Multivariate reciprocal differences for branched Thiele continued fraction expansions

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Introduction

For a univariate function given by its Taylor series expansion, one can construct a continued fraction expansion with the algorithm of Viscovatov. This continued fraction expansion can also be obtained as the limiting value of a Thiele interpolating continued fraction. For a multivariate function a Viscovatov-like algorithm for the construction of a branched continued fraction expansion was developed independently by Murphy and O'Donohoe [10] and by Kuchminskaya [5]. On the other hand, multivariate inverse differences to construct Thiele interpolating branched continued fractions were introduced independently by Kuchminskaya in [6] and by the authors in [3]. It is the purpose of the present paper to show the link between these two approaches in the multivariate case: we introduce multivariate reciprocal differences so as to obtain the branched continued fraction expansion as the limiting value of the Thiele interpolating branched continued fraction. Let us point out that multivariate reciprocal differences and their limiting values were already introduced by Siemaszko [12] for another type of branched continued fraction than the one we shall consider here. For a review of the different generalizations of the univariate interpolating and corresponding continued fraction to the multivariate case we refer the reader to [2] or [8].

1. Univariate case

Let us first briefly review the univariate theory. Let $f(x)$ be a univariate function and $x^{(\infty)} = \{ x_0, x_1, x_2, \ldots \}$ a sequence of distinct real points. If we compute inverse differences for

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$f(x)$:

$$\phi_0[x_0] = f(x_0),$$

$$\phi_l[x_0,\ldots,x_l] = \frac{x_l - x_{l-1}}{\phi_{l-1}[x_0,\ldots,x_{l-2},\,x_l] - \phi_{l-1}[x_0,\ldots,x_{l-1}]}, \quad l \geqslant 1, \tag{1}$$

then it is well-known that the continued fraction (CF)

$$\phi_0[x_0] + \sum_{l=1}^{\infty} \frac{x - x_{l-1}}{\left|\phi_l[x_0,\ldots,x_l]\right|} \tag{2}$$

is a Thiele interpolating CF for $f(x)$. Instead of computing inverse differences one can also compute reciprocal differences for $f(x)$:

$$\rho_0[x_0] = f(x_0), \qquad \rho_1[x_0, x_1] = (x_1 - x_0)/(f(x_1) - f(x_0)),$$

$$\rho_l[x_0,\ldots,x_l] = \frac{x_l - x_{l-1}}{\rho_{l-1}[x_0,\ldots,x_{l-2},\,x_l] - \rho_{l-1}[x_0,\ldots,x_{l-1}]} + \rho_{l-2}[x_0,\ldots,x_{l-2}],$$

$$l \geqslant 2.$$

The reciprocal differences are related to the inverse differences by

$$\phi_0[x_0] = \rho_0[x_0], \qquad \phi_1[x_0, x_1] = \rho_1[x_0, x_1],$$

$$\phi_l[x_0,\ldots,x_l] = \rho_l[x_0,\ldots,x_l] - \rho_{l-2}[x_0,\ldots,x_{l-2}], \quad l \geqslant 2,$$

and have the important property that they do not depend on the numbering of their arguments $x_0,\ldots,x_l$. As mentioned above, one way to obtain a CF expansion for $f(x)$ is as the limiting value of the CF (2):

$$\phi_0(u) + \sum_{l=1}^{\infty} \frac{x - u}{\left|\phi_l(u)\right|} \tag{3}$$

where

$$\phi_l(u) = \lim_{\substack{x_i \to u \\ i=0,\ldots,l}} \phi_l[x_0,\ldots,x_l], \quad l \geqslant 0.$$

The recursive scheme for the values $\phi_l(u)$ is well-known:

$$\phi_0(u) = f(x)\,|_{x=u} = \rho_0(u), \qquad \phi_1(u) = (\mathrm{d}f/\mathrm{d}x)^{-1}\big|_{x=u} = \rho_1(u), \tag{4a}$$

and for $l \geqslant 2$,

$$\phi_l(u) = l(\mathrm{d}\rho_{l-1}(x)/\mathrm{d}x)^{-1}\big|_{x=u}, \qquad \rho_l(x) = \phi_l(x) + \rho_{l-2}(x), \tag{4b}$$

where

$$\rho_l(x) = \lim_{\substack{x_i \to x \\ i=0,\ldots,l}} \rho_l[x_0,\ldots,x_l], \quad l \geqslant 0.$$

An alternative to this scheme for the construction of a Thiele CF expansion for $f(x)$ is based on Viscovatov's algorithm. If $f(x)$ is given by its Taylor series expansion around u

$$f(x) = c_0^{(0)} + c_1^{(0)}(x - u) + c_2^{(0)}(x - u)^2 + \cdots,$$

then the coefficients $\phi_l(u)$ in (3) can numerically be computed as follows [4]:

$$\phi_0(u) = c_0^{(0)}, \qquad \phi_1(u) = 1/c_1^{(0)},$$ (5a)

$$c_i^{(1)} = -\phi_1(u)c_{i+1}^{(0)}, \quad i \geqslant 1,$$

and for $l > 1$,

$$\phi_l(u) = c_1^{(l-2)}/c_1^{(l-1)},$$

$$c_i^{(l)} = c_{i+1}^{(l-2)} - \phi_l(u)c_{i+1}^{(l-1)}, \quad i \geqslant 1.$$ (5b)

2. Thiele interpolation and Viscovatov's algorithm for multivariate functions

We restrict ourselves to the bivariate case in order to simplify the notation. Given two sequences of distinct real points $x^{(\infty)} = \{x_0, x_1, x_2, \ldots\}$ and $y^{(\infty)} = \{y_0, y_1, y_2, \ldots\}$ and a bivariate function $f(x, y)$, many types of interpolating branched continued fractions (BCF) for $f(x, y)$ can be constructed, depending on the way in which $\mathbb{N}^2$ is enumerated [1,2,8]. If $\mathbb{N}^2$ is considered as a union of prongs
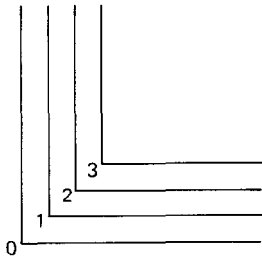
we can write in a purely formal way [3,6]

$$f(x, y) = \phi_{00}[x_0][y_0] + \sum_{l=1}^{\infty} \frac{x - x_{l-1}}{\phi_{l0}[x_0, \ldots, x_l][y_0]} + \sum_{l=1}^{\infty} \frac{y - y_{l-1}}{\phi_{0l}[x_0][y_0, \ldots, y_l]}$$

$$+ \sum_{k=1}^{\infty} \frac{(x - x_{k-1})(y - y_{k-1})}{B_k(x, y)}$$ (6)

with

$$B_k(x, y) = \phi_{kk}[x_0, \ldots, x_k][y_0, \ldots, y_k] + \sum_{l=k+1}^{\infty} \frac{x - x_{l-1}}{\phi_{lk}[x_0, \ldots, x_l][y_0, \ldots, y_k]}$$

$$+ \sum_{l=k+1}^{\infty} \frac{y - y_{l-1}}{\phi_{kl}[x_0, \ldots, x_k][y_0, \ldots, y_l]}$$

