NUMERICAL AND APPLIED MATHEMATICS
C. Brezinski (editor)
J.C. Baltzer AG, Scientific Publishing Co. © IMACS, 1989
pp. 415-429

RATIONAL INTERPOLATION ON GENERAL DATA SETS IN $\mathbb{C}^n$

Annie CUYT* and Brigitte VERDONK
Department of Mathematics and Computer Science, Universiteit Antwerpen (UIA), Universiteitsplein 1, B-2610 Wilrijk, Belgium

Abstract.
A problem is multidimensional when we are working with a k-tuple of univariate functions. A problem is multivariate when a complex-valued function depends on k variables. Generalizations of the concept of rational interpolant to the multivariate case have first been introduced for data lying on rectangular grids [7, 12, 15]. Here we shall extend these ideas to the case of general data sets, not necessarily on a rectangular grid [9]. In this presentation we omit the occurrence of coinciding interpolation points and refer therefore to [7]. When all interpolation points coincide, the problem reduces to a multivariate Padé approximation problem which has already extensively been studied in the past [6].

Our main aim is not to introduce another new approach for the solution of a multivariate rational interpolation problem but to establish a unifying theory that admits to see the wood for the trees again. We want to present to the reader a flowchart for when he or she is facing a multivariate rational interpolation problem. In our framework previously defined multivariate rational interpolation and Padé approximation techniques, introduced by several authors, can be described and completed [8] with a number of new techniques. The next sections mainly discuss the results obtained in [7] and [10]. We also simplify some results published in [9].

1. Multivariate rational interpolation.

Since the more general situation where f is a function of more than two variables is only notationally more difficult, the formulas will be given for bivariate functions even though we will often use the term "multivariate" in the text. Suppose we are given a bivariate function $f(x,y)$ in distinct points $(x_i, y_j) \in \mathbb{C}^2$. The set of data points can also be specified through a subset I of $\mathbb{N}^2$ as follows

$$I = \{(i,j) \in \mathbb{N}^2 |\ f(x_i, y_j) \text{ is given}\}$$

Sometimes the interpolation set I can be structured so that it satisfies the "inclusion property". This means that if a point belongs to the data set, then the rectangular subset of points emanating from the origin with the given point as its furthermost corner also lies in the data set. How this can be achieved in a lot of situations is explained in [11]. If we want to interpolate f in the points (x_i, y_j) by a bivariate rational function, i.e. by a quotient of bivariate polynomials, we must keep in mind that the "degree" of a bivariate polynomial is not uniquely determined. Therefore we introduce the concept of "degree set". The degree set S of a polynomial $s(x,y)$ is a finite subset of $\mathbb{N}^2$, associated with $s(x,y)$ in the following way

$$s(x,y) = \sum_{(i,j)\in S} a_{ij} x^i y^j$$

Special cases for the set S include

$$S = \{(i,j) \in \mathbb{N}^2 |\ 0 \le i \le n\ ,\ 0 \le j \le m\}$$

in which case the polynomial $s(x,y)$ has partial degree n in x and partial degree m in y. For

$$S = \{(i,j) \in \mathbb{N}^2 |\ 0 \le i + j \le n\}$$

the polynomial $s(x,y)$ is said to have homogeneous degree n.

One way to approach the multivariate rational interpolation problem is to specify the degree set N for the "numerator" polynomial and the degree set D for the "denominator" polynomial and to construct

$$p(x,y) = \sum_{(i,j)\in N} a_{ij} x^i y^j \qquad N \text{ from "numerator"}$$
$$\#N = n+1 \tag{1a}$$

and

$$q(x,y) = \sum_{(i,j)\in D} b_{ij} x^i y^j \qquad D \text{ from "denominator"}$$
$$\#D = m+1 \tag{1b}$$

such that

$$(fq-p)(x_k, y_\ell) = 0 \qquad (k,\ell) \in I$$
$$I \text{ from "interpolation conditions"}$$
$$\#I = n+m+1 \tag{1c}$$

It is clear that this problem always has a nontrivial solution for a_{ij} and b_{ij} since it is a homogeneous system of $n+m+1$ linear equations in $n+m+2$ unknowns. Hence at least one unknown can be chosen freely. This problem is called the $[N/D]_I$ rational interpolation problem for $f(x,y)$. We have mentioned that, given some data $f(x_i,y_j)$ of a bivariate function, either the data points (x_i,y_j) can be indexed such that the data set

$$I = \{(i,j) \in \mathbb{N}^2 \mid f(x_i,y_j) \text{ is given}\}$$

satisfies the inclusion property or they cannot be indexed in that way. The degree sets N and D of numerator and denominator can be chosen freely, but we must keep in mind that in order to use algorithms developed in the framework of "degree sets" either N is a subset of I when I satisfies the inclusion property or N and D satisfy the inclusion property when I doesn't. Of course it is always possible, in the former case, not to take advantage of the fact that I satisfies the inclusion property and use the algorithms for the latter case, for instance when one should want to take $N \not\subset I$. In any case it is not allowed for N and D not to satisfy the inclusion property when I doesn't. For more details we refer to the next sections.

When we are dealing with interpolation problems we must specify whether we are interested in an explicit formula for the interpolant or only in its value at some points different from the interpolation points. The former gives rise to a "coefficient problem" while the latter is a "value problem". What's more, different techniques exist for the computation of a rational interpolant: linear defining equations for the coefficient problem, recursive schemes and continued fraction representations for the coefficient and value problem.

In the case of multivariate rational interpolation the unknown numerator and denominator coefficients in the rational function can also be obtained from a linear system of equations. Depending on the structure of the data set the linear system to be solved is given in [11] in case I satisfies the inclusion property, and in [10, 12] in case I doesn't. For more information we refer to section 2 and the flowchart of section 5. When the multivariate rational function is written in continued fraction form then the partial denominators can be obtained using the definition of inverse differences given in [9]. Because the partial numerators have to be recomputed for every evaluation of the rational interpolant we discuss this technique with the value problem. So far for the coefficient problem.

On the other hand, the value of that rational interpolant can be computed recursively by means of the E-algorithm where the starting values depend upon the structure of the data set. Another recursive computation scheme is any forward algorithm for the calculation of a convergent when the multivariate rational function is written in continued fraction form. For I satisfying the inclusion property a generalized qd-algorithm recursively generates the partial numerators and denominators of the multivariate continued fraction [4]. When I does not satisfy the inclusion property a Thiele interpolating continued fraction can be constructed. For more details we refer to the sections 3 and 4 and the flowchart of section 5.

Before following the flowchart to pick the algorithm tailored to your multivariate rational interpolation problem, we must fix an enumeration $r(i,j)$ of the points in $\mathbb{N}^2$ (or $\mathbb{N}^k$). We assume throughout the text that this enumeration is such that

$$r(k,\ell) \le r(i,j) \text{ for all } k \le i \text{ and } \ell \le j \qquad (2)$$

Several numberings $r(i,j)$ for $\mathbb{N}^2$ satisfy this condition as for instance

(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),...

or

(0,0),(1,0),(0,1),(1,1),(2,0),(2,1),(0,2),(1,2),(2,2),...

The index pair that is the r^{th} point in $\mathbb{N}^2$ according to our enumeration is denoted (k_r,ℓ_r) and f_r is short for the function value $f(x_{k_r},y_{\ell_r})$ while f_{ij} is short for $f(x_i,y_j)$. The set of data we can work with is indexed by

$$I = \{(k_0,\ell_0),\ldots,(k_{n+m},\ell_{n+m})\}$$

and always $\#I = \#N + \#D - 1$. The points in the degree set N are called and numbered

$$N = \{(i_0,j_0),\ldots,(i_n,j_n)\}$$

and those in the degree set D are

$$D = \{(d_0,e_0),\ldots,(d_m,e_m)\}.$$

When $N \subset I$ the numbering is such that $(i_0,j_0) = (k_0,\ell_0),\ldots,(i_n,j_n) = (k_n,\ell_n)$. The numbering of the points within each set follows the numbering $r(i,j)$ chosen for the points (i,j) in $\mathbb{N}^2$ in the sense that the next point in the set is the next one in line in the intersection of that set with $\mathbb{N}^2$. The assumption (2) made about the numbering $r(i,j)$ implies the following two facts for a set N satisfying the inclusion property:

1) all the subsets $N_s^{(0)} = \{(i_0,j_0),\ldots,(i_s,j_s)\}$ of N also satisfy the inclusion property
2) the element with lowest rank number in N is the origin and so $(i_0,j_0) = (0,0)$

Hence from now on the numbering is always such that when a set satisfies the inclusion property, its subsets consisting of the first so many points preserve that property.

2. Determinant formulas.

2a. Inclusion property for I.

