A Note on Rational Interpolation

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Keywords
Rational Interpolation, Continued Fractions, Recursive Algorithms

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Introduction

Graduate and research students starting in rational approximation theory are soon confronted with different approaches to deal with a particular rational approximation problem. The aim of this classroom note is to explain and emphasize that several existing methods to construct a rational interpolant are only reformulations of one another. This is done by direct rewriting of each method into the other, rather than the classical proofs where one shows that each method essentially solves the same interpolation problem. In the process of this rewriting, knowledge is acquired on determinant identities [1], recursive computation schemes [9 Ch. 2, 3 Ch. 3], continued fractions [5, 7], equivalence transformations, etc. It is important to work one's way through this exercise because it is a good starting point to understand or develop any generalizations. Also for the Padé approximation problem and other rational approximation problems, similar links can be shown to hold between different approaches.

Rational Interpolation

The problem of constructing a rational function $p(x)/q(x)$ that interpolates a univariate function $f(x)$ at the data $(x_k, f(x_k))$ can be solved in various ways.

One can write down the linearized system of interpolation conditions $(fq-p)(x_k)=0$ and solve it using Cramer's rule to get determinant expressions for the unknown coefficients in $p(x)$ and $q(x)$ and hence also for $p(x)$ and $q(x)$ themselves.

One can start a resursive scheme, computing the rational interpolant of a certain degree from the knowledge of rational interpolants of lower degree.

One can use the fact that a rational function can be written as the convergent of a continued fraction and compute the interpolant in that way.

As the reference list indicates, several independent papers have been published on the subject. Of course all the techniques proposed for solving the rational interpolation problem are equivalent since they all essentially solve the same problem. In this classroom note we emphasize this by showing that any of the following formulas for $p(x)/q(x)$, whether it concerns the determinantal expressions or the recursive computation rules or the continued fraction representation, can be rewritten to yield any of the other listed formulas. It is important to see this interrelationship clearly because this is a good starting point for those who are interested in research on the rational interpolation problem (see e.g. [2] for generalizations to multivariate functions).

For the moment we concentrate on the following.

Let the univariate function $f(x)$ be given in the non-coincident interpolation points $\{x_0, x_1, x_2, \ldots\}$. We consider the next two problems :

find polynomials

$$\begin{aligned} P_j^{n,n}(x) &= \sum_{i=0}^{n} a_i x^i \\ Q_j^{n,n}(x) &= \sum_{i=0}^{n} b_i x^i \end{aligned} \tag{1a}$$

such that

$$(fQ_j^{n,n} - P_j^{n,n})(x_j) = 0 \quad \text{for } k = j, \ldots, j+2n \tag{1b}$$

and find polynomials

$$\begin{aligned} P_j^{n+1,n}(x) &= \sum_{i=0}^{n+1} a_i x^i \\ Q_j^{n+1,n}(x) &= \sum_{i=0}^{n} b_i x^i \end{aligned} \tag{2a}$$

such that

$$(fQ_j^{n+1,n} - P_j^{n+1,n})(x_k) = 0 \quad \text{for } k = j, \ldots, j+2n+1 \tag{2b}$$

It is well-known that all solutions of (1) are equivalent in the sense that if the polynomials P_1, Q_1 and P_2, Q_2 satisfy (1), then

$$P_1 Q_2 = P_2 Q_1$$

This implies that all solutions of (1) have the same irreducible form. This unique irreducible form is called the rational interpolant of order (n, n) for f. Similarly, all solutions of (2) have a unique irreducible form, which is then called the rational interpolant of order $(n+1, n)$ for f.

It is important to realize that the rational interpolant of order (n, n) (respectively order $(n+1, n)$) for f does not necessarily satisfy the conditions (1b) (respectively (2b)) because, by constructing the irreducible form, a polynomial may have been cancelled in numerator and denominator. If this common polynomial contained a factor $(x - x_k)$ with x_k an interpolation point, then it may be that (1b) (respectively (2b)) is no longer satisfied by the irreducible form of (1a) (respectively (2a)). If this is the case, the rational interpolant does not interpolate the function f at all the given data.

