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General Order Newton-Padé Approximants for Multivariate Functions

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Summary. Padé approximants are a frequently used tool for the solution of mathematical problems. One of the main drawbacks of their use for multivariate functions is the calculation of the derivatives of $f(x_1, \ldots, x_p)$. Therefore multivariate Newton-Padé approximants are introduced; their computation will only use the value of f at some points. In Sect. 1 we shall repeat the univariate Newton-Padé approximation problem which is a rational Hermite interpolation problem. In Sect. 2 we sketch some problems that can arise when dealing with multivariate interpolation. In Sect. 3 we define multivariate divided differences and prove some lemmas that will be useful tools for the introduction of multivariate Newton-Padé approximants in Sect. 4. A numerical example is given in Sect. 5, together with the proof that for $p=1$ the classical Newton-Padé approximants for a univariate function are obtained.

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1. Univariate Newton-Padé Approximants

The Newton-Padé approximation problem is a rational Hermite interpolation problem.

Let f be a real-valued function, whose derivatives $f_j^{(k)}$ $(k=0, \ldots, r_j)$ are given in distinct points x_j, $j=0, \ldots, l$. Let n and m be chosen such that

$$n+m+1=\sum_{j=0}^{l}(r_j+1)$$

Define

$$R_m^n(x)=\left\{\frac{p(x)}{q(x)}\middle| p(x)=\sum_{i=0}^{n} a_i x^i, q(x)=\sum_{j=0}^{m} b_j x^j\right\}.$$

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The rational Hermite interpolation problem consists in finding an element $\frac{p}{q}$ in $R_m^n(x)$ such that

$$(f\cdot q-p)^{(k)}(x_j)=0 \qquad k=0,\dots,r_j \qquad \text{for } j=0,\dots,l \tag{1a}$$

Problem (1a) is the way to interprete the rational interpolation problem

$$(f\cdot q-p)(x_i)=0 \qquad i=0,\dots,n+m$$

if some of the x_i coincide.

In [11] is proved that we have at least one nontrivial solution of (1a) and that two different solutions p_1, q_1 and p_2, q_2 are equivalent, i.e. that $p_1 q_2(x) = p_2 q_1(x)$.

In [3] the rational Hermite interpolation problem is reformulated as follows. In a formal manner we can construct for $f(x)$ the Newton interpolation series

$$f(x)=\sum_{i=0}^{\infty} f[x_0,\dots,x_i]\,B_i(x)$$

where $B_i(x)=\prod_{k=0}^{i-1}(x-x_k)$ and $f[x_0,\dots,x_i]$ is a divided difference with possible coalescence of points x_i.

Redefine

$$R_m^n(x)=\left\{\frac{p(x)}{q(x)}\,\middle|\,p(x)=\sum_{i=0}^{n} a_i B_i(x),\, q(x)=\sum_{j=0}^{m} b_j B_j(x)\right\}$$

and calculate $\frac{p}{q}$ such that

$$(f\cdot q-p)(x)=\sum_{k\geqq n+m+1} d_k B_k(x). \tag{1b}$$

It is easy to see that the problems (1a) and (1b) are equivalent. The Newton-Padé approximant is now defined as the irreducible form $\frac{p_0}{q_0}$ of a solution $\frac{p(x)}{q(x)}$ of (1a) or (1b). Now $\frac{p_0}{q_0}$ itself does not necessarily satisfy (1a) or (1b) anymore. More information about Newton-Padé approximants can be found in [3] and [2].

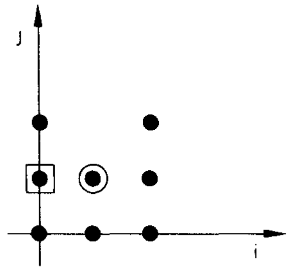
The following important theorem concerning the interpolation properties of $\frac{p}{q}$ with p and q satisfying (1a) or (1b) was proved in [8, p. 487]

Theorem 1.1. *If* $q(x_j)\neq 0$ *then* $f^{(k)}(x_j)=\left(\frac{p}{q}\right)^{(k)}(x_j)$ *for* $k=0,\dots,r_j$ *and* $j=0,\dots,l$.

This result will be generalized to the multivariate case in Sect. 4.