Consider the following set of basis functions for the real-valued polynomials in two variables

$$B_{ij}(x,y) = \prod_{k=0}^{i-1}(x-x_k)\prod_{\ell=0}^{j-1}(y-y_\ell)$$

instead of the basis functions x^iy^j. Clearly $B_{ij}(x,y)$ is a bivariate polynomial of degree set $([0,i]\times[0,j])\cap \mathbb{N}^2$. Given the $f_{ij}=f(x_i,y_j)$, we can write in a purely formal manner

$$f(x,y) = \sum_{(i,j)\in \mathbb{N}^2} f_{0i,0j}B_{ij}(x,y)$$

where $f_{0i,0j}$ are the bivariate divided differences

$$f_{0i,0j} = f[x_0,\dots,x_i][y_0,\dots,y_j]$$

given by

$$f[x_0,\dots,x_i][y_0,\dots,y_j] = \frac{f[x_1,\dots,x_i][y_0,\dots,y_j]-f[x_0,\dots,x_{i-1}][y_0,\dots,y_j]}{x_i-x_0}$$

or

$$f[x_0,\dots,x_i][y_0,\dots,y_j] = \frac{f[x_0,\dots,x_i][y_1,\dots,y_j]-f[x_0,\dots,x_i][y_0,\dots,y_{j-1}]}{y_j-y_0}$$

with

$$f[x_i][y_j]=f_{ij}.$$

When denoting divided differences, by default

$$f[x_{d_t},\dots,x_{k_u}][y_{e_t},\dots,y_{\ell_u}] = f_{d_tk_u,e_t\ell_u} = 0$$

$$\text{if } d_t>k_u \text{ or } e_t>\ell_u.$$

In order to construct rational interpolants for the given set I we choose two finite index sets N, a subset of I, and D, a subset of $\mathbb{N}^2$ and we put as in [10, 11]

$$p(x,y) = \sum_{(i,j)\in N} a_{ij}B_{ij}(x,y) \qquad (3a)$$

$$q(x,y) = \sum_{(i,j)\in D} b_{ij}B_{ij}(x,y) \qquad (3b)$$

$$(fq-p)(x,y) = \sum_{(i,j)\in \mathbb{N}^2\setminus I} c_{ij}B_{ij}(x,y) \qquad (3c)$$

It is easy to see that (3) implies (1). The fact that (1) implies (3) when I satisfies the inclusion property is proved in [9]. Now (3c) can be rewritten as

$$(fq)_{0i,0j} = p_{0i,0j} = a_{ij}, \qquad (i,j)\in N$$
$$(fq)_{0i,0j} = 0, \qquad (i,j)\in I\setminus N \qquad (4)$$

Let us assume for the sake of simplicity, that the interpolation set I is such that exactly m of the homogeneous equations (4) are linearly independent. Degenerate cases can be avoided by adding interpolation data to the set I until the rank of (4) is equal to m [7] but we omit these cases here. It is obvious that this condition guarantees the existence of a nontrivial solution of (4) which is now a homogeneous system in the $m+1$ unknown denominator coefficients. The polynomials $p(x,y)$ and $q(x,y)$ satisfying (3) are then respectively given by the folowing determinant expressions [10]

$$\begin{vmatrix} \sum_{(i,j)\in N} f_{d_0i,e_0j}B_{ij} & \cdots & \sum_{(i,j)\in N} f_{d_mi,e_mj}B_{ij} \\ f_{d_0k_{n+1},e_0\ell_{n+1}} & \cdots & f_{d_mk_{n+1},e_m\ell_{n+1}} \\ \vdots & & \vdots \\ f_{d_0k_{n+m},e_0\ell_{n+m}} & \cdots & f_{d_mk_{n+m},e_m\ell_{n+m}} \end{vmatrix} \qquad (5a)$$

and

$$\begin{vmatrix} B_{d_0e_0} & \cdots & B_{d_me_m} \\ f_{d_0k_{n+1},e_0\ell_{n+1}} & \cdots & f_{d_mk_{n+1},e_m\ell_{n+1}} \\ \vdots & & \vdots \\ f_{d_0k_{n+m},e_0\ell_{n+m}} & \cdots & f_{d_mk_{n+m},e_m\ell_{n+m}} \end{vmatrix} \qquad (5b)$$

where

$$f_{d_tk_u,e_t\ell_u} = f[x_{d_t},\dots,x_{k_u}][y_{e_t},\dots,y_{\ell_u}]$$

with

$$f_{d_ti_u,e_tj_u} = 0 \quad \text{if} \quad d_t>i_u \quad \text{or} \quad e_t>j_u.$$

In [8] these determinant formulas are given when all the interpolation points coincide and a lot of specific choices for N, D and I are described.

2b. No inclusion property for I.

Using the previous notation for the elements in the sets N, D and I, and assuming that the rank of the linear system (1c) is maximal, a solution $p(x,y)/q(x,y) = [N/D]_I$ of the system of $n+m+1$ homogeneous equations in the $n+m+2$ unknowns a_{ij} and b_{ij} is given by

$$\begin{vmatrix} x^{i_0}y^{j_0} & \dots & x^{i_n}y^{j_n} & 0 & \dots & 0 \\ x_{k_0}^{i_0}y_{\ell_0}^{j_0} & \dots & x_{k_0}^{i_n}y_{\ell_0}^{j_n} & f_0 x_{k_0}^{d_0}y_{\ell_0}^{e_0} & \dots & f_0 x_{k_0}^{d_m}y_{\ell_0}^{e_m} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{k_{n+m}}^{i_0}y_{\ell_{n+m}}^{j_0} & \dots & x_{k_{n+m}}^{i_n}y_{\ell_{n+m}}^{j_n} & f_{n+m} x_{k_{n+m}}^{d_0}y_{\ell_{n+m}}^{e_0} & \dots & f_{n+m} x_{k_{n+m}}^{d_m}y_{\ell_{n+m}}^{e_m} \end{vmatrix}$$
$$\overline{\begin{vmatrix} 0 & \dots & 0 & x^{d_0}y^{e_0} & \dots & x^{d_m}y^{e_m} \\ x_{k_0}^{i_0}y_{\ell_0}^{j_0} & \dots & x_{k_0}^{i_n}y_{\ell_0}^{j_n} & f_0 x_{k_0}^{d_0}y_{\ell_0}^{e_0} & \dots & f_0 x_{k_0}^{d_m}y_{\ell_0}^{e_m} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{k_{n+m}}^{i_0}y_{\ell_{n+m}}^{j_0} & \dots & x_{k_{n+m}}^{i_n}y_{\ell_{n+m}}^{j_n} & f_{n+m} x_{k_{n+m}}^{d_0}y_{\ell_{n+m}}^{e_0} & \dots & f_{n+m} x_{k_{n+m}}^{d_m}y_{\ell_{n+m}}^{e_m} \end{vmatrix}}$$

Since we take N and D satisfying the inclusion property when I doesn't, the subsets $N_s^{(0)}$ also satisfy the inclusion property and it is easy to see that for $r = 0,\dots,n+m$ the expression $(x-x_{k_r})^{i_s}(y-y_{\ell_r})^{j_s}$ is a linear combination of the $(s+1)$ first columns in the determinants above:

$$(x-x_{k_r})^{i_s}(y-y_{\ell_r})^{j_s} = \sum_{v=0}^{i_s}\sum_{w=0}^{j_s}(-1)^{v+w}\binom{i_s}{v}\binom{j_s}{w}x^{i_s-v}y^{j_s-w}x_{k_r}^{v}y_{\ell_r}^{w}$$

Since the sets $D_s^{(0)}$ also satisfy the inclusion property, we can take analogous linear combinations of the columns $n+2$ through $n+m+2$ and, with $(i_0,j_0)=(0,0)=(d_0,e_0)$, rewrite $p(x,y)/q(x,y)$ as

$$\frac{\begin{vmatrix} f_0 & \dots & f_{n+m} \\ (x-x_{k_0})^{i_1}(y-y_{\ell_0})^{j_1} & \dots & (x-x_{k_{n+m}})^{i_1}(y-y_{\ell_{n+m}})^{j_1} \\ \vdots & & \vdots \\ (x-x_{k_0})^{i_n}(y-y_{\ell_0})^{j_n} & \dots & (x-x_{k_{n+m}})^{i_n}(y-y_{\ell_{n+m}})^{j_n} \\ f_0(x-x_{k_0})^{d_1}(y-y_{\ell_0})^{e_1} & \dots & f_{n+m}(x-x_{k_{n+m}})^{d_1}(y-y_{\ell_{n+m}})^{e_1} \\ \vdots & & \vdots \\ f_0(x-x_{k_0})^{d_m}(y-y_{\ell_0})^{e_m} & \dots & f_{n+m}(x-x_{k_{n+m}})^{d_m}(y-y_{\ell_{n+m}})^{e_m} \end{vmatrix}}{\begin{vmatrix} 1 & \dots & 1 \\ (x-x_{k_0})^{i_1}(y-y_{\ell_0})^{j_1} & \dots & (x-x_{k_{n+m}})^{i_1}(y-y_{\ell_{n+m}})^{j_1} \\ \vdots & & \vdots \\ (x-x_{k_0})^{i_n}(y-y_{\ell_0})^{j_n} & \dots & (x-x_{k_{n+m}})^{i_n}(y-y_{\ell_{n+m}})^{j_n} \\ f_0(x-x_{k_0})^{d_1}(y-y_{\ell_0})^{e_1} & \dots & f_{n+m}(x-x_{k_{n+m}})^{d_1}(y-y_{\ell_{n+m}})^{e_1} \\ \vdots & & \vdots \\ f_0(x-x_{k_0})^{d_m}(y-y_{\ell_0})^{e_m} & \dots & f_{n+m}(x-x_{k_{n+m}})^{d_m}(y-y_{\ell_{n+m}})^{e_m} \end{vmatrix}} \tag{6}$$

3. Recurrence relations.

3a. Inclusion property for I.

If we denote $\#N = n+1$ then we can write

$$N = \bigcup_{s=0}^{n} N_s$$

with

$$\begin{aligned} \emptyset = N_{-1} \subset N_0 \subset N_1 \subset \dots \subset N_{n-1} \subset N_n = N \\ \#N_s = s+1 \\ N_s \setminus N_{s-1} = \{(i_s,j_s)\}; \qquad s = 0,\dots,n \\ r(i_s,j_s) > r(i_r,j_r); \qquad s > r. \end{aligned}$$

In other words, for each $s = 0,\dots,n$ we add to N_{s-1} the point (i_s,j_s) which is the next in line in $N \cap \mathbb{N}^2$ according to the enumeration given above. Denote $\#D = m+1$ and proceed in the same way. Then

$$D = \bigcup_{s=0}^{m} D_s$$

with for $s = 0, \ldots, m$

$$D_{-1} = \emptyset, \qquad \#D_s = s+1, \qquad D_s \backslash D_{s-1} = \{(d_s, e_s)\}.$$