The values $\phi_{kl}[x_0,\ldots,x_k[[y_0,\ldots,y_l]$ are *bivariate inverse differences* and can be computed according to the following scheme:

$$\phi[x_0][y_0] = f(x_0,\ y_0),$$

$$\phi[x_0,\ldots,x_l][y_0] = \frac{x_l - x_{l-1}}{\phi[x_0,\ldots,x_{l-2},\ x_l][y_0] - \phi[x_0,\ x_{l-2},\ x_{l-1}][y_0]}, \tag{7a}$$

$$\phi[x_0][y_0,\ldots,y_l] = \frac{y_l - y_{l-1}}{\phi[x_0][y_0,\ldots,y_{l-2},\ y_l] - \phi[x_0][y_0,\ldots,y_{l-2},\ y_{l-1}]},$$

$$\phi[x_0,\ldots,x_k][y_0,\ldots,y_k]$$

$$= \frac{(x_k - x_{k-1})(y_k - y_{k-1})}{\begin{array}{c}\phi[x_0,\ldots,x_{k-2},\ x_k][y_0,\ldots,y_{k-2},\ y_k] - \phi[x_0,\ldots,x_{k-2},\ x_{k-1}][y_0,\ldots,y_{k-2},\ y_k] \\ -\phi[x_0,\ldots,x_{k-2},\ x_k][y_0,\ldots,y_{k-2},\ y_{k-1}] + \phi[x_0,\ldots,x_{k-2},\ x_{k-1}][y_0,\ldots,y_{k-2},\ y_{k-1}]\end{array}}, \tag{7b}$$

and for $l > k$,

$$\phi[x_0,\ldots,x_l][y_0,\ldots,y_k]$$

$$= \frac{x_l - x_{l-1}}{\phi[x_0,\ldots,x_{l-2},\ x_l][y_0,\ldots,y_k] - \phi[x_0,\ldots,x_{l-2},\ x_{l-1}][y_0,\ldots,y_k]}, \tag{7c}$$

$$\phi[x_0,\ldots,x_k][y_0,\ldots,y_l]$$

$$= \frac{y_l - y_{l-1}}{\phi[x_0,\ldots,x_k][y_0,\ldots,y_{l-2},\ y_l] - \phi[x_0,\ldots,x_k][y_0,\ldots,y_{l-2},\ y_{l-1}]}. \tag{7d}$$

Let us now consider the limiting case

$$x_i \to u,\quad i \geqslant 0,\qquad y_j \to v,\quad j \geqslant 0,$$

$$\phi_{kl}(u,\ v) = \lim_{\substack{x_i \to u,\ y_j \to v \\ i=0,\ldots,k,\ j=0,\ldots,l}} \phi_{kl}[x_0,\ldots,x_k][y_0,\ldots,y_l].$$

The values $\phi_{kl}(u,\ v)$ are called *bivariate inverse derivatives*. The expression (6) then becomes

$$f(x,\ y) = \phi_{00}(u,\ v) + \sum_{l=1}^{\infty} \frac{x-u\ |}{|\phi_{l0}(u,\ v)} + \sum_{l=1}^{\infty} \frac{y-v\ |}{|\phi_{0l}(u,\ v)}$$

$$+ \sum_{k=1}^{\infty} \frac{(x-u)(y-v)\ |}{|\phi_{kk}(u,\ v) + \sum_{l=k+1}^{\infty} \frac{x-u\ |}{|\phi_{lk}(u,\ v)} + \sum_{l=k+1}^{\infty} \frac{y-v\ |}{|\phi_{kl}(u,\ v)}}. \tag{8}$$

If $f(x,\ y)$ is given by its Taylor series expansion around $(u,\ v)$

$$f(x,\ y) = \sum_{i,j=0}^{\infty} c_{ij}^{(0)}(x-u)^i(y-v)^j$$

we can compute the coefficients $\phi_{kl}(u,\ v)$ using a Viscovatov-like algorithm. To this end we adapt the formulas given in [5,10] so as to match the BCF (8). For a discussion of the existence and uniqueness of (8) we refer to [5,10]. We assume throughout the text that the conditions to

guarantee existence and unicity of the continued fraction expansion (8) are fulfilled. Let us introduce the notations

$$g_k(x) = \sum_{l=k+1}^{\infty} \frac{x-u}{|\phi_{lk}(u, v)|}, \quad k \geq 0,$$

$$h_k(y) = \sum_{l=k+1}^{\infty} \frac{y-v}{|\phi_{kl}(u, v)|}, \quad k \geq 0,$$

$$f_0(x, y) = f(x, y) - \phi_{00}(u, v) - g_0(x) - h_0(y).$$

In this way

$$f_0(x, y) = \sum_{k=1}^{\infty} \frac{(x-y)(y-v)}{|\phi_{kk}(u, v) + g_k(x) + h_k(y)|}$$

$$= \sum_{i,j=1}^{\infty} c_{ij}^{(0)}(x-u)^i(y-v)^j.$$

We define

$$f_1 = (x-u)(y-v) - (\phi_{11}(u, v) + g_1 + h_1)f_0, \tag{9a}$$

$$f_k = (x-u)(y-v)f_{k-2} - (\phi_{kk}(u, v) + g_k + h_k)f_{k-1}. \tag{9b}$$

As indicated further the coefficients $\phi_{kk}(u, v)$ can be chosen such that a series expansion for $f_k(x, y)$ is of the form

$$f_k(x, y) = (x-u)^{k+1}(y-v)^{k+1} \sum_{i,j=1}^{\infty} c_{ij}^{(k)}(x-u)^{i-1}(y-v)^{j-1}$$

while $g_k(x)$ and $h_k(y)$ can be written as

$$g_k(x) = \sum_{i=1}^{\infty} d_i^{(k)}(x-u)^i, \qquad h_k(y) = \sum_{j=1}^{\infty} e_j^{(k)}(y-v)^j. \tag{10a, b}$$