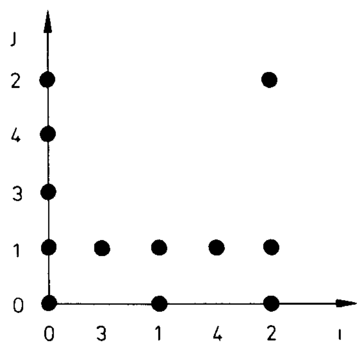
2. Multivariate Interpolation Problems

For the sake of simplicity we restrict ourselves to the case of two variables because the generalization to more than two variables is straightforward. Consider for instance the following set of data at points (x_i, y_j).



Where a circle indicates that $\frac{\partial f}{\partial x}$ is given and a square indicates that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y^2}$ are provided, in addition to $f_{ij} = f(x_i, y_j)$.

This situation is equivalent with



(2a)

where we let $x_3 \to x_0$, $x_4 \to x_1$, $y_3 \to y_1$ and $y_4 \to y_1$.

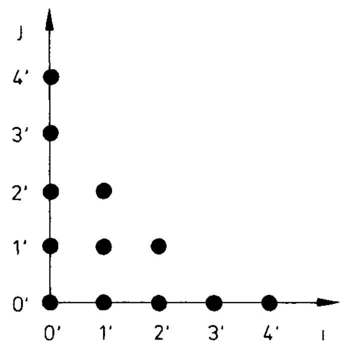
If we want to interpolate these (x_i, y_j, f_{ij}) by using tensor product methods following [10] then the data f_{ij} and the numbering of the x_i and y_j have to be given such that

a) x_0 is that x-coordinate for which the number of y-coordinates at which data are given is maximal, x_1 should be that one of the leftover points for which the same is true, and so on

b) y_0 is that y-coordinate for which the number of x-coordinates at which data are given is maximal, ...

c) the data set has the inclusion property, meaning that when a point belongs to the data set then the rectangular subset of points emanating from the origin with the given point as its furthermost corner also lies in the data set.

For the situation (2a) this is clearly not the case. So we try to reformulate the given problem by introducing a new numbering $(x_{i'}, y_{j'})$.



(2b)

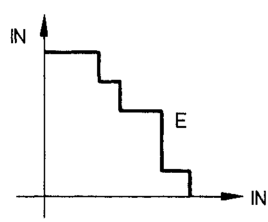
$$x_{0'}=x_0,\ x_{1'}=x_2,\ x_{2'}=x_1,\ x_{3'}=x_4,\ x_{4'}=x_3$$
$$y_{0'}=y_1,\ y_{1'}=y_0,\ y_{2'}=y_2,\ y_{3'}=y_4,\ y_{4'}=y_3.$$

The interpolation problems that can be reduced to situation (2b) are of course not the most general ones. But it is important to gain insight in these situations before generalizing to other sets of data.

In the sequel of the text we shall assume that the given interpolation problem is already structured as in (2b); this will enable us to adapt the notation (x_i, y_j) instead of $(x_{i'}, y_{j'})$.

3. Multivariate Divided Differences

Let the function values f_{ij} be given in the points (x_i, y_j) with $(i,j)\in E\subseteq \mathbb{N}^2$ where E has the inclusion property, i.e. if $(i,j)\in E$ then $(k,l)\in E$ for $k\leqq i$ and $l\leqq j$



We know from the previous section how to deal with coalescent interpolation points.

Consider the following set of basis functions for the real-valued polynomials in two variables:

$$B_{ij}(x,y)=\prod_{k=0}^{i-1}(x-x_k)\prod_{l=0}^{j-1}(y-y_l).$$

Clearly $B_{ij}(x,y)$ is a bivariate polynomial of degree $i+j$.