The notations N_s and D_s coincide with the previously introduced heavier notations $N_s^{(0)}$ and $D_s^{(0)}$. The formulas (5) can be rewritten so that they can be computed recursively. Multiplying the $(s+1)$th row in $p(x,y)$ and $q(x,y)$ by $B_{k_{n+s}\ell_{n+s}}(x,y)$ $(s = 1, \ldots, m)$, and dividing the $(s+1)$th column by $B_{d_s e_s}(x,y)$ $(s = 0, \ldots, m)$ respectively results in

$$\begin{vmatrix} \sum_{(i,j)\in N} f_{d_0 i, e_0 j} B_{d_0 i, e_0 j} & \cdots & \sum_{(i,j)\in N} f_{d_m i, e_m j} B_{d_m i, e_m j} \\ f_{d_0 k_{n+1}, e_0 \ell_{n+1}} B_{d_0 k_{n+1}, e_0 \ell_{n+1}} & \cdots & f_{d_m k_{n+1}, e_m \ell_{n+1}} B_{d_m k_{n+1}, e_m \ell_{n+1}} \\ \vdots & & \vdots \\ f_{d_0 k_{n+m}, e_0 \ell_{n+m}} B_{d_0 k_{n+m}, e_0 \ell_{n+m}} & \cdots & f_{d_m k_{n+m}, e_m \ell_{n+m}} B_{d_m k_{n+m}, e_m \ell_{n+m}} \end{vmatrix} \quad (7a)$$

$$\begin{vmatrix} 1 & \cdots & 1 \\ f_{d_0 k_{n+1}, e_0 \ell_{n+1}} B_{d_0 k_{n+1}, e_0 \ell_{n+1}} & \cdots & f_{d_m k_{n+1}, e_m \ell_{n+1}} B_{d_m k_{n+1}, e_m \ell_{n+1}} \\ \vdots & & \vdots \\ f_{d_0 k_{n+m}, e_0 \ell_{n+m}} B_{d_0 k_{n+m}, e_0 \ell_{n+m}} & \cdots & f_{d_m k_{n+m}, e_m \ell_{n+m}} B_{d_m k_{n+m}, e_m \ell_{n+m}} \end{vmatrix} \quad (7b)$$

where for $d \le k$ and $e \le \ell$

$$B_{dk,e\ell}(x,y) = \frac{B_{k\ell}(x,y)}{B_{de}(x,y)} = (x - x_d) \ldots (x - x_{k-1})(y - y_e) \ldots (y - y_{\ell-1})$$

and for $d > k$ or $e > \ell$

$$f_{dk,e\ell} = 0.$$

For such a quotient of determinants the E-algorithm is particularly suitable [2, 5]:

$$E_0^{(s)} = \sum_{(i,j)\in N_s} f_{d_0 i, e_0 j} B_{d_0 i, e_0 j}; \quad s = 0, \ldots, n+m$$

$$g_{0,r}^{(s)} = \sum_{(i,j)\in N_s} \left(f_{d_r i, e_r j} B_{d_r i, e_r j} - f_{d_{r-1} i, e_{r-1} j} B_{d_{r-1} i, e_{r-1} j} \right)$$
$$r = 1, \ldots, m; \quad s = 0, \ldots, n+m$$

$$E_r^{(s)} = \frac{E_{r-1}^{(s)} g_{r-1,r}^{(s+1)} - E_{r-1}^{(s+1)} g_{r-1,r}^{(s)}}{g_{r-1,r}^{(s+1)} - g_{r-1,r}^{(s)}}$$
$$s = 0, 1, \ldots, n; \quad r = 1, 2, \ldots, m \quad (8a)$$

$$g_{r,t}^{(s)} = \frac{g_{r-1,t}^{(s)} g_{r-1,r}^{(s+1)} - g_{r-1,t}^{(s+1)} g_{r-1,r}^{(s)}}{g_{r-1,r}^{(s+1)} - g_{r-1,r}^{(s)}}$$
$$s = r+1, r+2, \ldots \quad (8b)$$

The values $E_r^{(s)}$ and $g_{r,t}^{(s)}$ are stored as indicated in the tables 1 and 2.
As a result of these computations

$$[N/D]_I = E_m^{(n)}.$$

Since the solution $p(x,y)/q(x,y)$ of (4) is unique due to fact that the rank of (4) is m, the value $E_m^{(n)}$ itself does not depend upon the numbering of the points within the sets N, D and I. But this numbering affects the interpolation conditions satisfied by the intermediate E-values. For $s = 0, \ldots, n$ and $r = 0, \ldots, m$ [5]

$$E_r^{(s)} = [N_s/D_r]_{N_s \cup \{(k_{s+1}, \ell_{s+1}), \ldots, (k_{s+r}, \ell_{s+r})\}}.$$

3b. No inclusion property for I.

We shall now give a recursive computation scheme for the multivariate rational interpolants (6) under the sole conditions on the sets N, D, and I that N and D satisfy the inclusion property. Remember that the E-algorithm [2] computes a quotient of determinants of the following form

Table 1.

$E_0^{(0)}$			
	$E_1^{(0)}$		
$E_0^{(1)}$		$\ddots$	
	$E_1^{(1)}$	$E_m^{(0)}$	
$E_0^{(2)}$	$\vdots$	$\vdots$	$\ddots$
$\vdots$			$E_{n+m}^{(0)}$
		$E_m^{(n)}$	
	$E_1^{(n+m-1)}$		
$E_0^{(n+m)}$			

Table 2.

$$
\begin{array}{c|cc|c|ccccc|c|cccc}
g_{0,1}^{(0)} & g_{0,2}^{(0)} & & & g_{0,r}^{(0)} & & & & & & g_{0,m}^{(0)} & & & \\
 & & g_{1,2}^{(0)} & & & g_{1,r}^{(0)} & & & & & & \ddots & & \\
g_{0,1}^{(1)} & g_{0,2}^{(1)} & & & g_{0,r}^{(1)} & & \ddots & & & & g_{0,m}^{(1)} & & & \\
 & & g_{1,2}^{(1)} & & & g_{1,r}^{(1)} & & g_{r-1,r}^{(0)} & & & & \vdots & g_{m-1,m}^{(0)} & \\
g_{0,1}^{(2)} & g_{0,2}^{(2)} & \vdots & \cdots & g_{0,r}^{(2)} & \vdots & & \vdots & & \cdots & & & \vdots & \\
\vdots & \vdots & & & \vdots & & & g_{r-1,r}^{(n+m-r+1)} & & & & & g_{m-1,m}^{(n+1)} & \\
 & & & & & & \cdot^{\cdot^{\cdot}} & & & & & & & \\
 & & g_{1,2}^{(n+m-1)} & & & g_{1,r}^{(n+m-1)} & & & & & & \cdot^{\cdot^{\cdot}} & & \\
g_{0,1}^{(n+m)} & g_{0,2}^{(n+m)} & & & g_{0,r}^{(n+m)} & & & & & & g_{0,m}^{(n+m)} & & &
\end{array}
$$

$$E_s^{(r)} = E_s(S_r) = \frac{\begin{vmatrix} S_r & \dots & S_{r+s} \\ g_1(r) & \dots & g_1(r+s) \\ \vdots & & \vdots \\ g_s(r) & \dots & g_s(r+s) \end{vmatrix}}{\begin{vmatrix} 1 & \dots & 1 \\ g_1(r) & \dots & g_1(r+s) \\ \vdots & & \vdots \\ g_s(r) & \dots & g_s(r+s) \end{vmatrix}} \tag{9}$$

From (6) it is clear that $(p/q)(x,y) = [N/D]_I$ is of this form if we define for $r = 0,\dots,n+m$ and $s = 0,\dots,\min(n,m)$

$$\begin{aligned} S_r &= f_r \\ g_{2s-1}(r) &= (x - x_{k_r})^{i_s}(y - y_{\ell_r})^{j_s} \\ g_{2s}(r) &= f_r(x - x_{k_r})^{d_s}(y - y_{\ell_r})^{e_s} \end{aligned} \tag{10a}$$

and for $s = \min(n,m)+1,\dots,\max(n,m)$

$$\begin{aligned} g_{m+s}(r) &= (x - x_{k_r})^{i_s}(y - y_{\ell_r})^{j_s} \qquad \text{if } n \geq m \\ g_{n+s}(r) &= f_r(x - x_{k_r})^{d_s}(y - y_{\ell_r})^{e_s} \qquad \text{if } n < m \end{aligned} \tag{10b}$$

where $(i_s, j_s) \in N$ and $(d_s, e_s) \in D$. Then

$$(p/q)(x,y) = [N/D]_{I_{n+m}^{(0)}} = E_{n+m}^{(0)}$$

We remark that the rows in the numerator and denominator determinants of (6) have been permuted before defining the functions $g_s(r)$. To compute the multivariate rational interpolants recursively, the E-algorithm can now be applied. For $r = 0,\dots,n+m$ we have

$$\begin{aligned} E_0^{(r)} &= f_r \\ g_{0,s}^{(r)} &= g_s(r) \qquad s = 1,\dots,n+m \end{aligned}$$

$$\begin{aligned} E_s^{(r)} &= \frac{g_{s-1,s}^{(r+1)} E_{s-1}^{(r)} - g_{s-1,s}^{(r)} E_{s-1}^{(r+1)}}{g_{s-1,s}^{(r+1)} - g_{s-1,s}^{(r)}} \\ & s = 1,\dots,n+m \\ g_{s,t}^{(r)} &= \frac{g_{s-1,t}^{(r)} g_{s-1,s}^{(r+1)} - g_{s-1,t}^{(r+1)} g_{s-1,s}^{(r)}}{g_{s-1,s}^{(r+1)} - g_{s-1,s}^{(r)}} \\ & t = s+1, s+2, \dots \end{aligned}$$

with the E-values stored as in table 3.

As can be seen from table 3 the starting values for the E-algorithm are the function values in the data points while intermediate E-values $E_s^{(r)}$ are rational functions interpolating on subsets

$$I_s^{(r)} = \{(k_r, \ell_r), \dots, (k_{r+s}, \ell_{r+s})\}$$

Along a column the "degree" of numerator and denominator is constant and completely determined by the functions $g_s(r)$ appearing in the determinant expressions at that stage. Due to the row permutations performed above, advancing along a diagonal in the E-table alternatively increases the numerator and denominator degree sets until one of the sets is exhausted. Definition (10) of the functions $g_s(r)$ enables us to profit from the recursive scheme as much as possible when it has to be restarted for other sets N and D, since many intermediate values can usually be retained. When comparing the recursive technique for a general data set I with the one for a data set I satisfying the inclusion property, we remark that here we have to compute $n+m$ columns of the E-table while in the previous case we only had to compute m columns. The compensation for this phenomenon lies in the fact that this time the E-algorithm is started with the plain function values and the previous time the starting values contained divided differences which in their turn consume a number of recursive computations.