Equating coefficients in formula (9a),

$$(x-u)(y-v) \sum_{i,j=1}^{\infty} c_{ij}^{(1)}(x-u)^{i-1}(y-v)^{j-1}$$

$$= 1 - \left(\phi_{11}(u, v) + \sum_{i=1}^{\infty} d_i^{(1)}(x-u)^i + \sum_{j=1}^{\infty} e_j^{(1)}(y-v)^j \right)$$

$$\times \sum_{i,j=1}^{\infty} c_{ij}^{(0)}(x-u)^{i-1}(y-v)^{j-1},$$

we obtain for $i, j \geq 1$,

$$\phi_{11}(u, v) = \left(c_{11}^{(0)} \right)^{-1},$$

$$d_i^{(1)} = \frac{1}{c_{11}^{(0)}} \left(-\phi_{11}(u, v)c_{i+1,1}^{(0)} - \sum_{l=1}^{i-1} d_l^{(1)}c_{i+1-l,1}^{(0)} \right),$$

$$e_j^{(1)} = \frac{1}{c_{11}^{(0)}} \left(-\phi_{11}(u, v)c_{1,j+1}^{(0)} - \sum_{l=1}^{j-1} e_l^{(1)}c_{1,j+1-l}^{(0)} \right), \tag{11a}$$

$$c_{ij}^{(1)} = -\phi_{11}(u, v)c_{i+1,j+1}^{(0)} - \sum_{l=1}^{i} d_l^{(1)}c_{i+1-l,j+1}^{(0)} - \sum_{l=1}^{j} e_l^{(1)}c_{i+1,j+1-l}^{(0)};$$

and doing the same with (9b)

$$(x - u)(y - v) \sum_{i,j=1}^{\infty} c_{ij}^{(k)} (x - u)^{i-1} (y - v)^{j-1}$$

$$= \sum_{i,j=1}^{\infty} c_{ij}^{(k-2)} (x - u)^{i-1} (y - v)^{j-1}$$

$$- \left(\phi_{kk}(u, v) + \sum_{i=1}^{\infty} d_i^{(k)} (x - u)^i + \sum_{j=1}^{\infty} e_j^{(k)} (y - v)^j \right)$$

$$\times \sum_{i,j=1}^{\infty} c_{ij}^{(k-1)} (x - u)^{i-1} (y - v)^{j-1},$$

we find for $k \geq 2$ and $i, j \geq 1$,

$$\phi_{kk}(u, v) = c_{11}^{(k-2)} / c_{11}^{(k-1)},$$

$$d_i^{(k)} = \frac{1}{c_{11}^{(k-1)}} \left(c_{i+1,1}^{(k-2)} - \phi_{kk}(u, v) c_{i+1,1}^{(k-1)} - \sum_{l=1}^{i-1} d_l^{(k)} c_{i+1-l,1}^{(k-1)} \right),$$

$$e_j^{(k)} = \frac{1}{c_{11}^{(k-1)}} \left(c_{1,j+1}^{(k-2)} - \phi_{kk}(u, v) c_{1,j+1}^{(k-1)} - \sum_{l=1}^{j-1} e_l^{(k)} c_{1,j+1-l}^{(k-1)} \right), \tag{11b}$$

$$c_{ij}^{(k)} = c_{i+1,j+1}^{(k-2)} - \phi_{kk}(u, v) c_{i+1,j+1}^{(k-1)}$$

$$- \sum_{l=1}^{i} d_l^{(k)} c_{i+1-l,j+1}^{(k-1)} - \sum_{l=1}^{j} e_l^{(k)} c_{i+1,j+1-l}^{(k-1)}.$$

The values $\phi_{lk}(u, v)$ and $\phi_{kl}(u, v)$ for $l > k$, $k \geq 0$ can be computed from the knowledge of the $d_i^{(k)}$ and $e_j^{(k)}$, where $d_i^{(0)} = c_{i0}^{(0)}$ and $e_j^{(0)} = c_{0j}^{(0)}$ for $i, j \geq 1$, by applying the univariate Viscovatov-algorithm (5) to the series (10a) and (10b). To illustrate this technique we consider the following simple example. Take

$$f(x, y) = e^{x+y}$$

$$= 1 + x + y + \tfrac{1}{2} x^2 + xy + \tfrac{1}{2} y^2 + \tfrac{1}{6} x^3 + \tfrac{1}{2} x^2 y + \tfrac{1}{2} xy^2 + \tfrac{1}{6} y^3$$

$$+ \tfrac{1}{24} x^4 + \tfrac{1}{6} x^3 y + \tfrac{1}{4} x^2 y^2 + \tfrac{1}{6} xy^3 + \tfrac{1}{24} y^4 + \cdots$$

$$= \phi_{00}(u, v) + g_0(x) + h_0(y) + f_0(x, y),$$

where

$$\phi_{00}(u, v) = 1,$$

$$g_0(x) = x + \tfrac{1}{2} x^2 + \tfrac{1}{6} x^3 + \cdots,$$

$$h_0(y) = y + \tfrac{1}{2} y^2 + \tfrac{1}{6} y^3 + \cdots,$$

$$f_0(x, y) = xy \left(1 + \tfrac{1}{2} x + \tfrac{1}{2} y + \tfrac{1}{6} x^2 + \tfrac{1}{4} xy + \tfrac{1}{6} y^2 + \cdots \right).$$

In this case $(u, v) = (0, 0)$ and since the problem is completely symmetric we need only compute the coefficients $\phi_{lk}(u, v)$ for $l \geqslant k$ and $k \geqslant 0$. Using the formulas (11) we find

$$\phi_{11}(u, v) = 1,$$

$$d_1^{(1)} = -\tfrac{1}{2}, \quad d_2^{(1)} = \tfrac{1}{12}, \quad d_3^{(1)} = 0, \quad \ldots,$$

$$c_{11}^{(1)} = \tfrac{1}{4},$$

$$c_{21}^{(1)} = \tfrac{1}{12}, \qquad c_{12}^{(1)} = \tfrac{1}{12},$$

$$c_{31}^{(1)} = \tfrac{1}{48}, \qquad c_{22}^{(1)} = \tfrac{1}{36}, \qquad c_{13}^{(1)} = \tfrac{1}{48},$$

and

$$\phi_{22}(u, v) = 4,$$

$$d_1^{(2)} = \tfrac{2}{3}, \quad d_2^{(2)} = \tfrac{1}{9}, \quad \ldots.$$

Applying the univariate formulas (5) to the sequences

$$g_0(x) = x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots,$$

$$g_1(x) = -\tfrac{1}{2}x + \tfrac{1}{12}x^2 + 0x^3 + \cdots,$$

$$g_2(x) = \tfrac{2}{3}x + \tfrac{1}{9}x^2 + \cdots,$$

we obtain the BCF expansion for e^{x+y}:

$$1 + \left(\frac{x|}{|1} + \frac{x|}{|-2} + \frac{x|}{|-3} + \cdots \right) + \left(\frac{y|}{|1} + \frac{y|}{|-2} + \frac{y|}{|-3} + \cdots \right)$$

$$+ \cfrac{xy}{1 + \left(\frac{x|}{|-2} + \frac{x|}{|-3} + \frac{x|}{|2} + \cdots \right) + \left(\frac{y|}{|-2} + \frac{y|}{|-3} + \frac{y|}{|2} + \cdots \right)}$$

$$+ \cfrac{xy}{4 + \left(\frac{x|}{|3/2} + \frac{x|}{|-4} + \cdots \right) + \left(\frac{y|}{|3/2} + \frac{y|}{|-4} + \cdots \right)} + \cdots.$$