In order to write a formal bivariate Newton interpolation series

$$f(x,y)=\sum_{(i,j)\in\mathbb{N}^2} c_{ij}B_{ij}(x,y) \tag{3}$$

we introduce bivariate divided differences as follows

$$f[x_0][y_0]=f(x_0,y_0)$$

$$f[x_0][y_0,\dots,y_s]=\frac{f[x_0][y_1,\dots,y_s]-f[x_0][y_0,\dots,y_{s-1}]}{y_s-y_0}$$

$$f[x_0,\dots,x_r][y_0]=\frac{f[x_1,\dots,x_r][y_0]-f[x_0,\dots,x_{r-1}][y_0]}{x_r-x_0}$$

$$\begin{aligned}f[x_0,\dots,x_r][y_0,\dots,y_s]&\\ =\frac{f[x_0,\dots,x_r][y_1,\dots,y_s]-f[x_0,\dots,x_r][y_0,\dots,y_{s-1}]}{y_s-y_0}&\\ =\frac{f[x_1,\dots,x_r][y_0,\dots,y_s]-f[x_0,\dots,x_{r-1}][y_0,\dots,y_s]}{x_r-x_0}&.\end{aligned}$$

The last equality is not a demand of the definition but can easily be proved by induction.

From now on we shall most of the times use the abbreviated notation $f_{pr,qs}$ for $f[x_p,\dots,x_r][y_q,\dots,y_s]$ with the convention that $f_{pr,qs}=0$ if $p>r$ or $s>q$.

Lemma 3.1. *$f_{pr,qs}$ is independent of the order of the points $x_p,\dots,x_r$ and $y_q,\dots,y_s$.*

Proof. The proof is only a modification of the proof for univariate divided differences. □

When certain interpolation points in $f_{pr,qs}$ coincide, we must bear in mind the following remarks.

Let r_i be a positive integer indicating that r_i+1 of the x-coordinates in E coincide with x_i and let s_j indicate that s_j+1 of the y-coordinates in E coincide with y_j. These coalescent x- and y-coordinates are not necessarily consecutive. To indicate which x- or y-coordinates coincide respectively with x_i or y_j we introduce the following notation:

$i(0),\dots,i(r_i)$ denote the numbers of the x-coordinates coinciding with x_i

and analogously

$j(0),\dots,j(s_j)$ denote the numbers of the y-coordinates coinciding with y_j.

For the calculation of the divided differences we need then the starting values

$$f[x_{i(0)},\dots,x_{i(k)}][y_l]=\left.\frac{\partial^k f}{\partial x^k}\right|_{(x_i,y_l)} \qquad 0\leqq k\leqq r_i$$

$$f[x_k][y_{j(0)},\dots,y_{j(l)}]=\left.\frac{\partial^l f}{\partial y^l}\right|_{(x_k,y_j)} \qquad 0\leqq l\leqq s_j$$

and

$$f[x_{i(0)}, \ldots, x_{i(k)}][y_{j(0)}, \ldots, y_{j(l)}] = \left.\frac{\partial^{k+l} f}{\partial x^k \partial y^l}\right|_{(x_i, y_j)}$$

$$0 \leqq k \leqq r_i \quad \text{and} \quad 0 \leqq l \leqq s_j$$

The coefficients c_{ij} in (3) are now given by [1, 10]

$$c_{ij} = f[x_0, \ldots, x_i][y_0, \ldots, y_j] = f_{0i,0j}.$$

So we can write in a purely formal manner

$$f(x, y) = \sum_{(i,j) \in \mathbb{N}^2} f_{0i,0j} B_{ij}(x, y).$$

Before going on to the next section, let us formulate and prove the following two lemmas which will play an important role in our discussion: the first lemma is a generalization of the Leibniz rule for differentiating a product of functions and the second one concerns the basisfunctions $B_{ij}(x, y)$.

Lemma 3.2.

$$\begin{aligned}(f \cdot g)&[x_p, \ldots, x_r][y_q, \ldots, y_s] \\ &= \sum_{\mu=p}^{r} \sum_{\nu=q}^{s} f[x_p, \ldots, x_\mu][y_q, \ldots, y_\nu] \cdot g[x_\mu, \ldots, x_r][y_\nu, \ldots, y_s]\end{aligned}$$

Proof. The proof is by induction.

First we observe that for all $p \leqq r$ and for all q [9, p. 18]

$$(f \cdot g)[x_p, \ldots, x_r][y_q] = \sum_{\mu=p}^{r} f[x_p, \ldots, x_\mu][y_q] \cdot g[x_\mu, \ldots, x_r][y_q]$$

To proceed we assume that the product rule is valid for

$$(f \cdot g)[x_p, \ldots, x_r][y_q, \ldots, y_{s-1}]$$

with $q \leqq s-1$ and $p \leqq r$.