In the sequel of the text we shall assume that all linear systems of interpolation conditions have maximal rank. Under these conditions, the equivalence formulated in the theorem below can be proved for the solutions of the problems (1) and (2), which are then unique (but not necessarily irreducible) up to a constant multiplicative factor. Together with each formula we cite a reference where this formula is treated separately. However, it is our aim here to emphasize the interrelationship between the different formulas.

Let us denote $f(x_k)$ by f_k.

Theorem The statements (I), (II), (III), (IV) and (V) are equivalent :

(I) the polynomials $P_j^{n,n}, Q_j^{n,n}$ and $P_j^{n+1,n}, Q_j^{n+1,n}$ respectively satisfy (1) and (2)

(II) the following determinant expressions [10] for $P_j^{n,n}, Q_j^{n,n}$ and $P_j^{n+1,n}, Q_j^{n+1,n}$ can be written down :

$$\frac{P_j^{n,n}}{Q_j^{n,n}}(x) = \frac{\begin{vmatrix} f_j & x-x_j & f_j(x-x_j) & (x-x_j)^2 & f_j(x-x_j)^2 & \ldots & (x-x_j)^n & f_j(x-x_j)^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ f_{j+2n} & x-x_{j+2n} & & & & & & \end{vmatrix}}{\begin{vmatrix} 1 & x-x_j & f_j(x-x_j) & & \ldots & & (x-x_j)^n & f_j(x-x_j)^n \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots \\ 1 & x-x_{j+2n} & & & & & & \end{vmatrix}}$$

and

$$\frac{P_j^{n+1,n}}{Q_j^{n+1,n}}(x) = \frac{\begin{vmatrix} f_j & x-x_j & f_j(x-x_j) & \dots & (x-x_j)^n & f_j(x-x_j)^n & (x-x_j)^{n+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ f_{j+2n+1} & x-x_{j+2n+1} & & & & & \end{vmatrix}}{\begin{vmatrix} 1 & x-x_j & f_j(x-x_j) & \dots & & (x-x_j)^{n+1} \\ \vdots & \vdots & \vdots & & & \vdots \\ 1 & x-x_{j+2n+1} & & & & \end{vmatrix}}$$

(III) $P_j^{n,n}, Q_j^{n,n}$ and $P_j^{n+1,n}, Q_j^{n+1,n}$ satisfy the recursive computation rules [8] :

$$\begin{aligned} P_j^{n,n}(x) &= (x-x_j)p_j^{n,n-1}P_{j+1}^{n,n-1}(x) - (x-x_{j+2n})p_{j+1}^{n,n-1}P_j^{n,n-1}(x) \\ Q_j^{n,n}(x) &= (x-x_j)p_j^{n,n-1}Q_{j+1}^{n,n-1}(x) - (x-x_{j+2n})p_{j+1}^{n,n-1}Q_j^{n,n-1}(x) \end{aligned}$$

and

$$\begin{aligned} P_j^{n+1,n}(x) &= (x-x_j)q_j^{n,n}P_{j+1}^{n,n}(x) - (x-x_{j+2n+1})q_{j+1}^{n,n}P_j^{n,n}(x) \\ Q_j^{n+1,n}(x) &= (x-x_j)q_j^{n,n}Q_{j+1}^{n,n}(x) - (x-x_{j+2n+1})q_{j+1}^{n,n}Q_j^{n,n}(x) \end{aligned}$$

where

$$\begin{aligned} P_j^{n+1,n}(x) &= p_j^{n+1,n}x^{n+1} + \dots \\ Q_j^{n+1,n}(x) &= q_j^{n+1,n}x^{n} + \dots \\ P_j^{n,n}(x) &= p_j^{n,n}x^{n} + \dots \\ Q_j^{n,n}(x) &= q_j^{n,n}x^{n} + \dots \end{aligned}$$

and

$$\left. \begin{aligned} P_k^{0,0} &= f_k \\ Q_k^{0,0} &= 1 \end{aligned} \right. \quad \text{for } k = j, \dots, j+2n+1$$