3. Multivariate Thiele continued fraction expansion

As mentioned above we shall now present a scheme to derive the $\phi_{lk}(u, v)$ analytically using a method analogous to Thiele's method for the univariate case. We shall therefore first rewrite the bivariate inverse differences of $f(x, y)$ as univariate inverse differences of univariate functions related to $f(x, y)$. For the reader familiar with [12] we point out that there Siemaszko considers $\mathbb{N}^2$ as a union of horizontal (or vertical) lines and so his bivariate reciprocal differences are essentially univariate in nature. In our paper $\mathbb{N}^2$ is considered as a union of prongs. Going from one prong to the next one involves both coordinates and this will have its implications when introducing bivariate reciprocal differences related to the bivariate inverse differences (7). The

following notations will be used throughout the paper to indicate finite subsequences of $x^{(\infty)} = \{x_0, x_1, \ldots\}$ and $y^{(\infty)} = \{y_0, y_1, \ldots\}$:

$$x^{(k)} = (x_0, \ldots, x_k), \quad k \geqslant 0, \qquad y^{(k)} = (y_0, \ldots, y_k), \quad k \geqslant 0.$$

From the formulas (7) and the definition of univariate inverse differences, one can easily see that

$$\phi_{00}[x_0][y_0] = f(x_0, y_0) = \phi_0^{d_0(x; y_0)}[x_0], \quad d_0(x; y_0) = f(x, y_0),$$

$$= \phi_0^{e_0(y; x_0)}[y_0], \quad e_0(y; x_0) = f(x_0, y),$$

and by induction, for $l \geqslant 0$

$$\phi_{l,0}[x_0, \ldots, x_l][y_0] = \phi_l^{d_0(x; y_0)}[x_0, \ldots, x_l]$$

and

$$\phi_{0,l}[x_0][y_0, \ldots, y_l] = \phi_l^{e_0(y; x_0)}[y_0, \ldots, y_l]. \tag{12a}$$

In general we can state the following.

Theorem 1. *For $k \geqslant 1$ and $l \geqslant k$,*

$$\phi_{l,k}[x_0, \ldots, x_l][y_0, \ldots, y_k] = \phi_{l-k+1}^{d_k(x; x^{(k-2)}, y^{(k)})}[x_{k-1}, \ldots, x_l] \tag{12b}$$

and

$$\phi_{k,l}[x_0, \ldots, x_k][y_0, \ldots, y_l] = \phi_{l-k+1}^{e_k(y; x^{(k)}, y^{(k-2)})}[y_{k-1}, \ldots, y_l],$$

where

$$d_k(x; x^{(k-2)}, y^{(k)}) = \frac{1}{\phi_{k-1,k}[x_0, \ldots, x_{k-2}, x][y_0, \ldots, y_k]} \tag{13a}$$

and

$$e_k(y; x^{(k)}, y^{(k-2)}) = \frac{1}{\phi_{k,k-1}[x_0, \ldots, x_k][y_0, \ldots, y_{k-2}, y]}. \tag{13b}$$

Proof. We shall only give the proof for $\phi_{l,k}[x_0, \ldots, x_l][y_0, \ldots, y_k]$ with $l \geqslant k$ since it is completely analogous for $\phi_{k,l}[x_0, \ldots, x_k][y_0, \ldots, y_l]$. The proof is by induction on l. For $l = k$ we have

$$\phi_{k,k}[x_0, \ldots, x_k][y_0, \ldots, y_k]$$

$$= \frac{x_k - x_{k-1}}{\dfrac{\phi_{k-1,k-1}[x_0, \ldots, x_{k-2}, x_k][y_0, \ldots, y_{k-2}, y_k] - \phi_{k-1,k-1}[x_0, \ldots, x_{k-2}, x_k][y_0, \ldots, y_{k-1}]}{y_k - y_{k-1}} - \dfrac{\phi_{k-1,k-1}[x_0, \ldots, x_{k-1}][y_0, \ldots, y_{k-2}, y_k] - \phi_{k-1,k-1}[x_0, \ldots, x_{k-1}][y_0, \ldots, y_{k-1}]}{y_k - y_{k-1}}}$$

$$= \frac{x_k - x_{k-1}}{\dfrac{1}{\phi_{k-1,k}[x_0, \ldots, x_{k-2}, x_k][y_0, \ldots, y_k]} - \dfrac{1}{\phi_{k-1,k}[x_0, \ldots, x_{k-1}][y_0, \ldots, y_k]}}.$$

If we introduce the notation $d_k(x; x^{(k-2)}, y^{(k)})$ for the univariate function of the variable x

$$\frac{1}{\phi_{k-1,k}[x_0,\ldots,x_{k-2}, x][y_0,\ldots,y_k]},$$

depending on the parameters $x^{(k-2)}$ and $y^{(k)}$ then, from the definition of univariate inverse differences,

$$\phi_{k,k}[x_0,\ldots,x_k][y_0,\ldots,y_k] = \phi_1^{d_k(x;x^{(k-2)},y^{(k)})}[x_{k-1}, x_k].$$

Assume now that for $k \leqslant n < l$

$$\phi_{n,k}[x_0,\ldots,x_n][y_0,\ldots,y_k] = \phi_{n-k+1}^{d_k(x; x^{(k-2)},y^{(k)})}[x_{k-1},\ldots,x_n],$$

then

$$\phi_{l,k}[x_0,\ldots,x_l][y_0,\ldots,y_k]$$

$$= \frac{x_l - x_{l-1}}{\phi_{l-1,k}[x_0,\ldots,x_{l-2}, x_l][y_0,\ldots,y_k] - \phi_{l-1,k}[x_0,\ldots,x_{l-1}][y_0,\ldots,y_k]}$$

$$= \frac{x_l - x_{l-1}}{\phi_{l-k}^{d_k(x;x^{(k-2)},y^{(k)})}[x_{k-1},\ldots,x_{l-2}, x_l] - \phi_{l-k}^{d_k(x;x^{(k-2)},y^{(k)})}[x_{k-1},\ldots,x_{l-1}]}$$