Now

$$\begin{aligned}(f \cdot g)&[x_p, \ldots, x_r][y_q, \ldots, y_s] \\ &= \frac{(f \cdot g)_{pr,q+1s} - (f \cdot g)_{pr,qs-1}}{y_s - y_q} \\ &= \frac{1}{y_s - y_q}\left(\sum_{\mu=p}^{r} \sum_{\nu=q+1}^{s} f_{p\mu,q+1\nu} \cdot g_{\mu r,\nu s} - \sum_{\mu=p}^{r} \sum_{\nu=q}^{s-1} f_{p\mu,q\nu} \cdot g_{\mu r,\nu s-1}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{y_s - y_q} \left(\sum_{\mu=p}^{r} \sum_{\nu=q+1}^{s} (f_{p\mu,q+1\nu} - f_{p\mu,q\nu-1}) \cdot g_{\mu r,\nu s} \right.\\
&\quad \left. - \sum_{\mu=p}^{r} \sum_{\nu=q+1}^{s} f_{p\mu,q\nu-1} \cdot (g_{\mu r,\nu-1 s-1} - g_{\mu r,\nu s}) \right)\\
&= \frac{1}{y_s - y_q} \sum_{\mu=p}^{r} \sum_{\nu=q+1}^{s} (f_{p\mu,q\nu} \cdot g_{\mu r,\nu s} \cdot (y_\nu - y_q) + f_{p\mu,q\nu-1} \cdot g_{\mu r,\nu-1 s} \cdot (y_s - y_{\nu-1}))\\
&= \sum_{\mu=p}^{r} f_{p\mu,qs} \cdot g_{\mu r,ss}\\
&\quad + \frac{1}{y_s - y_q} \sum_{\mu=p}^{r} \sum_{\nu=q+1}^{s-1} (f_{p\mu,q\nu} \cdot g_{\mu r,\nu s} \cdot (y_\nu - y_q) + f_{p\mu,q\nu} \cdot g_{\mu r,\nu s} \cdot (y_s - y_\nu))\\
&\quad + \sum_{\mu=p}^{r} f_{p\mu,qq} \cdot g_{\mu r,qs}\\
&= \sum_{\mu=p}^{r} \sum_{\nu=q}^{s} f_{p\mu,q\nu} \cdot g_{\mu r,\nu s}. \quad \square
\end{aligned}
$$

Lemma 3.3. *For $k+l \geqq i+j$ the product*

$$B_{ij}(x,y) \cdot B_{kl}(x,y) = \sum_{\mu=0}^{i} \sum_{\nu=0}^{j} \lambda_{\mu\nu} B_{k+\mu,l+\nu}(x,y)$$

Proof. We write $B_{ij}(x,y) = B_{i0}(x,y) \cdot B_{0j}(x,y)$.

Since

$$B_{i0}(x,y) = \prod_{r=0}^{i-1} (x - x_r) = \sum_{\mu=0}^{i} \alpha_\mu \prod_{r=k}^{k+\mu-1} (x - x_r)$$

and

$$B_{0j}(x,y) = \prod_{r=0}^{j-1} (y - y_r) = \sum_{\nu=0}^{j} \beta_\nu \prod_{r=l}^{l+\nu-1} (y - y_r)$$

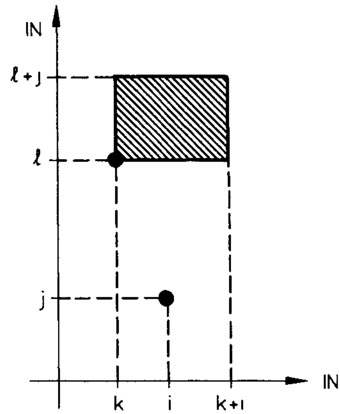
we have

$$
\begin{aligned}
B_{kl}(x,y) \cdot B_{ij}(x,y) &= (B_{kl}(x,y) \cdot B_{i0}(x,y)) \cdot B_{0j}(x,y)\\
&= \left(\sum_{\mu=0}^{i} \alpha_\mu B_{k+\mu,l}(x,y) \right) \cdot B_{0j}(x,y)\\
&= \sum_{\nu=0}^{j} \sum_{\mu=0}^{i} \alpha_\mu \beta_\nu B_{k+\mu,l+\nu}(x,y)
\end{aligned}
$$

which gives the desired formula for $\lambda_{\mu\nu} = \alpha_\mu \cdot \beta_\nu$. $\square$

A figure in $\mathbb{N}^2$ will clarify the meaning of this lemma.