Table 3.

$$f_0 = E_0^{(0)} = [N_0/D_0]_{I_0^{(0)}}$$

$$E_1^{(0)} = [N_1/D_0]_{I_1^{(0)}}$$

$$f_1 = E_0^{(1)} = [N_0/D_0]_{I_0^{(1)}} \qquad \ddots$$

$$E_1^{(1)} = [N_1/D_0]_{I_1^{(1)}}$$

$$f_2 = E_0^{(2)} = [N_0/D_0]_{I_0^{(2)}} \qquad \vdots \qquad E_{n+m}^{(0)} = [N_n/D_m]_{I_{n+m}^{(0)}}$$

$$\vdots \qquad \qquad \cdot^{\cdot^{\cdot}}$$

$$E_1^{(n+m-1)} = [N_1/D_0]_{I_1^{(n+m-1)}}$$

$$f_{n+m} = E_0^{(n+m)} = [N_0/D_0]_{I_0^{(n+m)}}$$

4. Continued fraction representation.

4a. Inclusion property for I.

Let us again suppose for the sake of simplicity that the homogeneous system of equations (4) has maximal rank. As a consequence the set I with $\#I = n+m+1$ is large enough to provide a nondegenerate solution. Hence we can write

$$I = \bigcup_{s=0}^{n+m} I_s$$

with

$$\begin{aligned} I_s &= N_s; & s &= 0,\dots,n \\ I_{n+s}\backslash I_{n+s-1} &= \{(k_{n+s}, \ell_{n+s})\}; & s &= 1,\dots,m \\ r(k_{n+s}, \ell_{n+s}) &> r(k_r, \ell_r); & n+s &> r \geq n+1 \end{aligned}$$

With the subsets N_s, D_r and I_{s+r} rational interpolants

$$[N_s/D_r]_{I_{s+r}}$$

can be constructed which satisfy only part of the interpolation conditions and which are of lower "degree". To this end it is again necessary that the numbering $r(i_r, j_r)$ of the points in $\mathbb{N}^2$ is such that the inclusion property of the set I is carried over to the subsets I_s. We can now fill a table with rational interpolants.

Table 4.

$[N_0/D_0]_{I_0}$	$[N_0/D_1]_{I_1}$	$[N_0/D_2]_{I_2}$	$\dots$
$[N_1/D_0]_{I_1}$	$[N_1/D_1]_{I_2}$	$[N_1/D_2]_{I_3}$	$\dots$
$[N_2/D_0]_{I_2}$	$[N_2/D_1]_{I_3}$	$[N_2/D_2]_{I_4}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	

where

$$[N/D]_I = [N_n/D_m]_{I_{n+m}}.$$

Our aim is to consider descending staircases in this table of multivariate rational functions:

$$\begin{array}{llll} [N_s/D_0]_{I_s} & & & \\ [N_{s+1}/D_0]_{I_{s+1}} & [N_{s+1}/D_1]_{I_{s+2}} & & \\ & [N_{s+2}/D_1]_{I_{s+3}} & [N_{s+2}/D_2]_{I_{s+4}} & \\ & & \vdots & \dots \end{array} \tag{11}$$

and to construct continued fractions of which the rth convergent equals the rth interpolant on the staircase. We restrict ourselves to the case where every three successive elements in (11) are different. In [4] we prove that it is possible to construct a continued fraction of the form

$$[N_s/D_0]_{I_s} + \frac{[N_{s+1}/D_0]_{I_{s+1}} - [N_s/D_0]_{I_s}|}{|1} + \frac{-q_1^{(s+1)}|}{|1+q_1^{(s+1)}} + \frac{-e_1^{(s+1)}|}{|1+e_1^{(s+1)}} + \frac{-q_2^{(s+1)}|}{|1+q_2^{(s+1)}} + \frac{-e_2^{(s+1)}|}{|1+e_2^{(s+1)}} + \dots \tag{12}$$

of which the successive convergents equal the successive elements on the descending staircase (11). Here

$$[N_s/D_0]_{I_s} = \sum_{(i,j)\in N_s} f_{d_0 i, e_0 j} B_{d_0 i, e_0 j}(x, y)$$

$$[N_{s+1}/D_0]_{I_{s+1}} = \sum_{(i,j)\in N_{s+1}} f_{d_0 i, e_0 j} B_{d_0 i, e_0 j}(x,y)$$

and the coefficients $q_r^{(s+1)}$ and $e_r^{(s+1)}$ can be computed using the following rhombus-rules. For $r \geq 2$

$$\frac{q_r^{(s+1)} e_{r-1}^{(s+1)}}{e_{r-1}^{(s+2)} q_{r-1}^{(s+2)}} = \frac{g_{r-2,r-1}^{(s+r)} - g_{r-2,r-1}^{(s+r-1)}}{g_{r-2,r-1}^{(s+r-1)}} \frac{g_{r-1,r}^{(s+r)}}{g_{r-1,r}^{(s+r+1)} - g_{r-1,r}^{(s+r)}} \tag{13}$$

and for $r \geq 1$

$$\frac{e_r^{(s+1)} + 1}{q_r^{(s+2)} + 1} = -\frac{g_{r-1,r}^{(s+r+1)} - g_{r-1,r}^{(s+r)}}{g_{r-1,r}^{(s+r)}} \tag{14}$$

If we arrange the values $q_r^{(s+1)}$ and $e_r^{(s+1)}$ in a table as follows

Table 5.

$$\begin{array}{ccccc}
q_1^{(1)} & & & & \\
 & e_1^{(1)} & & & \\
q_1^{(2)} & & q_2^{(1)} & & \\
 & e_1^{(2)} & & e_2^{(1)} & \\
q_1^{(3)} & & q_2^{(2)} & & \ddots \\
 & e_1^{(3)} & & e_2^{(2)} & \\
q_1^{(4)} & & q_2^{(3)} & & \ddots \\
\vdots & e_1^{(4)} & \vdots & e_2^{(3)} & \\
 & \vdots & & \vdots &
\end{array}$$

where subscripts indicate columns and superscripts indicate downward sloping diagonals, then (13) links the elements in the rhombus

$$\begin{array}{ccc}
 & e_{r-1}^{(s+1)} & \\
q_{r-1}^{(s+2)} & & q_r^{(s+1)} \\
 & e_{r-1}^{(s+2)} &
\end{array}$$

and (14) links two elements on an upward sloping diagonal

$$\begin{array}{cc}
 & e_r^{(s+1)} \\
q_r^{(s+2)} &
\end{array}$$

If starting values for $q_r^{(s+1)}$ were known, all the values in the multivariate qd-table could be computed. These starting values are given by

$$q_1^{(s+1)} = \frac{E_1^{(s+1)} - E_0^{(s+1)}}{E_0^{(s+1)} - E_0^{(s)}} = \frac{-f_{d_0 k_{s+2}, e_0 \ell_{s+2}} B_{d_0 k_{s+2}, e_0 \ell_{s+2}}}{f_{d_0 k_{s+1}, e_0 \ell_{s+1}} B_{d_0 k_{s+1}, e_0 \ell_{s+1}}} \frac{g_{0,1}^{(s+1)}}{g_{0,1}^{(s+2)} - g_{0,1}^{(s+1)}} \tag{15}$$

Since the qd-table given in table 5 needs the help-entries $g_{r,t}^{(s)}$ from table 2 we have baptised the rules (13–15) the qdg-algorithm. This new algorithm coincides with Rutishauser's qd-algorithm for the computation of univariate Padé approximants and with Claessens' generalized qd-algorithm for the computation of univariate rational interpolants [3]. In analogy with the univariate Padé approximation case and the univariate rational interpolation case it is also possible to give explicit determinant formulas for the partial numerators in (12). For these formulas we refer to [7].

4b. No inclusion property for I.

In order to complete the collection of techniques to construct the multivariate rational interpolant $[N/D]_I$ we shall now show that this rational interpolant can also be obtained as the convergent of a Thiele interpolating continued fraction. We therefore first reconsider the univariate formulas. The inverse differences in the univariate Thiele interpolating continued fraction are related to the univariate reciprocal differences by the well-known formulas

$$\begin{aligned}
\varphi[x_j] &= \rho[x_j] \\
\varphi[x_j, x_{j+1}] &= \rho[x_j, x_{j+1}] \\
\varphi[x_j, \ldots, x_{j+k}] &= \rho[x_j, \ldots, x_{j+k}] - \rho[x_j, \ldots, x_{j+k-2}] \\
k &\geq 2
\end{aligned}$$

In [14, p. 111] explicit determinant formulas are given, separately for odd and even numbered reciprocal differences. If we consider the univariate functions $g_s(r)$, $r = 0, \ldots, 2n$ and $s = 0, \ldots, n$ given by

$$\begin{aligned}
S_r &= f_r = f(x_r) \\
g_{2s-1}(r) &= (x - x_r)^s \\
g_{2s}(r) &= f_r (x - x_r)^s
\end{aligned} \tag{16}$$

then these formulas for the reciprocal differences can be joined into

$$\rho_s^{(r)}[x_r, \ldots, x_{r+s}] = (-1)^s \frac{\begin{vmatrix} g_{-1}(r) & \ldots & g_{-1}(r+s) \\ g_0(r) & \ldots & g_0(r+s) \\ g_1(r) & \ldots & g_1(r+s) \\ \vdots & & \vdots \\ g_{s-3}(r) & & \\ g_{s-2}(r) & & \\ g_s(r) & \ldots & g_s(r+s) \end{vmatrix}}{\begin{vmatrix} g_{-1}(r) & \ldots & g_{-1}(r+s) \\ g_0(r) & \ldots & g_0(r+s) \\ g_1(r) & \ldots & g_1(r+s) \\ \vdots & & \vdots \\ g_{s-3}(r) & & \\ g_{s-2}(r) & & \\ g_{s-1}(r) & \ldots & g_{s-1}(r+s) \end{vmatrix}} \tag{17}$$