$$= \phi_{l-k+1}^{d_k(x;x^{(k-2)},y^{(k)})}[x_{k-1},\ldots,x_l]$$

which completes the proof. $\square$

With Theorem 1 in mind, we introduce *bivariate reciprocal differences* for $f(x, y)$ as follows. For $k \geqslant 1$ and $l \geqslant k$

$$\rho_{l,k}[x_0,\ldots,x_l][y_0,\ldots,y_k] = \rho_{l-k+1}^{d_k(x;x^{(k-2)},y^{(k)})}[x_{k-1},\ldots,x_l]$$

and

$$\rho_{k,l}[x_0,\ldots,x_k][y_0,\ldots,y_l] = \rho_{l-k+1}^{e_k(y;x^{(k)},y^{(k-2)})}[y_{k-1},\ldots,y_l],$$ (14a)

and for $l \geqslant 0$

$$\rho_{l,0}[x_0,\ldots,x_l][y_0] = \rho_l^{d_0(x;y_0)}[x_0,\ldots,x_l]$$

and

$$\rho_{0,l}[x_0][y_0,\ldots,y_l] = \rho_l^{e_0(y;x_0)}[y_0,\ldots,y_l].$$ (14b)

The bivariate reciprocal differences are thus defined by means of univariate reciprocal differences for the functions $d_k(x; x^{(k-2)}, y^{(k)})$ and $e_k(y; x^{(k)}, y^{(k-2)})$ given by (13). The following properties can therefore easily be checked.

Theorem 2. (a) *For* $l \geqslant 2$,

$$\rho_{00}[x_0][y_0] = \phi_{00}[x_0][y_0],$$

$$\rho_{10}[x_0, x_1][y_0] = \phi_{10}[x_0, x_1][y_0],$$

$$\rho_{l,0}[x_0,\ldots,x_l][y_0] = \phi_{l,0}[x_0,\ldots,x_l][y_0] + \rho_{l-2,0}[x_0,\ldots,x_{l-2}][y_0],$$

$$\rho_{01}[x_0][y_0, y_1] = \phi_{01}[x_0][y_0, y_1],$$

$$\rho_{0,l}[x_0][y_0,\ldots,y_l] = \phi_{0,l}[x_0][y_0,\ldots,y_l] + \rho_{0,l-2}[x_0][y_0,\ldots,y_{l-2}],$$

and for $k \geqslant 1$ and $l \geqslant k + 2$,

$$\rho_{k,k}[x_0, \ldots, x_k][y_0, \ldots, y_k] = \phi_{k,k}[x_0, \ldots, x_k][y_0, \ldots, y_k],$$

$$\rho_{k+1,k}[x_0, \ldots, x_{k+1}][y_0, \ldots, y_k]$$

$$= \phi_{k+1,k}[x_0, \ldots, x_{k+1}][y_0, \ldots, y_k] + d_k(x_{k-1}; x^{(k-2)}, y^{(k)}),$$

$$\rho_{l,k}[x_0, \ldots, x_l][y_0, \ldots, y_k]$$

$$= \phi_{l,k}[x_0, \ldots, x_l][y_0, \ldots, y_k] + \rho_{l-2,k}[x_0, \ldots, x_{l-2}][y_0, \ldots, y_k],$$

$$\rho_{k,k+1}[x_0, \ldots, x_k][y_0, \ldots, y_{k+1}]$$

$$= \phi_{k,k+1}[x_0, \ldots, x_k][y_0, \ldots, y_{k+1}] + e_k(y_{k-1}; x^{(k)}, y^{(k-2)}),$$

$$\rho_{k,l}[x_0, \ldots, x_k][y_0, \ldots, y_l]$$

$$= \phi_{k,l}[x_0, \ldots, x_k][y_0, \ldots, y_l] + \rho_{k,l-2}[x_0, \ldots, x_k][y_0, \ldots, y_{l-2}].$$

(b) *For $l \geqslant 0$ the bivariate reciprocal differences $\rho_{l,0}[x_0, \ldots, x_l][y_0]$ and $\rho_{0,l}[x_0][y_0, \ldots, y_l]$ are independent of the order of the points $x_0, \ldots, x_l$ and $y_0, \ldots, y_l$ while for $k \geqslant 1$ and $l \geqslant k$ the bivariate reciprocal differences $\rho_{l,k}[x_0, \ldots, x_l][y_0, \ldots, y_k]$ and $\rho_{k,l}[x_0, \ldots, x_k][y_0, \ldots, y_l]$ are independent of the order of points $x_{k-1}, \ldots, x_l$ and $y_{k-1}, \ldots, y_l$ respectively.*

The proof is not given since it is based in a straightforward way on the definition of the $\rho_{l,k}[x_0, \ldots, x_l][y_0, \ldots, y_k]$ and the properties of univariate reciprocal differences.

Let us now again turn to the problem of computing the bivariate inverse derivatives $\phi_{l,k}(u, v)$ in (8). We still assume that the function $f(x, y)$ is formally given by its bivariate Taylor series expansion. We can write, for $k \geqslant 1$ and $l \geqslant k$

$$\phi_{l,k}(u, v) = \lim_{\substack{x_i \to u \, y_j \to v \\ i = 0, \ldots, l \, j = 0, \ldots, k}} \phi_{l,k}[x_0, \ldots, x_l][y_0, \ldots, y_k]$$

$$= \lim_{\substack{x_i \to u \\ i = k-1, \ldots, l}} \lim_{\substack{x_0, \ldots, x_{k-2} \to u \\ y_0, \ldots, y_k \to v}} \phi_{l-k+1}^{d_k(x; x^{(k-2)}, y^{(k)})}[x_{k-1}, \ldots, x_l].$$

By induction we can show the following lemma. This result will enable us to write down the bivariate inverse derivative as a univariate inverse derivative, but computed for a limit function.

Lemma.

$$\lim_{\substack{x_0, \ldots, x_{k-2} \to u \\ y_0, \ldots, y_k \to v}} \phi_{l-k+1}^{d_k(x; x^{(k-2)}, y^{(k)})}[x_{k-1}, \ldots, x_l] = \phi_{l-k+1}^{d_k(x; u, v)}[x_{k-1}, \ldots, x_l] \qquad (15)$$

where

$$d_k(x; u, v) = \lim_{\substack{x_0, \ldots, x_{k-2} \to u \\ y_0, \ldots, y_k \to v}} d_k(x; x^{(k-2)}, y^{(k)})$$

$$= \lim_{\substack{x_0, \ldots, x_{k-2} \to u \\ y_0, \ldots, y_k \to v}} \frac{1}{\phi_{k-1,k}[x_0, \ldots, x_{k-2}, x][y_0, \ldots, y_k]}.$$