(IV) Aitken–Neville–like formulas are satisfied by $P_j^{n,n}/Q_j^{n,n}$ and $P_j^{n+1,n}/Q_j^{n+1,n}$ [6] :

$$\frac{P_j^{n,n}}{Q_j^{n,n}} = \frac{P_{j+1}^{n-1,n-1}}{Q_{j+1}^{n-1,n-1}} + \frac{x_{j+2n}-x_j}{\dfrac{x-x_j}{\dfrac{P_{j+1}^{n,n-1}}{Q_{j+1}^{n,n-1}} - \dfrac{P_{j+1}^{n-1,n-1}}{Q_{j+1}^{n-1,n-1}}} + \dfrac{x_{j+2n}-x}{\dfrac{P_j^{n,n-1}}{Q_j^{n,n-1}} - \dfrac{P_{j+1}^{n-1,n-1}}{Q_{j+1}^{n-1,n-1}}}}$$

and

$$\frac{P_j^{n+1,n}}{Q_j^{n+1,n}} = \frac{P_{j+1}^{n,n-1}}{Q_{j+1}^{n,n-1}} + \frac{x_{j+2n+1}-x_j}{\dfrac{x-x_j}{\dfrac{P_{j+1}^{n,n}}{Q_{j+1}^{n,n}} - \dfrac{P_{j+1}^{n,n-1}}{Q_{j+1}^{n,n-1}}} + \dfrac{x_{j+2n+1}-x}{\dfrac{P_j^{n,n}}{Q_j^{n,n}} - \dfrac{P_{j+1}^{n,n-1}}{Q_{j+1}^{n,n-1}}}}$$

where

$$\frac{P_k^{0,0}}{Q_k^{0,0}} = f_k \qquad \text{for } k = j, \dots, j+2n+1$$

and

$$\frac{P_k^{1,0}}{Q_k^{1,0}} = \frac{(x-x_k)f_{k+1} + (x_{k+1}-x)f_k}{x_{k+1}-x_k} \qquad \text{for } k = j, \dots, j+2n$$

(V) $P_j^{n,n}, Q_j^{n,n}$ and $P_j^{n+1,n}, Q_j^{n+1,n}$ are the numerator and denominator of the $(2n)^{th}$ (respectively $(2n+1)^{th}$) convergent of the continued fraction [4]

$$\varphi[x_j] + \sum_{k=0}^{\infty} \left. \frac{x - x_{j+k}}{\varphi[x_j, \ldots, x_{j+k+1}]} \right| = \varphi[x_j] + \cfrac{x - x_j}{\varphi[x_j, x_{j+1}] + \cfrac{x - x_{j+1}}{\varphi[x_j, x_{j+1}, x_{j+2}] + \cfrac{x - x_{j+2}}{\cdots}}}$$

where

$$\begin{aligned}
&\varphi[x_j] = f_j \\
&\varphi[x_j, \ldots, x_{j+k+1}] = \frac{x_{j+k+1} - x_{j+k}}{\varphi[x_j, \ldots, x_{j+k-1}, x_{j+k+1}] - \varphi[x_j, \ldots, x_{j+k-1}, x_{j+k}]}
\end{aligned}$$

Proof

In contrast to the proofs found in [10],[8],[6] and [4], where it is shown that the respective statements (II) through (V) are each equivalent with (I), the construction of the proof given here will be as follows : we shall demonstrate that (I) can be rewritten as (II), (II) can be rewritten as (III) and (III) implies (I); then we shall prove that (IV) is an equivalent reformulation of (III); as for a proof of the equivalence between (V) and (I) we refer to [4]. Also we shall prove the theorem only for $P_j^{n+1,n}, Q_j^{n+1,n}$ because the reasoning for $P_j^{n,n}, Q_j^{n,n}$ is completely analogous.