A definition of multivariate reciprocal differences is now at hand if we take for the functions $g_s(r)$ the bivariate analogon of the univariate functions (16). With respect to this definition we would like however to point out a few things. First it is clear that the bivariate analogon of the functions (16) is not uniquely determined. It is a natural choice to take for $r = 0,1,\dots$ and $s = 0,1,\dots$

$$\begin{aligned} g_{2s-1}(r) &= (x - x_{k_r})^{i_s}(y - y_{\ell_r})^{j_s} \qquad (i_s, j_s) \in N \\ g_{2s}(r) &= f_r(x - x_{k_r})^{d_s}(y - y_{\ell_r})^{e_s} \qquad (d_s, e_s) \in D \end{aligned} \tag{18}$$

since this is a special case of definition (10) in that we require

$$\#N = \#D \qquad \text{or} \qquad \#N = \#D + 1 \tag{19}$$

In the sequel of the text we shall assume that the bivariate functions $g_s(r)$ are given by (18) with the sets N and D satisfying (19). Then definition (10) and (18) coincide. Note that definition (18) and condition (19) for the sets N and D imply that

$$E_s^{(r)} = [N_{\lfloor (s+1)/2 \rfloor}/D_{\lfloor s/2 \rfloor}]_{I_s^{(r)}}$$

or equivalently, that the multivariate rational interpolant is located on the main staircase

$$\begin{array}{lll} [N_0^{(0)}/D_0^{(0)}]_{I_0^{(0)}} & & \\ [N_1^{(0)}/D_0^{(0)}]_{I_1^{(0)}} & [N_1^{(0)}/D_1^{(0)}]_{I_2^{(0)}} & \\ & [N_2^{(0)}/D_1^{(0)}]_{I_3^{(0)}} & \cdots \\ & & \vdots \end{array}$$

in the "table" of multivariate rational interpolants. This is not a restriction. In [9] and at the end of section 5 we show how multivariate rational interpolants located on other staircases in the "table" can also be obtained as convergents of multivariate Thiele interpolating continued fractions. If we introduce the notation

$$\gamma_s = i_{\lfloor \frac{s+1}{2} \rfloor} + j_{\lfloor \frac{s+1}{2} \rfloor} + d_{\lfloor \frac{s}{2} \rfloor} + e_{\lfloor \frac{s}{2} \rfloor}$$

then we can define bivariate reciprocal differences by plugging (18) into (17) and replacing s by γ_s.

$$\rho_s^{(r)}[(x_{k_r}, y_{\ell_r}), \dots, (x_{k_{r+s}}, y_{\ell_{r+s}})] = (-1)^{\gamma_s} \frac{\begin{vmatrix} g_{-1}(r) & \dots & g_{-1}(r+s) \\ g_0(r) & \dots & g_0(r+s) \\ g_1(r) & \dots & g_1(r+s) \\ \vdots & & \vdots \\ g_{s-3}(r) & & \\ g_{s-2}(r) & & \\ g_s(r) & \dots & g_s(r+s) \end{vmatrix}}{\begin{vmatrix} g_{-1}(r) & \dots & g_{-1}(r+s) \\ g_0(r) & \dots & g_0(r+s) \\ g_1(r) & \dots & g_1(r+s) \\ \vdots & & \vdots \\ g_{s-3}(r) & & \\ g_{s-2}(r) & & \\ g_{s-1}(r) & \dots & g_{s-1}(r+s) \end{vmatrix}} \tag{20}$$

Note that this formula is completely analogous to (17) for the univariate case since then $\gamma_s = s$. The reciprocal differences (20) are, as in the univariate case, independent of the order of the points. Also, if we look back at the explicit determinant formula (6) in the special case that $p/q(x,y) = [N_m/D_m]_{I_{2m}^{(0)}}$ then it is easy to see from (20) that

$$\rho[(x_{k_0}, y_{\ell_0}), \dots, (x_{k_{2m}}, y_{\ell_{2m}})] = \frac{\text{coeff of } x^{i_m}y^{j_m}}{\text{coeff of } x^{d_m}y^{e_m}}$$

and analogously if $p/q(x,y) = [N_m/D_{m-1}]_{I_{2m-1}^{(0)}}$ then

$$\rho[(x_{k_0}, y_{\ell_0}), \dots, (x_{k_{2m-1}}, y_{\ell_{2m-1}})] = \frac{\text{coeff of } x^{d_{m-1}}y^{e_{m-1}}}{\text{coeff of } x^{i_m}y^{j_m}}$$

This is completely analogous to the univariate situation. For $[N_m/D_m]_{I_{2m}^{(0)}}$ the powers $x^{i_m}y^{j_m}$ and $x^{d_m}y^{e_m}$ can be considered as the "highest degree" terms in $p(x,y)$ and $q(x,y)$ respectively in the sense that they are the last added and since N and D satisfy the inclusion property all powers x^iy^j with $i \le i_m$ and $j \le j_m$ already occur in $p(x,y)$ and $q(x,y)$. In analogy with the univariate formulas we now define

$$\begin{aligned} \varphi_0^{(r)} &= \varphi[(x_{k_r}, y_{\ell_r})] = f_r \\ \varphi_1^{(r)} &= \varphi[(x_{k_r}, y_{\ell_r}), (x_{k_{r+1}}, y_{\ell_{r+1}})] \\ &= \rho_1^{(r)}[(x_{k_r}, y_{\ell_r}), (x_{k_{r+1}}, y_{\ell_{r+1}})] \\ \varphi_s^{(r)} &= \varphi[(x_{k_r}, y_{\ell_r}), \dots, (x_{k_{r+s}}, y_{\ell_{r+s}})] \\ &= \rho_s^{(r)}[(x_{k_r}, y_{\ell_r}), \dots, (x_{k_{r+s}}, y_{\ell_{r+s}})] - \\ &\qquad \rho_{s-2}^{(r)}[(x_{k_r}, y_{\ell_r}), \dots, (x_{k_{r+s-2}}, y_{\ell_{r+s-2}})] \quad r \ge 2 \end{aligned} \tag{21}$$

Now consider the main staircase of multivariate rational interpolants

$$[N_{\lfloor \frac{s+1}{2} \rfloor}/D_{\lfloor \frac{s}{2} \rfloor}]_{I_s^{(0)}} = E_s^{(0)} \qquad s = 0,1,\dots$$

given previously. If the functions $g_s(r)$ are given by (18) then we proved in [9] that the successive convergents of the continued fraction

$$\varphi[(x_{k_0}, y_{\ell_0})] + \sum_{u=1}^{\infty} \left. \frac{A_u}{\varphi[(x_{k_0}, y_{\ell_0}), \ldots, (x_{k_u}, y_{\ell_u})]} \right| \quad (22)$$

are the multivariate rational interpolants on the "main staircase"

$$\begin{aligned} E_{2r}^{(0)} &= \left[N_r^{(0)}/D_r^{(0)}\right]_{I_{2r}^{(0)}} \\ E_{2r+1}^{(0)} &= \left[N_{r+1}^{(0)}/D_r^{(0)}\right]_{I_{2r+1}^{(0)}} \end{aligned} \qquad r = 0, 1, \ldots$$

if the partial numerators A_u are given by

$$A_1 = \varphi_1^{(0)} \frac{f_0 g_{0,1}^{(1)} - f_1 g_{0,1}^{(0)}}{g_{0,1}^{(1)} - g_{0,1}^{(0)}}$$

$$A_2 = \varphi_2^{(0)} \varphi_1^{(0)} \frac{g_{1,2}^{(0)}}{(-1)^{\gamma_2} \rho_2^{(0)} g_{0,1}^{(0)} - g_{0,2}^{(0)}}$$

$$A_u = \varphi_u^{(0)} \varphi_{u-1}^{(0)} \frac{g_{u-1,u}^{(0)}}{(-1)^{\gamma_u} \rho_u^{(0)} g_{u-2,u-1}^{(0)} - g_{u-2,u}^{(0)}} \left(1 + \frac{g_{u-2,u-1}^{(0)}}{(-1)^{\gamma_{u-1}} \rho_{u-1}^{(0)} g_{u-3,u-2}^{(0)} - g_{u-3,u-1}^{(0)}}\right) \quad u \geq 3$$

When using the continued fraction representation (22) to obtain multivariate rational interpolants, it is necessary to compute the inverse and reciprocal differences and the auxiliary values $g_{s,u}^{(r)}$ and with these values compute the partial numerator and denominator coefficients. The multivariate rational interpolant is then obtained as a convergent of (22). The computation of the values $g_{s,u}^{(r)}$ is already discussed. We shall now give a computation scheme for the bivariate reciprocal differences. The inverse differences can then be constructed from the reciprocal differences according to (21). The reciprocal differences $\rho_s^{(r)}$ satisfy the following recursion [9]:

$$\rho_0^{(r)} = f_r$$

$$\rho_1^{(r)} = (-1)^{\gamma_1} \frac{g_{0,1}^{(r+1)} - g_{0,1}^{(r)}}{f_{r+1} - f_r}$$

$$\rho_2^{(r)} = (-1)^{\gamma_2} \left(\frac{g_{0,2}^{(r)} - g_{1,2}^{(r)}}{g_{0,1}^{(r)}} + \frac{g_{1,2}^{(r)} - g_{1,2}^{(r+1)}}{g_{0,1}^{(r+1)}} \frac{\rho_1^{(r+1)}}{\rho_1^{(r+1)} - \rho_1^{(r)}} \right)$$

$$\rho_s^{(r)} = (-1)^{\gamma_s} \left(\frac{g_{s-2,s}^{(r)} - g_{s-1,s}^{(r)}}{g_{s-2,s-1}^{(r)}} + \frac{g_{s-1,s}^{(r)} - g_{s-1,s}^{(r+1)}}{g_{s-2,s-1}^{(r+1)}} \times \right.$$

$$\left. \frac{\rho_{s-1}^{(r+1)} - (-1)^{\gamma_{s-1}} \frac{g_{s-3,s-1}^{(r+1)} - g_{s-2,s-1}^{(r+1)}}{g_{s-3,s-2}^{(r+1)}}}{\left[\begin{array}{l} \rho_{s-1}^{(r+1)} - (-1)^{\gamma_{s-1}} \frac{g_{s-3,s-1}^{(r+1)} - g_{s-2,s-1}^{(r+1)}}{g_{s-3,s-2}^{(r+1)}} \\ - \rho_{s-1}^{(r)} + (-1)^{\gamma_{s-1}} \frac{g_{s-3,s-1}^{(r)} - g_{s-2,s-1}^{(r)}}{g_{s-3,s-2}^{(r)}} \\ \qquad + \frac{(-1)^{\gamma_{s-1}} g_{s-2,s-1}^{(r)} - g_{s-2,s-1}^{(r+1)}}{g_{s-3,s-2}^{(r+1)}} \end{array} \right]} \right) \quad (s \geq 3)$$

(23)

When we compare this technique with the recursive computation of $[N/D]_I$ in the previous section we remark the following. The continued fraction of which the partial numerators and denominators are given above generates only rational interpolants $E_s^{(0)}$. The recursive scheme also explicitly computes intermediate values $E_s^{(r)}$ with $r \neq 0$. So different algorithms will serve different purposes.