If we multiply $B_{ij}(x, y)$ with $B_{kl}(x, y)$ and $k+l \geqq i+j$, then in the product the only occurring $B_{rs}(x, y)$ are those with (r, s) lying in the shaded rectangle.



4. Multivariate General Order Newton-Padé Approximants

Because Levin's introduction of multivariate Padé approximants in [6] is perhaps the most general one, we intend to generalize it for multivariate Newton-Padé approximants. We will again restrict ourselves to the case of two variables, because the generalization to more than two variables is only notationally more difficult.

With any finite subset D of $\mathbb{N}^2$ we will associate a polynomial

$$\sum_{(i,j)\in D} b_{ij} B_{ij}(x, y)$$

and we will call D the rank of the polynomial.

Given the double Newton series

$$f(x, y) = \sum_{(i,j)\in \mathbb{N}^2} f_{0i,0j} B_{ij}(x, y)$$

we choose three subsets N, D and E of $\mathbb{N}^2$ and construct an $[N/D]_E$ Newton-Padé approximant to $f(x, y)$ as follows:

$$p(x, y) = \sum_{(i,j)\in N} a_{ij} B_{ij}(x, y) \quad (N \text{ from "numerator"}) \tag{3a}$$

$$q(x, y) = \sum_{(i,j)\in D} b_{ij} B_{ij}(x, y) \quad (D \text{ from "denominator"}) \tag{3b}$$

$$(f \cdot q - p)(x, y) = \sum_{(i,j)\in \mathbb{N}^2 \setminus E} d_{ij} B_{ij}(x, y) (E \text{ from "equations"}) \tag{3c}$$

We select N, D and E such that

D has $m+1$ elements, numbered $(i_0, j_0), \ldots, (i_m, j_m)$

$N \subset E$

E has the inclusion property
$E \setminus N$ has at least m elements
Clearly the coefficients d_{ij} in

$$(f \cdot q - p)(x, y) = \sum_{(i,j) \in \mathbb{N}^2} d_{ij} B_{ij}(x, y)$$

are

$$d_{ij} = (f \cdot q - p)_{0i, 0j}.$$

So the conditions (3c) are equivalent with

$$(f \cdot q - p)_{0i, 0j} = 0 \qquad \text{for } (i,j) \text{ in } E \tag{4}$$

Now the system of Eqs. (4) can be divided into a nonhomogeneous and a homogeneous part:

$$(f \cdot q)_{0i, 0j} = p_{0i, 0j} \qquad \text{for } (i,j) \text{ in } N \tag{4a}$$

$$(f \cdot q)_{0i, 0j} = 0 \qquad \text{for } (i,j) \text{ in } E \setminus N. \tag{4b}$$

Let's take a look at the conditions (4b).

Suppose that E is such that exactly m of the homogeneous Eqs. (4b) are linearly independent; we number the respective m elements in $E \setminus N$ with $(h_1, k_1), \ldots, (h_m, k_m)$ and define the set

$$H = \{(h_1, k_1), \ldots, (h_m, k_m)\} \subseteq E \setminus N \qquad (H \text{ from "homogeneous equations"}).$$

By means of Lemma 3.2 we have

$$(f \cdot q)_{0i, 0j} = (q \cdot f)_{0i, 0j} = \sum_{r=0}^{i} \sum_{s=0}^{j} q_{0r, 0s} \cdot f_{ri, sj}.$$

Since the only nontrivial $q_{0r, 0s}$ are the ones with

$$(r, s) \in D = \{(i_0, j_0), \ldots, (i_m, j_m)\}$$
$$(f \cdot q)_{0i, 0j} = \sum_{(r,s) \in D} b_{rs} f_{ri, sj}$$

So the homogeneous system of m equations in $m+1$ unknowns is

$$\begin{pmatrix} f_{i_0 h_1, j_0 k_1} & \cdots & f_{i_m h_1, j_m k_1} \\ \vdots & & \vdots \\ f_{i_0 h_m, j_0 k_m} & \cdots & f_{i_m h_m, j_m k_m} \end{pmatrix} \begin{pmatrix} b_{i_0, j_0} \\ \vdots \\ b_{i_m, j_m} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5}$$