The polynomials

$$P_j^{n+1,n}(x) = \sum_{i=0}^{n+1} a_i x^i$$
$$Q_j^{n+1,n}(x) = \sum_{i=0}^{n} b_i x^i$$

together clearly contain $2n+3$ unknown coefficients a_i, b_i. Because one parameter can always be chosen by a normalization of numerator and denominator, these a_i, b_i are uniquely determined by imposing the $2n+2$ interpolating conditions

$$(fQ_j^{n+1,n} - P_j^{n+1,n})(x_k) = 0 \quad k = j, \ldots, j+2n+1$$

More explicitly these linear equations can be written as

$$f_k b_0 - a_0 - a_1 x_k + b_1 f_k x_k - a_2 x_k^2 + b_2 f_k x_k^2 \ldots - a_n x_k^n + b_n f_k x_k^n - a_{n+1} x_k^{n+1} = 0 \quad k = j, \ldots, j+2n+1$$

Since this homogeneous system of equations is assumed to have maximal rank, its solution is given by the following determinants of size $(2n+3)$:

$P_j^{n+1,n}(x) =$

$$\begin{vmatrix}
0 & -1 & -x & 0 & -x^2 & 0 & \ldots & -x^n & 0 & -x^{n+1} \\
-f_j & -1 & -x_j & f_j x_j & -x_j^2 & f_j x_j^2 & \ldots & -x_j^n & f_j x_j^n & -x_j^{n+1} \\
-f_{j+1} & -1 & -x_{j+1} & f_{j+1} x_{j+1} & -x_{j+1}^2 & f_{j+1} x_{j+1}^2 & \ldots & -x_{j+1}^n & f_{j+1} x_{j+1}^n & -x_{j+1}^{n+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
-f_{j+2n+1} & -1 & -x_{j+2n+1} & & & & & & &
\end{vmatrix}$$

and

$Q_j^{n+1,n}(x) =$

$$\left|\begin{array}{cccccccccc} 1 & 0 & 0 & -x & 0 & -x^2 & \dots & 0 & -x^n & 0 \\ -f_j & -1 & -x_j & f_j x_j & -x_j^2 & f_j x_j^2 & \dots & -x_j^n & f_j x_j^n & -x_j^{n+1} \\ -f_{j+1} & -1 & -x_{j+1} & f_{j+1}x_{j+1} & -x_{j+1}^2 & f_{j+1}x_{j+1}^2 & \dots & -x_{j+1}^n & f_{j+1}x_{j+1}^n & -x_{j+1}^{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ -f_{j+2n+1} & -1 & -x_{j+2n+1} & & & & & & & \end{array}\right| \quad (3)$$

By making a suitable combination of the rows and colums in the above determinants, we easily obtain the form given in (II) for $P_j^{n+1,n}(x)/Q_j^{n+1,n}(x)$.

From the expression (3) for $Q_j^{n+1,n}(x)$ we can also immediately see that its highest degree coefficient $q_j^{n+1,n}$ is given by

$$q_j^{n+1,n} = \left|\begin{array}{cccccccc} 1 & f_j & -x_j & f_j x_j & -x_j^2 & \dots & f_j x_j^{n-1} & -x_j^n & -x_j^{n+1} \\ 1 & f_{j+1} & -x_{j+1} & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & & & & & \\ 1 & f_{j+2n+1} & -x_{j+2n+1} & & & & & & \end{array}\right| \quad (4)$$

The analogous expression

$$q_j^{n,n} = \left|\begin{array}{cccccccc} 1 & f_j & -x_j & f_j x_j & -x_j^2 & \dots & f_j x_j^{n-1} & -x_j^n \\ 1 & f_{j+1} & -x_{j+1} & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & & & & \\ 1 & f_{j+2n} & -x_{j+2n} & & & & & \end{array}\right| \quad (5)$$

is used further on in this proof.

To find the recurrence given in (III) from the determinantal formulas (II) we shall make use of Jacobi's identity [1]. Therefore we first write

$$\frac{P_j^{n+1,n}(x)}{Q_j^{n+1,n}(x)} = \frac{\left|\begin{array}{cccccc} f_j & (x-x_j)^{n+1} & (x-x_j) & f_j(x-x_j) & \dots & f_j(x-x_j)^n \\ f_{j+2n+1} & & & & & \\ & & & A & & \\ f_{j+1} & & & & & \\ \vdots & & & & & \\ f_{j+2n} & & & & & \end{array}\right| * G}{\left|\begin{array}{cccccc} 1 & (x-x_j)^{n+1} & (x-x_j) & f_j(x-x_j) & \dots & f_j(x-x_j)^n \\ \vdots & & & A & & \\ 1 & & & & & \end{array}\right| * G}$$