5. Flowchart.

Let us summarize the previous three sections. In the case of multivariate rational interpolation the unknown numerator and denominator coefficients in the rational function can be obtained from a linear system of equations. Depending on the structure of the data set the linear system to be solved is given below by LINSYS_INCLUSION in case I satisfies the inclusion property, and by LINSYS_GENERAL in case I doesn't. For more information we refer to the flowchart.

On the other hand, the value of that rational interpolant can be computed recursively by means of the E-algorithm where the starting values depend upon the structure of the data set. If I satisfies the inclusion property and N is a subset of I then $[N/D]_I = E_m^{(n)}$ with the starting values and recursion given in RECURS_INCLUSION below. If I does not satisfy the inclusion property but N and D do then $[N/D]_I = E_{n+m}^{(0)}$ with the starting values and recursion given in RECURS_GENERAL. In these recursive computation schemes intermediate rational interpolants $[N_s/D_r]_{I_{s+r}^{(s)}}$ are generated. The complete set of these interpolants is given in table 6. The routine RECURS_INCLUSION in each step links three entries in the lower level, for instance those that are boxed, while the routine RECURS_GENERAL in each step links two entries in the lower level and one entry in a higher level, namely those that are circled.

Another recursive computation scheme is any forward algorithm for the calculation of a convergent when the multivariate rational function is written in continued fraction form. For I satisfying the inclusion property a generalized qd-algorithm recursively generates the partial numerators and denominators of the multivariate continued fraction [4]. The details can be found in CONTFR_INCLUSION. When I does not satisfy the inclusion property a Thiele interpolating continued fraction can be constructed of which the partial numerators and denominators are computed recursively as in CONTFR_GENERAL. This algorithm extends the formulas (21–23) below the main descending staircase. The continued fraction (22) is rediscovered by putting $n = m$. So the routines CONTFR_INCLUSION and CONTFR_GENERAL both have as successive convergents the elements on the descending staircase (11) with $s = n - m$ in the lower level of table 6. They can as well be used to generate elements on a staircase in a higher level.

The complete flowchart consists of the figures 1, 2 and 3 where figure 1 is the main figure.

LINSYS_INCLUSION:

$$\begin{cases} a_{i_0 j_0} - \sum_{u=0}^{m} b_{d_u e_u} f_{d_u i_0, e_u j_0} = 0 \\ \quad \vdots \\ a_{i_n j_n} - \sum_{u=0}^{m} b_{d_u e_u} f_{d_u i_n, e_u j_n} = 0 \\ \\ b_{d_0 e_0} f_{d_0 k_{n+1}, e_0 \ell_{n+1}} + \ldots + b_{d_m e_m} f_{d_m k_{n+1}, e_m \ell_{n+1}} = 0 \\ \quad \vdots \\ b_{d_0 e_0} f_{d_0 k_{n+m}, e_0 \ell_{n+m}} + \ldots + b_{d_m e_m} f_{d_m k_{n+m}, e_m \ell_{n+m}} = 0 \end{cases}$$

DEF_GFUNC_INCLUSION:

$$g_s(r) = \sum_{u=0}^{r} f_{d_s k_u, e_s \ell_u} B_{d_s k_u, e_s \ell_u} - \sum_{u=0}^{r} f_{d_{s-1} k_u, e_{s-1} \ell_u} B_{d_{s-1} k_u, e_{s-1} \ell_u}$$
$$s = 1, \ldots, m \qquad r = 0, \ldots, n+m$$

RECURS_INCLUSION:

$$E_0^{(s)} = \sum_{u=0}^{s} f_{d_0 k_u, e_0 \ell_u} B_{d_0 k_u, e_0 \ell_u}$$
$$s = 0, \ldots, n+m$$

$$g_{0,r}^{(s)} = g_r(s) \qquad r = 1, \ldots, m \qquad s = 0, \ldots, n+m$$

$$E_r^{(s)} = \frac{g_{r-1,r}^{(s+1)} E_{r-1}^{(s)} - g_{r-1,r}^{(s)} E_{r-1}^{(s+1)}}{g_{r-1,r}^{(s+1)} - g_{r-1,r}^{(s)}}$$
$$r = 1, \ldots, m \qquad s = 0, \ldots, n+m-r$$

$$g_{r,t}^{(s)} = \frac{g_{r-1,t}^{(s)} g_{r-1,r}^{(s+1)} - g_{r-1,t}^{(s+1)} g_{r-1,r}^{(s)}}{g_{r-1,r}^{(s+1)} - g_{r-1,r}^{(s)}}$$
$$t = r+1, r+2, \ldots$$

CONTFR_INCLUSION:

$n \geq m$:

$$q_1^{(s+1)} = \frac{-f_{d_0 k_{s+2}, e_0 \ell_{s+2}} B_{d_0 k_{s+2}, e_0 \ell_{s+2}}}{f_{d_0 k_{s+1}, e_0 \ell_{s+1}} B_{d_0 k_{s+1}, e_0 \ell_{s+1}}} \frac{g_{0,1}^{(s+1)}}{g_{0,1}^{(s+2)} - g_{0,1}^{(s+1)}}$$
$$s = 0, \ldots, n+m-2$$

$$e_r^{(s+1)} + 1 = -\frac{g_{r-1,r}^{(s+r+1)} - g_{r-1,r}^{(s+r)}}{g_{r-1,r}^{(s+r)}} \left(q_r^{(s+2)} + 1 \right)$$
$$r \geq 1 \qquad s = 0, \ldots, n+m-2r-1$$

$$q_r^{(s+1)} = \frac{e_{r-1}^{(s+2)} q_{r-1}^{(s+2)}}{e_{r-1}^{(s+1)}} \times \frac{g_{r-2,r-1}^{(s+r)} - g_{r-2,r-1}^{(s+r-1)}}{g_{r-2,r-1}^{(s+r-1)}} \frac{g_{r-1,r}^{(s+r)}}{g_{r-1,r}^{(s+r+1)} - g_{r-1,r}^{(s+r)}}$$
$$r \geq 2 \qquad s = 0, \ldots, n+m-2r$$

$$[N/D]_I = \sum_{u=0}^{n-m} f_{i_0 i_u, j_0 j_u} B_{i_0 i_u, j_0 j_u} + \frac{f_{i_0 i_{n-m+1}, j_0 j_{n-m+1}} B_{i_0 i_{n-m+1}, j_0 j_{n-m+1}} \rfloor}{\lfloor 1} + \frac{-q_1^{(n-m+1)} \rfloor}{\lfloor 1 + q_1^{(n-m+1)}} + \frac{-e_1^{(n-m+1)} \rfloor}{\lfloor 1 + e_1^{(n-m+1)}} + \ldots + \frac{-q_m^{(n-m+1)} \rfloor}{\lfloor 1 + q_m^{(n-m+1)}}$$

$n < m$: reciprocal covariance

LINSYS_GENERAL:

$$\begin{cases} f_0 b_{d_0 e_0} + f_0 b_{d_1 e_1} x_{k_0}^{d_1} y_{\ell_0}^{e_1} + \ldots + f_0 b_{d_m e_m} x_{k_0}^{d_m} y_{\ell_0}^{e_m} \\ \quad - a_{i_0 j_0} - a_{i_1 j_1} x_{k_0}^{i_1} y_{\ell_0}^{j_1} - a_{i_n j_n} x_{k_0}^{i_n} y_{\ell_0}^{j_n} = 0 \\ \quad \vdots \\ \\ f_{n+m} b_{d_0 e_0} + f_{n+m} b_{d_1 e_1} x_{k_{n+m}}^{d_1} y_{\ell_{n+m}}^{e_1} + \ldots \\ \quad - a_{i_0 j_0} - a_{i_1 j_1} x_{k_{n+m}}^{i_1} y_{\ell_{n+m}}^{j_1} - \ldots = 0 \end{cases}$$

DEF_GFUNC_GENERAL:

$n \geq m:$

$$
\begin{aligned}
g_s(r) &= (x - x_{k_r})^{i_s}(y - y_{\ell_r})^{j_s} \\
&\qquad s = 1, \ldots, n - m \\
g_{n-m+2s-1}(r) &= (x - x_{k_r})^{i_{n-m+s}}(y - y_{\ell_r})^{j_{n-m+s}} \\
&\qquad s = 1, \ldots, m \\
g_{n-m+2s}(r) &= f_r(x - x_{k_r})^{d_s}(y - y_{\ell_r})^{e_s} \\
&\qquad s = 1, \ldots, m
\end{aligned}
$$

$n < m:$

$$
\begin{aligned}
g_s(r) &= f_r(x - x_{k_r})^{d_s}(y - y_{\ell_r})^{e_s} \\
&\qquad s = 1, \ldots, m - n \\
g_{m-n+2s-1}(r) &= (x - x_{k_r})^{i_s}(y - y_{\ell_r})^{j_s} \\
&\qquad s = 1, \ldots, n \\
g_{m-n+2s}(r) &= f_r(x - x_{k_r})^{d_{m-n+s}}(y - y_{\ell_r})^{e_{m-n+s}} \\
&\qquad s = 1, \ldots, n
\end{aligned}
$$