As we suppose the rank of the coefficient matrix to be maximal, a solution $q(x, y)$ is given by

$$q(x, y) = \begin{vmatrix} B_{i_0 j_0}(x, y) & \cdots & B_{i_m j_m}(x, y) \\ f_{i_0 h_1, j_0 k_1} & \cdots & f_{i_m h_1, j_m k_1} \\ \vdots & & \vdots \\ f_{i_0 h_m, j_0 k_m} & \cdots & f_{i_m h_m, j_m k_m} \end{vmatrix}$$

By the conditions (4a) and Lemma 3.2 we find

$$\begin{aligned}
p(x,y) &= \sum_{(i,j)\in N} a_{ij} B_{ij}(x,y) \\
&= \sum_{(i,j)\in N} p_{0i,0j} B_{ij}(x,y) \\
&= \sum_{(i,j)\in N} (q\cdot f)_{0i,0j} B_{ij}(x,y) \\
&= \sum_{(i,j)\in N} \sum_{r=0}^{i} \sum_{s=0}^{j} q_{0r,0s} f_{ri,sj} B_{ij}(x,y) \\
&= \sum_{(r,s)\in D} b_{rs} \Big(\sum_{(i,j)\in N} f_{ri,sj} B_{ij}(x,y) \Big).
\end{aligned}$$

Consequently a determinant representation for $p(x,y)$ is given by

$$p(x,y) = \begin{vmatrix} \sum_{(i,j)\in N} f_{i_0 i, j_0 j} B_{ij}(x,y) & \cdots & \sum_{(i,j)\in N} f_{i_m i, j_m j} B_{ij}(x,y) \\ f_{i_0 h_1, j_0 k_1} & \cdots & f_{i_m h_1, j_m k_1} \\ \vdots & & \vdots \\ f_{i_0 h_m, j_0 k_m} & \cdots & f_{i_m h_m, j_m k_m} \end{vmatrix}$$

Remark the fact that if all the interpolation points coincide with the origin, then these general order Newton-Padé approximants reduce to Levin's general order Padé approximants because in that case

$$B_{ij}(x,y) = x^i y^j$$

and

$$f_{ih,jk} = \left. \frac{\partial^{h-i+k-j} f}{\partial x^{h-i} \partial y^{k-j}} \right|_{(0,0)}$$

If $q(x_k, y_l) \neq 0$ for $(k,l)\in E$ then $\frac{1}{q}(x,y)$ can be written as

$$\frac{1}{q}(x,y) = \sum_{(i,j)\in \mathbb{N}^2} e_{ij} B_{ij}(x,y)$$

Hence by the use of Lemma 3.2

$$\left(f - \frac{p}{q}\right)(x,y) = \left(\frac{1}{q}\cdot(f\cdot q - p)\right)(x,y) = \sum_{(i,j)\in \mathbb{N}^2 \setminus E} \tilde{d}_{ij} B_{ij}(x,y)$$

Theorem 4.1 describes which interpolation properties are now satisfied by $\frac{p}{q}$.

Theorem 4.1. *If* $q(x_k, y_l) \neq 0$ *for* $(k,l)\in E$ *then*

$$\frac{\partial^{\mu+\nu} f}{\partial x^\mu \partial y^\nu}(x_k, y_l) = \frac{\partial^{\mu+\nu} \left(\frac{p}{q}\right)}{\partial x^\mu \partial y^\nu}(x_k, y_l)$$

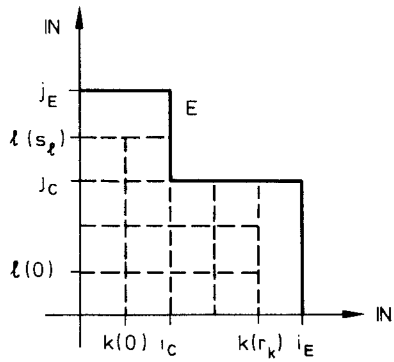
for

$$(\mu,\nu)\in I=\{(\mu,\nu)\,|\,0\leqq\mu\leqq r_k,\ 0\leqq\nu\leqq s_l\}\cap\{(\mu,\nu)\,|\,(k(\mu),l(\nu))\in E\}$$

(*where* $x_k=x_{k(\mu)}$ *for* $\mu=0,\ldots,r_k$ *and* $y_l=y_{l(\nu)}$ *for* $\nu=0,\ldots,s_l$).