with

$$A = \left|\begin{array}{ccccc} (x-x_{j+2n+1})^{n+1} & x-x_{j+2n+1} & f_{j+2n+1}(x-x_{j+2n+1}) & \dots & f_{j+2n+1}(x-x_{j+2n+1})^n \\ (x-x_{j+1})^{n+1} & x-x_{j+1} & & & \\ \vdots & \vdots & \vdots & & \vdots \\ (x-x_{j+2n})^{n+1} & x-x_{j+2n} & & & \end{array}\right|$$

and

$$G = \left|\begin{array}{cccc} x-x_{j+1} & f_{j+1}(x-x_{j+1}) & \dots & f_{j+1}(x-x_{j+1})^n \\ \vdots & \vdots & & \vdots \\ x-x_{j+2n} & f_{j+2n}(x-x_{j+2n}) & \dots & \end{array}\right|$$

The determinant G is obtained from the rearranged determinant expressions for $P_j^{n+1,n}(x)$ and $Q_j^{n+1,n}(x)$ by omitting the first two rows and columns. It is assumed that $G \neq 0$.

Applying Jacobi's identity yields

$$\frac{P_j^{n+1,n}(x)}{Q_j^{n+1,n}(x)} = \frac{A * B - C * D}{A * E - C * F} \tag{6}$$

with

$$B = \begin{vmatrix} f_j & x - x_j & f_j(x - x_j) & \dots & f_j(x - x_j)^n \\ f_{j+1} & x - x_{j+1} & f_{j+1}(x - x_{j+1}) & \dots & f_{j+1}(x - x_{j+1})^n \\ \vdots & \vdots & \vdots & & \vdots \\ f_{j+2n} & x - x_{j+2n} & & & \end{vmatrix} = P_j^{n,n}(x)$$

$$C = \begin{vmatrix} (x - x_j)^{n+1} & x - x_j & f_j(x - x_j) & \dots & f_j(x - x_j)^n \\ (x - x_{j+1})^{n+1} & x - x_{j+1} & f_{j+1}(x - x_{j+1}) & \dots & f_{j+1}(x - x_{j+1})^n \\ \vdots & \vdots & \vdots & & \vdots \\ (x - x_{j+2n})^{n+1} & & & & \end{vmatrix}$$

$$D = \begin{vmatrix} f_{j+2n+1} & x - x_{j+2n+1} & f_{j+2n+1}(x - x_{j+2n+1}) & \dots & f_{j+2n+1}(x - x_{j+2n+1})^n \\ f_{j+1} & x - x_{j+1} & f_{j+1}(x - x_{j+1}) & & \vdots \\ f_{j+2} & \vdots & \vdots & & \\ \vdots & & & & \\ f_{j+2n} & x - x_{j+2n} & f_{j+2n}(x - x_{j+2n}) & & \end{vmatrix} = P_{j+1}^{n,n}(x)$$

$$E = \begin{vmatrix} 1 & x - x_j & f_j(x - x_j) & \dots & f_j(x - x_j)^n \\ 1 & x - x_{j+1} & \vdots & & \vdots \\ \vdots & \vdots & & & \\ 1 & x - x_{j+2n} & & & \end{vmatrix} = Q_j^{n,n}(x)$$

$$F = \begin{vmatrix} 1 & x - x_{j+2n+1} & f_{j+2n+1}(x - x_{j+2n+1}) & \dots & f_{j+2n+1}(x - x_{j+2n+1})^n \\ 1 & x - x_{j+1} & \vdots & & \vdots \\ \vdots & \vdots & & & \\ 1 & x - x_{j+2n} & & & \end{vmatrix} = Q_{j+1}^{n,n}(x)$$

Here we have taken into account the formulas (II) for B, D, E and F. To find an explicit expression for the quantities A and C we rewrite e.g. A as :

$$A = (x - x_{j+1}) \dots (x - x_{j+2n+1}) \times \begin{vmatrix} 1 & f_{j+1} & x - x_{j+1} & f_{j+1}(x - x_{j+1}) & \dots & f_{j+1}(x - x_{j+1})^{n-1} & (x - x_{j+1})^n \\ 1 & f_{j+2} & x - x_{j+2} & & & & \\ \vdots & \vdots & \vdots & & & & \vdots \\ 1 & f_{j+2n+1} & x - x_{j+2n+1} & & & & (x - x_{j+2n+1})^n \end{vmatrix}$$

