bar{f}\|_\infty = \infty$$

on every poly-interval I around the origin. Nevertheless, the following continuity property can be proved.

Let f and $\bar{f}$ be multivariate power series and let n and m be fixed.

Theorem 2.1. *If $d_{n,m} = 0$ and $q(x) \neq 0$ for all x in a suitably chosen poly-interval I, then*

$$\forall\epsilon, \exists\delta: \|(f-\bar{f})(x_1,\dots,x_p)\|_{n+m} < \delta \Rightarrow \|T_{n,m}f - T_{n,m}\bar{f}\|_\infty < \epsilon.$$

Proof. If $d_{n,m} = 0$ then $\partial_0(f \cdot q_\star - p_\star) \geq \partial_0 q_\star + n + m + 1$ because of (2c). So for

$$p_\star(x_1, \ldots, x_p) = \sum_{i=0}^{n} A_{\star i} x^{i+\partial_0 q_\star} \quad \text{and} \quad q_\star(x_1, \ldots, x_p) = \sum_{j=0}^{m} B_{\star j} x^{j+\partial_0 q_\star},$$

where $A_{\star i}$ and $B_{\star j}$ are respectively $(i + \partial_0 q_\star)$-linear and $(j + \partial_0 q_\star)$-linear operators on $\mathbb{R}^p$, we have

$$\begin{aligned}
&(C_0 \cdot B_{\star 0})(x) = A_{\star 0}(x), \\
&(C_0 \cdot B_{\star 1} + C_1 \cdot B_{\star 0})(x) = A_{\star 1}(x), \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } x \text{ in } \mathbb{R}^p, \\
&(C_0 \cdot B_{\star n} + \cdots + C_n \cdot B_{\star 0})(x) = A_{\star n}(x), \\
&(C_{n+1} \cdot B_{\star 0} + \cdots + C_{n+1-m} \cdot B_{\star m})(x) = 0, \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } x \text{ in } \mathbb{R}^p, \\
&(C_{n+m} \cdot B_{\star 0} + \cdots + C_n \cdot B_{\star m})(x) = 0,
\end{aligned} \tag{3}$$

where $C_k \cdot B_{\star j}(x) = C_k x^k \cdot B_{\star j} x^{j+\partial_0 q_\star}$.

Let us normalize $q_\star$(x) such that the $B_{\star j}$ are unique. Now (3) has a nontrivial solution for the $B_{\star j}$ $(j = 0, \ldots, m)$, since $B_{\star 0} x^{\partial_0 q_\star} \not\equiv 0$ because of the definition of $\partial_0 q_\star$ and because of the nontriviality of $q(x)$ in the Padé approximation problem (1).

Choose $\bar{x}$ in $\mathbb{R}^p$ such that $B_{\star 0} \bar{x}^{\partial_0 q_\star} \neq 0$ and write $C_k \bar{x}^k = c_k$ and $B_{\star j} \bar{x}^{j+\partial_0 q_\star} = b_j$. Then the homogeneous system

$$\begin{aligned}
&c_{n+1} b_0 + \cdots + c_{n+1-m} b_m = 0, \\
&\vdots \\
&c_{n+m} b_0 + \cdots + c_n b_m = 0,
\end{aligned}$$

has a unique solution for the given $b_0 \neq 0$. So the determinant

$$\begin{vmatrix} c_n & \cdots & c_{n+1-m} \\ \vdots & \ddots & \\ c_{n+m-1} & & c_n \end{vmatrix} \neq 0.$$

In other words,

$$D_{n,m}(C) = \begin{vmatrix} C_n x^n & \cdots & C_{n+1-m} x^{n+1-m} \\ \vdots & \ddots & \\ C_{n+m-1} x^{n+m-1} & & C_n x^n \end{vmatrix}$$

is not identically equal to 0.