If $r_k=0=s_l$ *this reduces to* $f(x_k,y_l)=\left(\frac{p}{q}\right)(x_k,y_l)$ *for* (k,l) *in* E.

Proof. Given r_k and s_l for fixed (x_k,y_l) in E, consider the following situation for the interpolation points with respect to E; the number of dotted points equals the number of elements in I.



(6)

We define

$$i_E=\max\{i\,|\,(i,j)\in E\}$$
$$j_E=\max\{j\,|\,(i,j)\in E\}$$
$$i_C=\max\{i\,|\,\forall j,0\leqq j\leqq j_E:(i,j)\in E\}$$
$$j_C=\max\{j\,|\,\forall i,0\leqq i\leqq i_E:(i,j)\in E\}$$

Using these definitions we rewrite E as

$$E=E_1\cup E_2$$

with

$$E_1=\{(i,j)\,|\,0\leqq i\leqq i_E,\ 0\leqq j\leqq j_C\}$$
$$E_2=\{(i,j)\,|\,0\leqq i\leqq i_C,\ 0\leqq j\leqq j_E\}$$

and I as

$$I=I_1\cup I_2$$

with

$$I_1=\{(\mu,\nu)\,|\,0\leqq\mu\leqq r_k,\ 0\leqq\nu\leqq s_l,\ (k(\mu),l(\nu))\in E_1\}$$
$$I_2=\{(\mu,\nu)\,|\,0\leqq\mu\leqq r_k,\ 0\leqq\nu\leqq s_l,\ (k(\mu),l(\nu))\in E_2\}$$

Because $q(x_k,y_l)\neq 0$ for (k,l) in E we have

$$\left(f-\frac{p}{q}\right)(x,y)=\sum_{(i,j)\in\mathbb{N}^2\setminus E}\tilde{d}_{ij}B_{ij}(x,y)$$

To check the interpolation conditions we write

$$\frac{\partial^{\mu+\nu} B_{ij}}{\partial x^\mu \partial y^\nu} = \frac{\partial^{\mu+\nu}(B_{i0} \cdot B_{0j})}{\partial x^\mu \partial y^\nu} = \frac{\partial^\mu B_{i0}\, \partial^\nu B_{0j}}{\partial x^\mu \partial y^\nu}$$

If we subdivide $\mathbb{N}^2 \setminus E$ in 3 regions

$$A = \{(i,j) \mid i > i_E\}$$
$$B = \{(i,j) \mid j > j_E\}$$
$$C = \{(i,j) \mid i_C < i \leqq i_E, j_C < j \leqq j_E\}$$

it is easy to see that

$$\left.\frac{\partial^\mu B_{i0}}{\partial x^\mu}\right|_{(x_k, y_l)} = 0 \quad \text{for } (i,j) \text{ in } A \text{ and } (\mu, \nu) \text{ in } I.$$

$$\left.\frac{\partial^\nu B_{0j}}{\partial y^\nu}\right|_{(x_k, y_l)} = 0 \quad \text{for } (i,j) \text{ in } B \text{ and } (\mu, \nu) \text{ in } I.$$

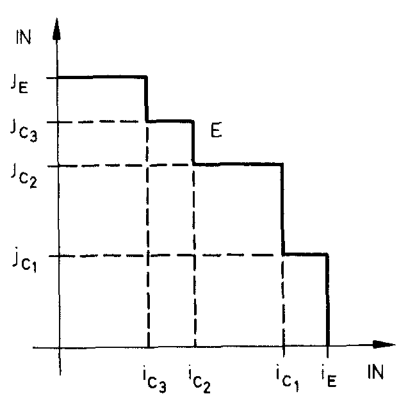
$$\left.\frac{\partial^\mu B_{i0}}{\partial x^\mu}\right|_{(x_k, y_l)} = 0 \quad \text{for } (i,j) \text{ in } C \text{ and } (\mu, \nu) \text{ in } I_2$$

$$\left.\frac{\partial^\nu B_{0j}}{\partial y^\nu}\right|_{(x_k, y_l)} = 0 \quad \text{for } (i,j) \text{ in } C \text{ and } (\mu, \nu) \text{ in } I_1$$