By a number of suitable combinations of rows and columns we find, using (4) and (5), that

$$A = (x - x_{j+1}) \dots (x - x_{j+2n+1}) q_{j+1}^{n,n}$$

and analogously

$$C = (x - x_j) \dots (x - x_{j+2n}) q_j^{n,n}$$

Substituting the expressions for A, B, C, D, E and F in (6) yields

$$\frac{P_j^{n+1,n}(x)}{Q_j^{n+1,n}(x)} = \frac{(x - x_{j+2n+1})q_{j+1}^{n,n} P_j^{n,n}(x) - (x - x_j)q_j^{n,n} P_{j+1}^{n,n}(x)}{(x - x_{j+2n+1})q_{j+1}^{n,n} Q_j^{n,n}(x) - (x - x_j)q_j^{n,n} Q_{j+1}^{n,n}(x)}$$

It is easy to verify that the numerator (respectively denominator) of the lefthand side and of the righthand side in the above expression are polynomials of the same degree $n+1$ (respectively n). This concludes the proof that (II) can be rewritten as (III).

In order to show that (III) implies (I), one can easily verify that the polynomials $P_j^{n+1,n}$ and $Q_j^{n+1,n}$ as given by (III) have degree $n+1$ and n respectively. We still need to check that the expressions (III) for numerator and denominator interpolate the given function :

for $x = x_j$ we have :

$$(fQ_j^{n+1,n} - P_j^{n+1,n})(x_j) = (x_j - x_{j+2n+1})q_{j+1}^{n,n}(fQ_j^{n,n} - P_j^{n,n})(x_j) = 0$$

for $x = x_k$ with $k = j+1, \ldots, j+2n$ we have

$$\begin{aligned}(fQ_j^{n+1,n} - P_j^{n+1,n})(x_k) =& (x_k - x_{j+2n+1})q_{j+1}^{n,n}(fQ_j^{n,n} - P_j^{n,n})(x_k) \\ & - (x_k - x_j)q_j^{n,n}(fQ_{j+1}^{n,n} - P_{j+1}^{n,n})(x_k) = 0\end{aligned}$$

for $x = x_{j+2n+1}$, we have

$$(fQ_j^{n+1,n} - P_j^{n+1,n})(x_{j+2n+1}) = (x_j - x_{j+2n+1})q_j^{n,n}(fQ_{j+1}^{n,n} - P_{j+1}^{n,n})(x_{j+2n+1}) = 0$$

So we are back at (I).

Let us now show that (III) can be reformulated as (IV) in order to obtain an algorithm similar to the polynomial Aitken–Neville interpolation scheme. To prove this equivalence we start from the expression (IV) for $P_j^{n+1,n}/Q_j^{n+1,n}$. It can be written as :

$$\frac{P_j^{n+1,n}}{Q_j^{n+1,n}} = \frac{P_{j+1}^{n,n-1}}{Q_{j+1}^{n,n-1}} + \frac{(x_{j+2n+1} - x_j)\Delta_1\Delta_2}{(x - x_j)\Delta_1 Q_{j+1}^{n,n} Q_{j+1}^{n,n-1} + (x_{j+2n+1} - x)\Delta_2 Q_{j+1}^{n,n-1} Q_j^{n,n}} \tag{7}$$

where

$$\begin{aligned}\Delta_1 &= P_j^{n,n} Q_{j+1}^{n,n-1} - P_{j+1}^{n,n-1} Q_j^{n,n} \\ \Delta_2 &= P_{j+1}^{n,n} Q_{j+1}^{n,n-1} - P_{j+1}^{n,n-1} Q_{j+1}^{n,n}\end{aligned}$$