RECURS_GENERAL:

$$
\begin{aligned}
E_0^{(s)} &= f_s \qquad s = 0, \ldots, n + m \\
g_{0,r}^{(s)} &= g_r(s) \qquad r = 1, \ldots, n + m \qquad s = 0, \ldots, n + m \\
E_r^{(s)} &= \frac{g_{r-1,r}^{(s+1)} E_{r-1}^{(s)} - g_{r-1,r}^{(s)} E_{r-1}^{(s+1)}}{g_{r-1,r}^{(s+1)} - g_{r-1,r}^{(s)}} \\
&\qquad r = 1, \ldots, n + m \qquad s = 0, \ldots, n + m - r \\
g_{r,t}^{(s)} &= \frac{g_{r-1,t}^{(s)} g_{r-1,r}^{(s+1)} - g_{r-1,t}^{(s+1)} g_{r-1,r}^{(s)}}{g_{r-1,r}^{(s+1)} - g_{r-1,r}^{(s)}} \\
&\qquad t = r + 1, r + 2, \ldots
\end{aligned}
$$

CONTFR_GENERAL:

$n \geq m:$

$$
\begin{aligned}
{}_{n-m}\sigma_1^{(s)} &= \frac{g_{0,1}^{(s+1)} - g_{0,1}^{(s)}}{f_{s+1} - f_s} \qquad s = 0, \ldots, n + m - 1 \\
{}_{n-m}\sigma_2^{(s)} &= \frac{g_{1,2}^{(s)} - g_{1,2}^{(s+1)}}{g_{0,1}^{(s+1)}} \, \frac{{}_{n-m}\sigma_1^{(s+1)}}{{}_{n-m}\sigma_1^{(s+1)} - {}_{n-m}\sigma_1^{(s)}} \\
&\qquad s = 0, \ldots, n + m - 2 \\
{}_{n-m}\sigma_r^{(s)} &= \frac{g_{r-1,r}^{(s)} - g_{r-1,r}^{(s+1)}}{g_{r-2,r-1}^{(s+1)}} \times \\
&\qquad \frac{{}_{n-m}\sigma_{r-1}^{(s+1)}}{{}_{n-m}\sigma_{r-1}^{(s+1)} - {}_{n-m}\sigma_{r-1}^{(s)} + \frac{g_{r-2,r-1}^{(s)} - g_{r-2,r-1}^{(s+1)}}{g_{r-3,r-2}^{(s+1)}}} \\
&\qquad r = 3, \ldots, n + m \qquad s = 0, \ldots, n + m - r
\end{aligned}
$$

$$
\begin{aligned}
\gamma(n - m, r) &= i_r + j_r \qquad r = 0, \ldots, n - m - 1 \\
\gamma(n - m, r) &= i_{n-m+\lfloor\frac{r-n+m+1}{2}\rfloor} + j_{n-m+\lfloor\frac{r-n+m+1}{2}\rfloor} + \\
&\qquad d_{\lfloor\frac{r-n+m}{2}\rfloor} + e_{\lfloor\frac{r-n+m}{2}\rfloor} \\
&\qquad r = n - m, \ldots, n + m
\end{aligned}
$$

$$
\begin{aligned}
{}_{n-m}\rho_0^{(s)} &= f_s \qquad s = 0, \ldots, n + m \\
{}_{n-m}\rho_1^{(s)} &= (-1)^{\gamma(n-m,1)} \; {}_{n-m}\sigma_1^{(s)} \\
&\qquad s = 0, \ldots, n + m - 1 \\
{}_{n-m}\rho_r^{(s)} &= (-1)^{\gamma(n-m,r)} \left(\frac{g_{r-2,r}^{(s)} - g_{r-1,r}^{(s)}}{g_{r-2,r-1}^{(s)}} + {}_{n-m}\sigma_r^{(s)} \right) \\
&\qquad r = 2, \ldots, n + m \qquad s = 0, \ldots, n + m - r
\end{aligned}
$$

$$
\begin{aligned}
{}_{n-m}\varphi_0^{(s)} &= {}_{n-m}\rho_0^{(s)} \qquad s = 0, \ldots, n + m \\
{}_{n-m}\varphi_1^{(s)} &= {}_{n-m}\rho_1^{(s)} \qquad s = 0, \ldots, n + m - 1 \\
{}_{n-m}\varphi_r^{(s)} &= {}_{n-m}\rho_r^{(s)} - {}_{n-m}\rho_{r-2}^{(s)} \\
&\qquad r = 2, \ldots, n + m \qquad s = 0, \ldots, n + m - r
\end{aligned}
$$

$$
\begin{aligned}
{}_{n-m}\tau_{n-m+r}^{(0)} = \Big(& g_{n-m+r-1,n-m+r}^{(0)} \; {}_{n-m}\varphi_{n-m+r}^{(0)} / \\
& [(-1)^{\gamma(n-m,n-m+r)} \; {}_{n-m}\rho_{n-m+r}^{(0)} g_{n-m+r-2,n-m+r-1}^{(0)} - \\
& \qquad g_{n-m+r-2,n-m+r}^{(0)}] \Big) \\
& \qquad r = 2, \ldots, 2m
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= {}_{n-m}\varphi_{n-m+1}^{(0)} (E_{n-m+1}^{(0)} - E_{n-m}^{(0)}) \\
A_2 &= {}_{n-m}\tau_{n-m+2}^{(0)} \; {}_{n-m}\varphi_{n-m+1}^{(0)} \\
A_u &= {}_{n-m}\tau_{n-m+u}^{(0)} \left({}_{n-m}\varphi_{n-m+u-1}^{(0)} + {}_{n-m}\tau_{n-m+u-1}^{(0)} \right) \\
&\qquad u = 3, \ldots, 2m
\end{aligned}
$$

$$
[N/D]_I = E_{n-m}^{(0)} + \sum_{u=1}^{2m} \frac{A_u \,\rfloor}{\lfloor\, {}_{n-m}\varphi_{n-m+u}^{(0)}}
$$

For a list of the notations used herein, we refer to the appendix.

Table 6.

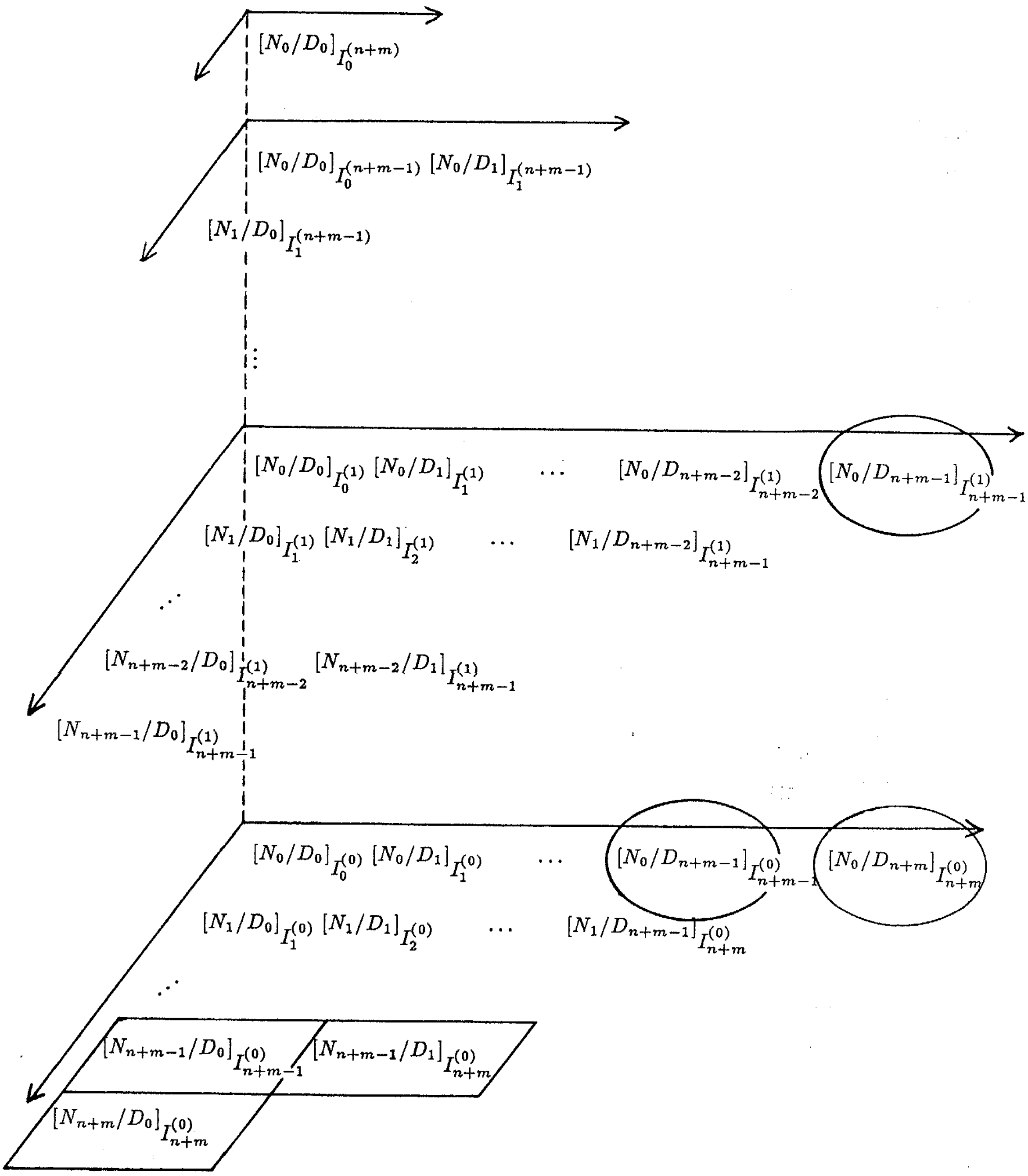

$[N_0/D_0]_{I_0^{(n+m)}}$
$[N_0/D_0]_{I_0^{(n+m-1)}}$ $[N_0/D_1]_{I_1^{(n+m-1)}}$
$[N_1/D_0]_{I_1^{(n+m-1)}}$
$\vdots$
$[N_0/D_0]_{I_0^{(1)}}$ $[N_0/D_1]_{I_1^{(1)}}$ $\cdots$ $[N_0/D_{n+m-2}]_{I_{n+m-2}^{(1)}}$ $[N_0/D_{n+m-1}]_{I_{n+m-1}^{(1)}}$
$[N_1/D_0]_{I_1^{(1)}}$ $[N_1/D_1]_{I_2^{(1)}}$ $\cdots$ $[N_1/D_{n+m-2}]_{I_{n+m-1}^{(1)}}$
$\cdot^{\cdot^{\cdot}}$
$[N_{n+m-2}/D_0]_{I_{n+m-2}^{(1)}}$ $[N_{n+m-2}/D_1]_{I_{n+m-1}^{(1)}}$
$[N_{n+m-1}/D_0]_{I_{n+m-1}^{(1)}}$
$[N_0/D_0]_{I_0^{(0)}}$ $[N_0/D_1]_{I_1^{(0)}}$ $\cdots$ $[N_0/D_{n+m-1}]_{I_{n+m-1}^{(0)}}$ $[N_0/D_{n+m}]_{I_{n+m}^{(0)}}$
$[N_1/D_0]_{I_1^{(0)}}$ $[N_1/D_1]_{I_2^{(0)}}$ $\cdots$ $[N_1/D_{n+m-1}]_{I_{n+m}^{(0)}}$
$\cdot^{\cdot^{\cdot}}$
$[N_{n+m-1}/D_0]_{I_{n+m-1}^{(0)}}$ $[N_{n+m-1}/D_1]_{I_{n+m}^{(0)}}$
$[N_{n+m}/D_0]_{I_{n+m}^{(0)}}$

Figure 1.