If $d_{n,m} = 0$ we also have $\partial_0 q = nm$ and $|A_n x^{nm+n}| + |B_m x^{nm+m}|$ is not identically zero (A_n and B_m contain the terms of degree $nm + n$ and $nm + m$ in $p(x)$ and $q(x)$ respectively). In [3] we showed that if $D_{n,m}(C) \not\equiv 0$, A_n and B_m can be given by

$$A_n x^{nm+n} = \begin{vmatrix} C_n x^n & C_{n-1} x^{n-1} & \cdots & C_{n-m} x^{n-m} \\ C_{n+1} x^{n+1} & C_n x^n & \cdots & C_{n+1-m} x^{n+1-m} \\ \vdots & & \ddots & \vdots \\ C_{n+m} x^{n+m} & & & C_n x^n \end{vmatrix},$$

$$B_m x^{nm+m} = \begin{vmatrix} C_n x^n & \cdots & C_{n+2-m} x^{n+2-m} & -C_{n+1} x^{n+1} \\ & \ddots & \vdots & \\ \vdots & & & \vdots \\ & & C_n x^n & \\ C_{n+m-1} x^{n+m-1} & & C_{n+1} x^{n+1} & -C_{n+m} x^{n+m} \end{vmatrix}.$$

So because $D_{n,m}(C)$, $A_n x^{nm+n}$, $B_m x^{nm+m}$, are determinants and thus continuous functions of the C_k, we have for $\overline{C}_k$ close to C_k

$$D_{n,m}(\overline{C}) \not\equiv 0, \qquad |\overline{A}_n x^{nm+n}| + |\overline{B}_m x^{nm+m}|$$

does not vanish identically, where $\overline{A}_n$ and $\overline{B}_m$ are the result of replacing C_k by $\overline{C}_k$ in the determinants above.

Thus, if we solve the Padé approximation problem for $f(x) = \sum_{k=0}^{\infty} C_k x^k$ and $\bar{f}(x) = \sum_{k=0}^{\infty} \overline{C}_k x^k$ with n and m fixed and $\overline{C}_k$ close to C_k, we get solutions $p_C(x)$, $q_C(x)$ and $p_{\overline{C}}(x)$, $q_{\overline{C}}(x)$ and we get irreducible rational forms $(p_{\star C}/q_{\star C})(x)$ and $(p_{\star \overline{C}}/q_{\star \overline{C}})$. We recall from [3] that if $D_{n,m}(C) \not\equiv 0$, we also have the following determinantal formulas for p_C and q_C:

$$p_C(x) = \begin{vmatrix} \sum_{k=0}^{n} C_k x^k & \cdots & \sum_{k=0}^{n-m} C_k x^k \\ C_{n+1} x^{n+1} & & \\ \vdots & & D_{n,m}(C) \\ C_{n+m} x^{n+m} & & \end{vmatrix},$$

$$q_C(x) = \begin{vmatrix} 1 & \cdots & 1 \\ C_{n+1} x^{n+1} & & \\ \vdots & & D_{n,m}(C) \\ C_{n+m} x^{n+m} & & \end{vmatrix}.$$

The fact that $q_C(x)$ is a continuous function of the C_k implies the existence of a constant δ such that for $\bar{f} = \sum_{k=0}^{\infty} \overline{C}_k x^k$ with $\|f - \bar{f}\|_{n+m} \leqslant \delta$, also $q_{\overline{C}}(x) \neq 0$ for all x in the poly-interval I. Hence

$$\begin{aligned} \left\| \frac{p_{\star C}}{q_{\star C}} - \frac{p_{\star \overline{C}}}{q_{\star \overline{C}}} \right\|_\infty &= \max_{x \in I} \left| \frac{p_{\star C}}{q_{\star C}}(x) - \frac{p_{\star \overline{C}}}{q_{\star \overline{C}}}(x) \right| \\ &= \left\| \frac{p_{\star C}}{q_{\star C}} - \frac{p_{\star \overline{C}}}{q_{\star \overline{C}}} + \frac{p_C}{q_C} - \frac{p_C}{q_C} + \frac{p_{\overline{C}}}{q_{\overline{C}}} - \frac{p_{\overline{C}}}{q_{\overline{C}}} \right\|_\infty \\ &\leqslant \left\| \frac{p_C}{q_C} - \frac{p_{\overline{C}}}{q_{\overline{C}}} \right\|_\infty + \left\| \frac{p_{\star C}}{q_{\star C}} - \frac{p_C}{q_C} \right\|_\infty + \left\| \frac{p_{\star \overline{C}}}{q_{\star \overline{C}}} - \frac{p_{\overline{C}}}{q_{\overline{C}}} \right\|_\infty, \end{aligned}$$