Proof. For the first step of the induction we have

$$\lim_{\substack{x_0,\ldots,x_{k-2}\to u \\ y_0,\ldots,y_k\to v}} \phi_1^{d_k(x;x^{(k-2)},y^{(k)})}[x_{k-1}, x_k]$$

$$= \lim_{\substack{x_0,\ldots,x_{k-2}\to u \\ y_0,\ldots,y_k\to v}} \frac{x_k - x_{k-1}}{d_k(x_k; x^{(k-2)}, y^{(k)}) - d_k(x_{k-1}; x^{(k-2)}, y^{(k)})}$$

$$= \frac{x_k - x_{k-1}}{d_k(x_k; u, v) - d_k(x_{k-1}; u, v)} = \phi_1^{d_k(x;u,v)}[x_{k-1}, x_k].$$

Assume now that for $k \leqslant n < l$

$$\lim_{\substack{x_0,\ldots,x_{k-2}\to u \\ y_0,\ldots,y_k\to v}} \phi_{n-k+1}^{d_k(x;x^{(k-2)},y^{(k)})}[x_{k-1},\ldots, x_n] = \phi_{n-k+1}^{d_k(x;u,v)}[x_{k-1},\ldots, x_n].$$

Then

$$\lim_{\substack{x_0,\ldots,x_{k-2}\to u \\ y_0,\ldots,y_k\to v}} \phi_{l-k+1}^{d_k(x;x^{(k-2)},y^{(k)})}[x_{k-1},\ldots, x_l]$$

$$= \lim_{\substack{x_0,\ldots,x_{k-2}\to u \\ y_0,\ldots,y_k\to v}} \frac{x_l - x_{l-1}}{\phi_{l-k}^{d_k(x;x^{(k-2)},y^{(k)})}[x_{k-1},\ldots, x_{l-2}, x_l] - \phi_{l-k}^{d_k(x;x^{(k-2)},y^{(k)})}[x_{k-1},\ldots, x_{l-1}]}$$

$$= \frac{x_l - x_{l-1}}{\phi_{l-k}^{d_k(x;u,v)}[x_{k-1},\ldots, x_{l-2}, x_l] - \phi_{l-k}^{d_k(x;u,v)}[x_{k-1},\ldots, x_{l-1}]}$$

$$= \phi_{l-k+1}^{d_k(x;u,v)}[x_{k-1},\ldots, x_l]. \qquad \square$$

The existence of

$$\lim_{\substack{x_0,\ldots,x_{k-2}\to u \\ y_0,\ldots,y_k\to v}} \phi_{k-1,k}[x_0,\ldots, x_{k-2}, x][y_0,\ldots, y_k]$$

is guaranteed by arguments similar to those that guarantee the existence of

$$\lim_{\substack{x_i\to u \\ i=0,\ldots,l}} \phi_l[x_0,\ldots, x_l]$$

in the univariate case. So we can write for $k \geqslant 1$ and $l \geqslant k$:

$$\phi_{l,k}(u, v) = \lim_{x_{k-1},\ldots,x_l\to u} \phi_{l-k+1}^{d_k(x;u,v)}[x_{k-1},\ldots, x_l]$$

$$= \phi_{l-k+1}^{d_k(x;u,v)}(u), \tag{16a}$$

and in a completely analogous way

$$\phi_{k,l}(u, v) = \lim_{y_{k-1},\ldots,y_l\to v} \phi_{l-k+1}^{e_k(y;u,v)}[y_{k-1},\ldots, y_l]$$

$$= \phi_{l-k+1}^{e_k(y;u,v)}(v) \tag{16b}$$

where

$$e_k(y; u, v) = \lim_{\substack{x_0,\ldots,x_k \to u \\ y_0,\ldots,y_{k-2} \to v}} e_k\left(y; x^{(k)}, y^{(k-2)}\right)$$

$$= \lim_{\substack{x_0,\ldots,x_k \to u \\ y_0,\ldots,y_{k-2} \to v}} \frac{1}{\phi_{k,k-1}[x_0,\ldots,x_k][y_0,\ldots,y_{k-2}, y]}.$$

For $k = 0$, we have from the formulas (12a)

$$\phi_{l,0}(u, v) = \phi_l^{d_0(x;u,v)}(u), \qquad d_0(x; u, v) = f(x, v),$$
$$\phi_{0,l}(u, v) = \phi_l^{e_0(y;u,v)}(v), \qquad e_0(y; u, v) = f(u, y). \tag{16c}$$

If the functions $d_k(x; u, v)$ and $e_k(y; u, v)$ are known, the relations (16) say that the computation of the $\phi_{l,k}(u, v)$ is essentially reduced to a univariate problem. Indeed, if we apply the univariate scheme (4) to $f(x) = d_k(x; u, v)$ we find for $k \geqslant 1$

$$\phi_{k,k}(u, v) = \phi_1^{d_k(x;u,v)}(u) = \left(\frac{\mathrm{d}}{\mathrm{d}x} d_k(x; u, v)\right)^{-1}_{|x=u},$$

and for $l > k$

$$\phi_{l,k}(u, v) = \phi_{l-k+1}^{d_k(x;u,v)}(u)$$
$$= (l - k + 1)\left(\frac{\mathrm{d}}{\mathrm{d}x} \rho_{l-k}^{d_k(x;u,v)}(x)\right)^{-1}_{|x=u}.$$

It would be nice if we could rewrite this last expression in terms of bivariate reciprocal differences. By a reasoning completely analogous to the one used to derive Lemma 1, we have

$$\rho_{l-k}^{d_k(x;u,v)}(x) = \lim_{x_{k-1},\ldots,x_{l-1} \to x} \rho_{l-k}^{d_k(x;u,v)}[x_{k-1},\ldots,x_{l-1}]$$

$$= \lim_{\substack{x_i \to x \\ i=k-1,\ldots,l-1}} \lim_{\substack{x_0,\ldots,x_{k-2} \to u \\ y_0,\ldots,y_k \to v}} \rho_{l-k}^{d_k(x;x^{(k-2)},y^{(k)})}[x_{k-1},\ldots,x_{l-1}].$$

Hence, by definition (14) of bivariate reciprocal differences

$$\rho_{l-k}^{d_k(x;u,v)}(x) = \lim_{\substack{x_0,\ldots,x_{k-2} \to u \\ y_0,\ldots,y_k \to v \\ x_{k-1},\ldots,x_{l-1} \to x}} \rho_{l-1,k}[x_0,\ldots,x_{l-1}][y_0,\ldots,y_k]$$

$$= \rho_{l-1,k}\big[\underbrace{u,\ldots,u}_{k-1}, \underbrace{x,\ldots,x}_{l-k+1}\big][v].$$