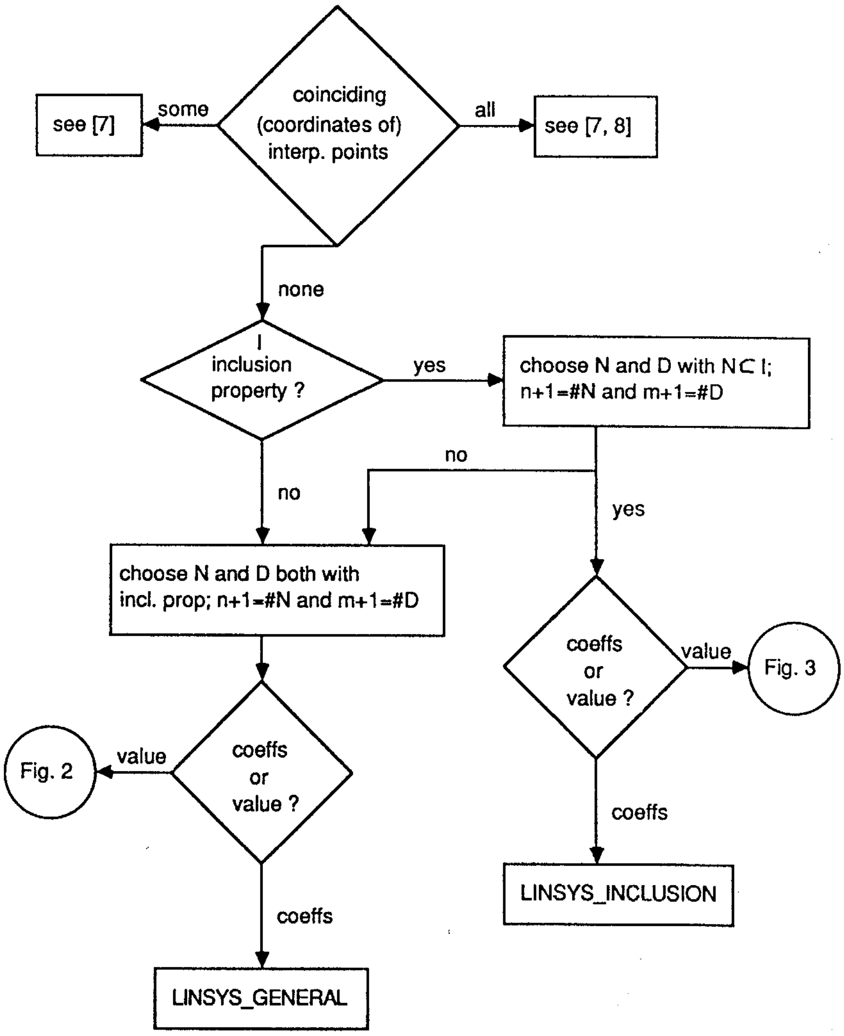


Figure 2.

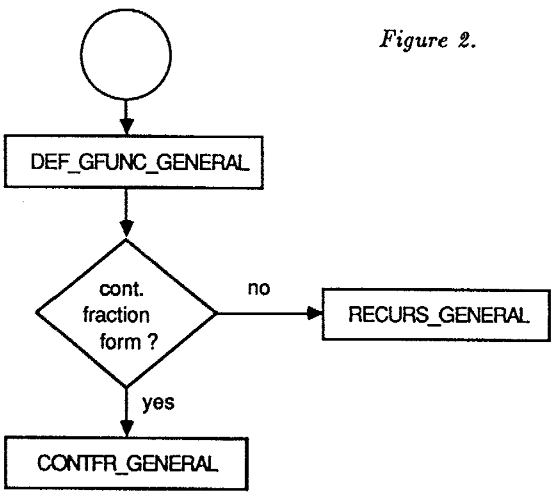
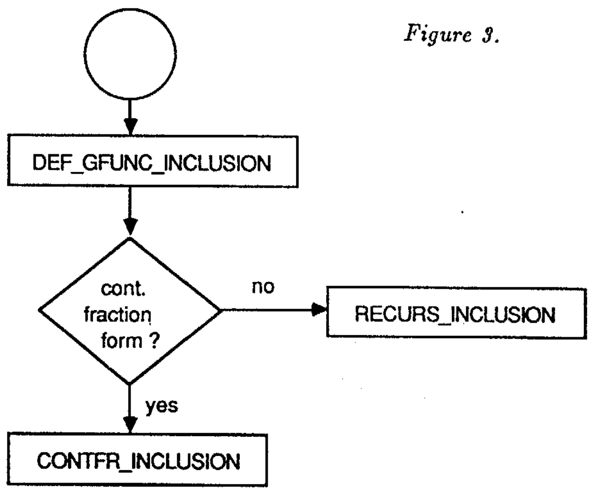


Figure 3.



Appendix.

We list here a number of notations which are frequently used throughout the paper.

$$B_{ij}(x,y) = \prod_{s=0}^{i-1}(x-x_s)\prod_{t=0}^{j-1}(y-y_t)$$

$$N = \{(i_0,j_0),\dots,(i_n,j_n)\} \subset \mathbb{N}^2$$

$$p(x,y) = \sum_{(i,j)\in N} a_{ij}x^iy^j$$

or

$$p(x,y) = \sum_{(i,j)\in N} a_{ij}B_{ij}(x,y)$$

$$D = \{(d_0,e_0),\dots,(d_m,e_m)\} \subset \mathbb{N}^2$$

$$q(x,y) = \sum_{(i,j)\in D} b_{ij}x^iy^j$$

or

$$q(x,y) = \sum_{(i,j)\in D} b_{ij}B_{ij}(x,y)$$

$$I = \{(k_0,\ell_0),\dots,(k_{n+m},\ell_{n+m})\}$$

$$N_s = \{(i_0,j_0),\dots,(i_s,j_s)\} = N_s^{(0)}$$

$$N_s^{(r)} = \{(i_r,j_r),\dots,(i_{r+s},j_{r+s})\}$$

$$f_{ik,j\ell} = f[x_i,\dots,x_k][y_j,\dots,y_\ell]$$

$$B_{ik,j\ell}(x,y) = B_{k\ell}(x,y)/B_{ij}(x,y)$$

$$\gamma_s = i_{\lfloor\frac{s+1}{2}\rfloor} + j_{\lfloor\frac{s+1}{2}\rfloor} + d_{\lfloor\frac{s}{2}\rfloor} + e_{\lfloor\frac{s}{2}\rfloor}$$

$$\rho_s^{(r)} = \rho[(x_{k_r},y_{\ell_r}),\dots,(x_{k_{r+s}},y_{\ell_{r+s}})] = {}_0\rho_s^{(r)}$$

$$\varphi_s^{(r)} = \varphi[(x_{k_r},y_{\ell_r}),\dots,(x_{k_{r+s}},y_{\ell_{r+s}})] = {}_0\varphi_s^{(r)}$$

References.

1. Brezinski C., "Accélération de la convergence en Analyse Numérique", LNM **584**, Springer Verlag, Berlin, 1977.
2. Brezinski C., *A general extrapolation algorithm*, Numer. Math. **35** (1980), 175–187.
3. Claessens G., *A generalization of the qd-algorithm*, J. Comput. Appl. Math. **7** (1981), 237–247.
4. Cuyt A., *A multivariate qd-like algorithm*, BIT **28** (1988), 98–112.
5. Cuyt A., *A recursive computation scheme for multivariate rational interpolants*, SIAM J. Num. Anal. **24** (1987), 228–238.
6. Cuyt A., *A review of multivariate Padé approximation theory*, J. Comput. Appl. Math. **12** & **13** (1985), 221–232.
7. Cuyt A., "General order multivariate rational Hermite interpolants", Monograph, University of Antwerp (UIA), 1986.
8. Cuyt A., *Multivariate Padé approximants revisited*, BIT **26** (1986), 71–79.
9. Cuyt A. and Verdonk B., *Different techniques for the construction of multivariate rational interpolants*, in "Nonlinear Numerical Methods and Rational Approximation" (Cuyt A. ed.), Reidel (1988), 167–190.
10. Cuyt A. and Verdonk B., "Different techniques for the construction of multivariate rational interpolants and Padé approximants", Monograph, University of Antwerp (UIA), 1988.
11. Cuyt A. and Verdonk B., *General order Newton Padé approximants for multivariate functions*, Numer. Math. **43** (1984), 293–307.
12. Cuyt A. and Verdonk B., *Multivariate rational interpolation*, Computing **34** (1985), 41–61.
13. Levin D., *General order Padé type rational approximants defined from double power series*, J. Inst. Math. Appl. **18** (1976), 1–8.
14. Milne-Thomson L., "The calculus of finite differences", Mac Millan, London, 1951.
15. Siemaszko W., *Thiele-type branched continued fractions for two-variable functions*, J. Comput. Appl. Math. **9** (1983), 137–153.