where

$$\left\| \frac{p_{\star\bar{C}}}{q_{\star\bar{C}}} - \frac{p_{\bar{C}}}{q_{\bar{C}}} \right\|_\infty = 0 = \left\| \frac{p_{\star C}}{q_{\star C}} - \frac{p_C}{q_C} \right\|_\infty,$$

because $p_{\star\bar{C}} q_{\bar{C}} = p_{\bar{C}} q_{\star\bar{C}}$ and $p_{\star C} q_C = p_C q_{\star C}$.

Now

$$|p_C(x) - p_{\bar{C}}(x)| \leqslant \sum_{i=0}^{n} \|A_i - \bar{A}_i\| \cdot \|x\|^{nm+i},$$

$$|q_C(x) - q_{\bar{C}}(x)| \leqslant \sum_{j=0}^{m} \|B_j - \bar{B}_j\| \cdot \|x\|^{nm+j}.$$

Since $\|A_i - \bar{A}_i\| \leqslant L \cdot \|f - \bar{f}\|_{n+m}$ and since I is a finite poly-interval, we can write

$$\|p_C - p_{\bar{C}}\| \leqslant M \|f - \bar{f}\|_{n+m}.$$

Analogously,

$$\|q_C - q_{\bar{C}}\| \leqslant M \|f - \bar{f}\|_{n+m}.$$

Since $q_{\bar{C}}(x) \neq 0$ in I, we get a constant E such that

$$|q_{\bar{C}}(x)| > E \quad \text{for all } x \text{ in } I.$$

So

$$\left\| \frac{p_C}{q_C} - \frac{p_{\bar{C}}}{q_{\bar{C}}} \right\|_\infty = \left\| \frac{(p_C - p_{\bar{C}}) q_C + (q_{\bar{C}} - q_C) p_C}{q_C q_{\bar{C}}} \right\|_\infty$$

$$\leqslant \left\| \frac{1}{q_C} \right\|_\infty \cdot \frac{1}{E} (\|q_C\|_\infty + \|p_C\|_\infty) M \|f - \bar{f}\|_{n+m} \leqslant K \cdot \|f - \bar{f}\|_{n+m},$$

and this terminates the proof, for we already had

$$\|T_{n,m} f - T_{n,m} \bar{f}\|_\infty = \left\| \frac{p_{\star C}}{q_{\star C}} - \frac{p_{\star\bar{C}}}{q_{\star\bar{C}}} \right\|_\infty \leqslant \left\| \frac{p_C}{q_C} - \frac{p_{\bar{C}}}{q_{\bar{C}}} \right\|_\infty. \qquad \square$$

We have defined the defect $d_{n,m} = \min(n - n', m - m')$, when $(p_\star / q_\star)$ is the (n, m) multivariate Padé approximant for $f(x_1, \ldots, x_p)$ and $n' = \partial p_\star - \partial_0 p_\star$, $m' = \partial q_\star - \partial_0 q_\star$. Let us now take $\bar{f}(x_1, \ldots, x_p)$ close to $f(x_1, \ldots, x_p)$, i.e. $\|f - \bar{f}\|_{n+m}$ small, and denote the defect for the (n, m) multivariate Padé approximant for $\bar{f}$ by $\bar{d}_{n,m}$. Then we can prove the following property.