If we compute the numerator in the righthand side of (7), we have

$$\begin{aligned}N(x) \equiv\ & P_{j+1}^{n,n-1} Q_{j+1}^{n,n} Q_{j+1}^{n,n-1}(x - x_j)\Delta_1 + P_{j+1}^{n,n-1} Q_j^{n,n} Q_{j+1}^{n,n-1}(x_{j+2n+1} - x)\Delta_2 \\ & + Q_{j+1}^{n,n-1} P_{j+1}^{n,n} Q_{j+1}^{n,n-1}(x_{j+2n+1} - x_j)\Delta_1 - Q_{j+1}^{n,n-1} P_{j+1}^{n,n-1} Q_{j+1}^{n,n}(x_{j+2n+1} - x_j)\Delta_1\end{aligned}$$

which yields, after some easy computations,

$$N(x) = (x - x_{j+2n+1})(Q_{j+1}^{n,n-1})^2 P_j^{n,n}(-\Delta_2) + (x - x_j)(Q_{j+1}^{n,n-1})^2 P_{j+1}^{n,n}\Delta_1$$

Because

$$\begin{aligned}&(fQ_{j+1}^{n,n-1} - P_{j+1}^{n,n-1})(x_k) = 0 \quad && k = j+1, \ldots, j+2n \\ &(fQ_{j+1}^{n,n} - P_{j+1}^{n,n})(x_k) = 0 && k = j+1, \ldots, j+2n+1\end{aligned}$$

we have that

$$\begin{aligned}\Delta_2 &= P_{j+1}^{n,n} Q_{j+1}^{n,n-1} - P_{j+1}^{n,n-1} Q_{j+1}^{n,n} \\ &= (f Q_{j+1}^{n,n-1} - P_{j+1}^{n,n-1}) Q_{j+1}^{n,n} - (f Q_{j+1}^{n,n} - P_{j+1}^{n,n}) Q_{j+1}^{n,n-1} \\ &= u(x - x_{j+1}) \dots (x - x_{j+2n})\end{aligned}$$

where, taking into account the degrees of $P_{j+1}^{n,n-1}, Q_{j+1}^{n,n}, P_{j+1}^{n,n}$ and $Q_{j+1}^{n,n-1}$

$$u = -q_{j+1}^{n,n} p_{j+1}^{n,n-1}$$

In an analogous way we find that

$$\Delta_1 = P_j^{n,n} Q_{j+1}^{n,n-1} - P_{j+1}^{n,n-1} Q_j^{n,n} = v(x - x_{j+1}) \dots (x - x_{j+2n})$$

with

$$v = -p_{j+1}^{n,n-1} q_j^{n,n}$$

So,

$$\begin{aligned}N(x) = {} & p_{j+1}^{n,n-1}(x - x_{j+1}) \dots (x - x_{j+2n})(Q_{j+1}^{n,n-1})^2 q_{j+1}^{n,n}(x - x_{j+2n+1}) P_j^{n,n} \\ & - p_{j+1}^{n,n-1}(x - x_{j+1}) \dots (x - x_{j+2n})(Q_{j+1}^{n,n-1})^2 q_j^{n,n}(x - x_j) P_{j+1}^{n,n}\end{aligned}$$

For the denominator of the righthand side of (7), we have

$$D(x) \equiv (Q_{j+1}^{n,n-1})^2 \Delta_1 (x - x_j) Q_{j+1}^{n,n} + (Q_{j+1}^{n,n-1})^2 \Delta_2 (x_{j+2n+1} - x) Q_j^{n,n}$$

Substituting the expressions for Δ_1 and Δ_2 yields

$$\begin{aligned}D(x) = & - p_{j+1}^{n,n-1}(x - x_{j+1}) \dots (x - x_{j+2n})(Q_{j+1}^{n,n-1})^2 q_j^{n,n}(x - x_j) Q_{j+1}^{n,n} \\ & + p_{j+1}^{n,n-1}(x - x_{j+1}) \dots (x - x_{j+2n})(Q_{j+1}^{n,n-1})^2 q_{j+1}^{n,n}(x - x_{j+2n+1}) Q_j^{n,n}\end{aligned}$$

By dividing out the common factor in $N(x)$ and $D(x)$, we obtain polynomials of degree $n+1$ and n respectively, and hence (7) reduces to the formulas (III) for $P_j^{n+1,n}$ and $Q_j^{n+1,n}$. ■