In this way, for $k \geqslant 1$ and $l \geqslant k + 1$,

$$\phi_{l,k}(u, v) = (l - k + 1)\left(\frac{\partial}{\partial x} \rho_{l-1,k}[u,\ldots,u, x,\ldots,x][y]\right)^{-1}_{|(x,y)=(u,v)},$$

and more generally we can write

$$\phi_{l,k}\big[\underbrace{u,\ldots,u}_{k-1}, \underbrace{x,\ldots,x}_{l-k+1}\big][y] = (l - k + 1)\left(\frac{\partial}{\partial x} \rho_{l-1,k}[u,\ldots,u, x,\ldots,x][y]\right)^{-1}. \tag{17}$$

For $k \geqslant 1$ and $l \geqslant k$, we introduce the notations

$$\phi_{lk}[x;\, u][y] = \phi_{lk}\left[\underbrace{u,\ldots,u}_{k-1},\, \underbrace{x,\ldots,x}_{l-k+2}\right]\left[\underbrace{y,\ldots,y}_{k+1}\right],$$

$$\rho_{lk}[x;\, u][y] = \rho_{lk}\left[\underbrace{u,\ldots,u}_{k-1},\, \underbrace{x,\ldots,x}_{l-k+2}\right]\left[\underbrace{y,\ldots,y}_{k+1}\right],$$

$$\phi_{kl}[x][y;\, v] = \phi_{kl}\left[\underbrace{x,\ldots,x}_{k+1}\right]\left[\underbrace{v,\ldots,v}_{k-1},\, \underbrace{y,\ldots,y}_{l-k+2}\right],$$

$$\rho_{kl}[x][y;\, v] = \rho_{kl}\left[\underbrace{x,\ldots,x}_{k+1}\right]\left[\underbrace{v,\ldots,v}_{k-1},\, \underbrace{y,\ldots,y}_{l-k+2}\right],$$

and we remark that $\phi_{l,k}[x;\, u][y]$, $\rho_{l,k}[x;\, u][y]$ are bivariate functions of the variables x and y depending on the parameter u, where u occurs as many times as indicated by the length of the parameter $x^{(k-2)}$ in the function $d_k(x;\, x^{(k-2)},\, y^{(k)})$. A similar remark holds for the functions $\phi_{kl}[x][y;\, v]$ and $\rho_{kl}[x][y;\, v]$.

If we evaluate the bivariate functions $\phi_{lk}[x;\, u][y]$ and $\phi_{kl}[x][y;\, v]$ at $(u,\, v)$ we get the bivariate inverse derivatives we are looking for to construct the Thiele BCF expansion. Note that this evaluation is performed by taking limits using de l'Hôpital's rule. By Theorem 2 we can compute $\rho_{l-1,k}[x;\, u][y]$ in (17) as

$$\rho_{k,k}[x;\, u][y] = \phi_{k,k}[x;\, u][y],$$

$$\rho_{k+1,k}[x;\, u][y] = \phi_{k+1,k}[x;\, u][y] + d_k(x;\, u,\, y),$$

$$\rho_{l,k}[x;\, u][y] = \phi_{l,k}[x;\, u][y] + \rho_{l-2,k}[x;\, u][y], \quad l > k+1.$$

Grouping all these formulas yields the complete computation scheme for the bivariate inverse derivatives:

0*th prong*:

$$d_0(x;\, u,\, y) = f(x,\, y),$$

$$\phi_{0,0}(u,\, v) = f(u,\, v) = \rho_{0,0}(u,\, v),$$

$$\phi_{1,0}(u,\, v) = \left(\frac{\partial f}{\partial x}(x,\, y)\right)^{-1}_{|(x,y)=(u,v)} = \rho_{1,0}(u,\, v);$$

$l \geqslant 2$:

$$\phi_{l,0}(u,\, v) = l\left(\frac{\partial \rho_{l-1,0}(x,\, y)}{\partial x}\right)^{-1}_{|(x,y)=(u,v)},$$

$$\rho_{l,0}(x,\, y) = \phi_{l,0}(x,\, y) + \rho_{l-2,0}(x,\, y).$$

$$e_0(y;\, x,\, v) = f(x,\, y),$$

$$\phi_{0,0}(u,\, v) = f(u,\, v) = \rho_{0,0}(u,\, v), \qquad \phi_{0,1}(u,\, v) = \left(\frac{\partial f}{\partial y}(x,\, y)\right)^{-1}_{|(x,y)=(u,v)} = \rho_{0,1}(u,\, v);$$

$l \geqslant 2$:

$$\phi_{0,l}(u, v) = l \left(\frac{\partial \rho_{0,l-1}(x, y)}{\partial y} \right)_{|(x,y)=(u,v)},$$

$$\rho_{0,l}(x, y) = \phi_{0,l}(x, y) + \rho_{0,l-2}(x, y).$$

kth prong:

$$d_k(x; u, y) = \frac{1}{\phi_{k-1,k}[u,\ldots, u, x][y,\ldots, y]},$$

$$e_k(y; x, v) = \frac{1}{\phi_{k,k-1}[x,\ldots, x][v,\ldots, v, y]},$$

with the usual recursion for inverse differences where it must be clear from above how to deal with inverse differences on the $(k-1)$th prong that have a number of coinciding interpolation points

$$\phi_{k,k}[x; u][y] = \left(\frac{\mathrm{d}}{\mathrm{d}x} d_k(x; u, y) \right)^{-1} = \rho_{k,k}[x; u][y],$$

$$\phi_{k,k}(u, v) = \phi_{k,k}[x; u][y]_{|(x,y)=(u,v)};$$

$l \geqslant k+1$:

$$\phi_{l,k}[x; u][y] = (l - k + 1) \left(\frac{\partial}{\partial x} \rho_{l-1,k}[x; u][y] \right)^{-1},$$

$$\phi_{l,k}(u, v) = \phi_{l,k}[x; u][y]_{|(x,y)=(u,v)},$$

$$\rho_{k+1,k}[x; u][y] = \phi_{k+1,k}[x; u][y] + d_k(x; u, y),$$

$$\rho_{l,k}[x; u][y] = \phi_{l,k}[x; u][y] + \rho_{l-2,k}[x; u][y].$$

$$\phi_{k,k}[x][y; v] = \left(\frac{\mathrm{d}}{\mathrm{d}y} e_k(y; x, v) \right)^{-1} = \rho_{k,k}[x][y; v],$$

$$\phi_{k,k}(u, v) = \phi_{k,k}[x][y; v]_{|(x,y)=(u,v)};$$

$l \geqslant k+1$:

$$\phi_{k,l}[x][y; v] = (l - k + 1) \left(\frac{\partial}{\partial y} \rho_{k,l-1}[x][y; v] \right)^{-1},$$

$$\phi_{k,l}(u, v) = \phi_{k,l}[x][y; v]_{|(x,y)=(u,v)},$$

$$\rho_{k,k+1}[x][y; v] = \phi_{k,k+1}[x][y; v] + e_k(y; x, v),$$

$$\rho_{k,l}[x][y; v] = \phi_{k,l}[x][y; v] + \rho_{k,l-2}[x][y; v].$$

With this computation scheme for the coefficients $\phi_{k,l}(u, v)$ it is possible to construct the BCF expansion (8) for $f(x, y)$ as the limiting value of the interpolating BCF (6). However, it should be obvious that there is no guarantee that the constructed BCF expansion will actually converge to the function $f(x, y)$. For convergence results we refer the reader to [7].