Corollary 2.2. *If $d_{n,m} = 0$ for f, then $\bar{d}_{n,m} = 0$ for $\bar{f}$ close to f.*

Proof. If $d_{n,m} = 0$ then $D_{n,m}(C)$ is nontrivial and thus $D_{n,m}(\bar{C})$ is nontrivial for $\bar{C}_k$ close to C_k. The set $D = \{x \mid D_{n,m}(\bar{C})(x) \neq 0\}$ is a dense set in $\mathbb{R}^p$ because $D_{n,m}(\bar{C})$ is a polynomial in x. Take $\bar{x} \in D$. Then the system

$$c_{n+1} b_0 + \cdots + c_{n+1-m} b_m = 0,$$

$$\vdots$$

$$c_{n+m} b_0 + \cdots + c_n b_m = 0,$$

where $c_k = \overline{C}_k \overline{x}^k$ and where $b_0, \ldots, b_m$ are unknown, has for $b_0 = D_{n,m}(\overline{C})(\overline{x})$ a unique solution $b_1, \ldots, b_m$ where b_j is the result of an $(nm+j)$-linear operator evaluated at $\overline{x}^{nm+j}$, because

$$b_j = \begin{vmatrix} c_n & & \boxed{\begin{matrix} -c_{n+1} \\ \vdots \\ -c_{n+m} \end{matrix}} & & c_{n+1-m} \\ & \ddots & & & \\ \vdots & & & & \vdots \\ & & & \ddots & \\ c_{n+m-1} & & & & c_n \end{vmatrix}$$

$\uparrow$ jth column in $D_{n,m}(\overline{C})(\overline{x})$ replaced by this column

$$= \begin{vmatrix} \overline{C}_n \overline{x}^n & & -\overline{C}_{n+1} \overline{x}^{n+1} & & \overline{C}_{n+1-m} \overline{x}^{n+1-m} \\ & \ddots & & & \\ \vdots & & \vdots & & \vdots \\ & & & \ddots & \\ \overline{C}_{n+m-1} \overline{x}^{n+m-1} & & -\overline{C}_{n+m} \overline{x}^{n+m} & & \overline{C}_n \overline{x}^n \end{vmatrix}$$

Let us denote this by $b_j = \overline{B}_{nm+j} \overline{x}^{nm+j}$.

For x in $\mathbb{R}^p \setminus D$, the value $\overline{B}_{nm+j} x^{nm+j}$ can uniquely be defined by continuity because D is dense. So there is only one solution

$$\overline{p}(x_1, \ldots, x_p) = \sum_{i=0}^{n} \overline{A}_{nm+i} x^{nm+i},$$

$$\overline{q}(x_1, \ldots, x_p) = \sum_{j=0}^{m} \overline{B}_{nm+j} x^{nm+j},$$

of the (n, m) Padé approximation problem with $\overline{B}_{nm} = D_{n,m}(\overline{C})$.

Let $\overline{p}_\star$ and $\overline{q}_\star$ be numerator and denominator of the irreducible form of $\overline{p}(x_1, \ldots, x_p)/\overline{q}(x_1, \ldots, x_p)$. Then from the polynomial $u(x_1, \ldots, x_p) = \sum_{k=\partial_0 u}^{\partial u} U_k x^k$ such that

$$\overline{p} = \overline{p}_\star \cdot u, \qquad \overline{q} = \overline{q}_\star \cdot u,$$

we have $\partial_0 u = \partial u = nm - \partial_0 q_\star$, otherwise

$$\overline{p}_\star \cdot U_{nm-\partial_0 q_\star} \quad \text{and} \quad \overline{q}_\star \cdot U_{nm-\partial_0 q_\star}$$

would be another solution of the (n, m) Padé approximation problem with the same term of lowest degree in the denominator as $\overline{p}$ and $\overline{q}$.

So for $\overline{C}_k$ close to C_k we have

$$\partial \overline{p}_\star - \partial_0 q_\star = (\partial \overline{p} - \partial u) - \partial_0 q_\star = \partial \overline{p} - nm,$$

$$\partial \overline{q}_\star - \partial_0 q_\star = (\partial \overline{q} - \partial u) - \partial_0 q_\star = \partial \overline{q} - nm,$$

$$|\overline{A}_{nm+n} x^{nm+n}| + |\overline{B}_{nm+m} x^{nm+m}| \quad \text{nontrivial},$$

and thus

$$\overline{d}_{n,m} = 0. \qquad \square$$

The similarity of these results with the ones obtained for univariate Padé approximants is remarkable.

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