The interpolation problems (1) and (2) were formulated for specific choices of the degrees of numerator and denominator. They are special cases of the following univariate rational interpolation problem :

find polynomials

$$\begin{aligned} P_j^{n,m}(x) &= \sum_{i=0}^{n} a_i x^i \\ Q_j^{n,m}(x) &= \sum_{i=0}^{m} b_i x^i \end{aligned} \tag{8a}$$

such that

$$(f Q_j^{n,m} - P_j^{n,m})(x_k) = 0 \qquad \text{for } k = j, j+1, \dots, j+n+m \tag{8b}$$

If the functions $P_j^{n,m}/Q_j^{n,m}$ are ordered for different values of n and m in the table

$$\begin{array}{c|ccc} \text{n} \setminus \text{m} & 0 & 1 & 2 \\ \hline 0 & \frac{P_j^{0,0}}{Q_j^{0,0}} & \frac{P_j^{0,1}}{Q_j^{0,1}} & \dots \\ 1 & \frac{P_j^{1,0}}{Q_j^{1,0}} & \frac{P_j^{1,1}}{Q_j^{1,1}} & \dots \\ 2 & \vdots & & \end{array}$$

Figure 1 *Figure 2*

On a staircase as drawn in *Figure 1*, the functions are of the form $P_j^{t+n,n}/Q_j^{t+n,n}$ and $P_j^{t+n+1,n}/Q_j^{t+n+1,n}$, $t > 0$ fixed and $n \geq 0$. To use the formulas (II) through (V) to compute these rational interpolating functions, we write

$$\begin{aligned}\frac{P_j^{t+n,n}}{Q_j^{t+n,n}}(x) &= P_j^{t-1,0}(x) + S(x) \cdot \frac{\tilde{P}_{j+t}^{n,n}}{\tilde{Q}_{j+t}^{n,n}}(x) \\ \frac{P_j^{t+n+1,n}}{Q_j^{t+n+1,n}}(x) &= P_j^{t-1,0}(x) + S(x) \cdot \frac{\tilde{P}_{j+t}^{n+1,n}}{\tilde{Q}_{j+t}^{n+1,n}}(x)\end{aligned} \tag{9}$$

Here $P_j^{t-1,0}$ is the Newton interpolating polynomial of degree $t-1$

$$P_j^{t-1,0} = \sum_{i=0}^{t-1} f[x_j, \ldots, x_{j+i}] \prod_{k=1}^{i} (x - x_{j+k-1})$$

and

$$S(x) = f[x_j, \ldots, x_{j+t}] \prod_{k=1}^{t} (x - x_{j+k-1})$$

The $f[x_j, \ldots, x_{j+i}]$ are divided differences defined by

$$\begin{aligned}f[x_j] &= f(x_j) = f_j \\ f[x_j, \ldots, x_{j+i}] &= \frac{f[x_{j+1}, \ldots, x_{j+i}] - f[x_j, \ldots, x_{j+i-1}]}{x_{j+i} - x_j}\end{aligned}$$

If the functions $\tilde{P}^{n,n}_{j+t}/\tilde{Q}^{n,n}_{j+t}$ and $\tilde{P}^{n+1,n}_{j+t}/\tilde{Q}^{n+1,n}_{j+t}$ in (9) respectively solve the rational interpolation problem (1) and (2) for the function

$$\tilde{f}(x) = \frac{f(x) - P_j^{t-1,0}(x)}{S(x)}$$

interpolating $\tilde{f}$ in $x_{j+t+\ell}$, $\ell = 0, \ldots 2n$ (respectively $\ell = 0, \ldots, 2n+1$), then it can easily be verified that the functions $P_j^{t+n,n}/Q_j^{t+n,n}$ and $P_j^{t+n+1,n}/Q_j^{t+n+1,n}$, as given by (9), solve the rational interpolation problem (8) of order $(t+n, n)$, respectively $(t+n+1, n)$.

Hence the rational interpolating functions on descending staircases as in *Figure 1* can be computed using one of the formulas (II) through (V). For situations like in *Figure 2*, one proceeds in a similar way.

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