We shall illustrate the technique introduced here by applying it to the function $f(x, y) = e^{x+y}$. On the 0th prong we find with $(u, v) = (0, 0)$:

$$d_0(x; 0, y) = e^{x+y},$$

$$\phi_{0,0}(x, y) = e^{x+y} = \rho_{0,0}(x, y), \qquad \phi_{0,0}(0, 0) - 1,$$

$$\phi_{1,0}(x, y) = e^{-(x+y)} = \rho_{1,0}(x, y), \quad \phi_{1,0}(0, 0) = 1,$$

$$\phi_{2,0}(x, y) = -2 e^{x+y}, \qquad\qquad \phi_{2,0}(0, 0) = -2,$$

$$\rho_{2,0}(x, y) = -e^{x+y},$$

$$\rho_{3,0}(x, y) = -3 e^{-(x+y)}, \qquad\qquad \phi_{3,0}(0, 0) = -3.$$

By the symmetric nature of the function $f(x, y)$ we need only to compute the bivariate inverse derivatives $\phi_{l,k}(u, v)$ for $l \geq k$. So we can immediately go on to the first prong:

$$d_1(x; 0, y) = \frac{1}{\phi_{0,1}[x][y, y]} = e^{x+y},$$

$$\phi_{1,1}[x; 0][y] = e^{-(x+y)} = \rho_{1,1}[x; 0][y], \qquad \phi_{1,1}(0, 0) = 1,$$

$$\phi_{2,1}[x; 0][y] = -2 e^{x+y}, \qquad\qquad \phi_{2,1}(0, 0) = -2,$$

$$\rho_{2,1}[x; 0][y] = -e^{x+y},$$

$$\phi_{3,1}[x; 0][y] = -3 e^{-(x+y)}, \qquad\qquad \phi_{3,1}(0, 0) = -3,$$

$$\rho_{3,1}[x; 0][y] = -2 e^{-(x+y)},$$

$$\phi_{4,1}[x; 0][y] = 2e^{x+y}, \qquad\qquad \phi_{4,1}(0, 0) = 2.$$

As mentioned above the values $\phi_{l,1}[x; u][y]$ do not actually depend on the parameter u since in $d_1(x; x^{(-1)}, y^{(1)})$ the subsequence $x^{(-1)}$ is empty. On the 2nd prong we have

$$d_2(x; 0, y) = 1/\phi_{1,2}[0, x][y, y, y],$$

where by Theorem 1

$$\phi_{1,2}[0, x][y, y, y] = \phi_2^{e_1(y;0,x)}(y)$$

with

$$e_1(y; 0, x) = (\phi_{1,0}[0, x][y])^{-1}.$$

Since $(e_1(y; 0, x))^{-1} = (e^{x+y} - e^y)/x$ we can apply the univariate scheme to compute $\phi_{1,2}[0, x][y, y, y]$ and find

$$d_2(x; 0, y) = -\tfrac{1}{2}x/(e^{x+y} - e^y),$$

$$\phi_{2,2}[x; 0][y] = -2\frac{(e^{x+y} - e^y)^2}{e^{x+y} - e^y - x\, e^{x+y}} = \rho_{2,2}[x; 0][y], \qquad \phi_{2,2}(0, 0) = 4,$$

$$\phi_{3,2}[x; 0][y] = -\frac{(e^{x+y} - e^y - x\, e^{x+y})^2}{e^{x+y}(e^{x+y} - e^y)(2\, e^{x+y} - 2\, e^y - x\, e^{x+y} - x\, e^y)},$$

$$\phi_{3,2}(0, 0) = \tfrac{3}{2}.$$

We note that automatic differentiation can be used to compute and evaluate, by de l'Hôpital's rule, the functions $\phi_{l,k}[x; 0][y]$ in order to increase the usefulness of this scheme and to avoid laborious and possibly erroneous work [9,11].

With the coefficients $\phi_{k,l}(0, 0)$ obtained in this way we again find the BCF expansion for e^{x+y} which was given at the end of Section 2.

References

[1] A. Cuyt, A review of multivariate Padé approximation theory, *J. Comput. Appl. Math.* **12** (1985) 221–232.

[2] A. Cuyt and B. Verdonk, A review of branched continued fraction theory for the construction of multivariate rational approximants, *Appl. Numer. Math.*, to appear.

[3] A. Cuyt and B. Verdonk, Multivariate rational interpolation, *Computing* **34** (1985) 41–61.

[4] A. Cuyt and L. Wuytack, *Nonlinear Methods in Numerical Analysis* (North-Holland, Amsterdam, 1986).

[5] K. Kuchminskaya, Corresponding and associated branched continued fractions for double power series (in Russian), *Dokl. Akad. Nauk Ukr. SSR, Ser. A* **7** (1978) 614–617.

[6] K. Kuchminskaya, On approximation of functions by continued and branched continued fractions, *Mat. Met. Fiz. Meh. Polya* **12** (1980) 3–10.

[7] K. Kuchminskaya, On the convergence of two-dimensional continued fractions, in: *Constructive Theory of Functions* (Publ. House Bulgarian Acad. Sci., Sofia, 1984) 501–506.

[8] K. Kuchminskaya and W. Siemaszko, Rational approximation and interpolation of functions by branched continued fractions, in: *Rational Approximation and Its Applications in Mathematics and Physics*, J. Gilewicz, M. Pindor and W. Siemaszko, Eds., Lecture Notes Math. **1237** (Springer, Berlin, 1987) 24–40.

[9] U. Kulisch, ed., *PASCAL-SC: A PASCAL Extension for Scientific Computation*, Wiley-Teubner Ser. Comput. Sci. (Wiley/Teubner, London/Stuttgart, 1987).

[10] J. Murphy and M. O'Donohoe, A two-variable generalization of the Stieltjes-type continued fraction, *J. Comput. Appl. Math.* **4** (1978) 181–190.

[11] L. Rall, *Automatic Differentiation: Techniques and Applications*, Lecture Notes Comput. Sci. **120** (Springer, Berlin, 1981).

[12] W. Siemaszko, Thiele-type branched continued fractions for two-variable functions, *J. Comput. Appl. Math.* **9** (1983) 137–153.