Finally

$$\left.\frac{\partial^{\mu+\nu}\left(f - \frac{p}{q}\right)}{\partial x^\mu \partial y^\nu}\right|_{(x_k, y_l)} = 0 \quad \text{for } (\mu, \nu) \text{ in } I \text{ and } (x_k, y_l) \text{ in } E$$

The most general situation for the interpolation points with respect to E is slightly more complicated but completely analogous to the one given in (6); we illustrate this remark by means of the following figure



The proof in this case is performed in the same way as above. □

It is our intention to define these Newton-Padé approximants also for more general sets of interpolation points, since the Newton interpolating formula (3) can be written down for more general sets of data; we refer to [5] and [7].

5. Examples

a) Univariate Newton-Padé Approximant

Consider the Newton interpolation series for $f(x,0)$ and choose

$$D=\{(j,0)\mid 0\leqq j\leqq m\}$$
$$N=\{(i,0)\mid 0\leqq i\leqq n\}$$
$$E\supseteq\{(k,0)\mid 0\leqq k\leqq n+m\}$$

If the points in $\{(k,0)\mid n+1\leqq k\leqq n+m\}$ supply linearly independent equations, then the determinant representations for $p(x,0)$ and $q(x,0)$ are

$$q(x,0)=\begin{vmatrix} 1 & (x-x_0) & \cdots & \prod_{k=0}^{m-1}(x-x_k) \\ f_{0n+1,00} & f_{1n+1,00} & \cdots & f_{mn+1,00} \\ \vdots & & & \vdots \\ f_{0n+m,00} & f_{1n+m,00} & \cdots & f_{mn+m,00} \end{vmatrix}$$

$$p(x,0)=\begin{vmatrix} \sum_{i=0}^{n} f_{0i,00}\prod_{k=0}^{i-1}(x-x_k) & \cdots & \sum_{i=0}^{n} f_{mi,00}\prod_{k=0}^{i-1}(x-x_k) \\ f_{0n+1,00} & \cdots & f_{mn+1,00} \\ \vdots & & \vdots \\ f_{0n+m,00} & \cdots & f_{mn+m,00} \end{vmatrix}$$

which coincide with the formulas given in [4, p. 36] for the univariate Newton-Padé approximant.

b) Numerical Example

Consider

$$f(x,y)=1+\frac{x}{0\cdot 1-y}+\sin(xy)$$

and

$$x_k=k\cdot\pi \qquad k=0,1,\ldots$$
$$y_l=(l-1)\cdot\pi \qquad l=0,1,\ldots$$

The Newton interpolating series looks like

$$f(x,y)=1+\frac{1}{0\cdot 1+\sqrt{\pi}}x+\frac{10}{0\cdot 1+\sqrt{\pi}}x(y+\sqrt{\pi})+\frac{10}{0\cdot 01-\pi}x(y+\sqrt{\pi})y+\ldots$$

Choose

$$D=\{(0,0),(1,0),(0,1)\}$$
$$N=\{(0,0),(1,0),(0,1),(1,1)\}$$
$$E=N\cup\{(2,0),(2,1),(0,2),(1,2)\}$$

Writing down the system of equations (4b), it is easy to check that

$$H=\{(2,1),(1,2)\}$$

The determinantal formulas for $p(x,y)$ and $q(x,y)$ yield

$$q(x,y)=\begin{vmatrix} 1 & x & y+\sqrt{\pi} \\ f_{02,01} & f_{12,01} & f_{02,11} \\ f_{01,02} & f_{11,02} & f_{01,12} \end{vmatrix}$$
$$=\frac{100}{0\cdot 01-\pi}\left(1-\frac{1}{0\cdot 1+\sqrt{\pi}}(y+\sqrt{\pi})\right)$$

$$p(x,y)=\begin{vmatrix} \sum_{i=0}^{1}\sum_{j=0}^{1} f_{0i,0j}B_{ij}(x,y) & \sum_{i=0}^{1}\sum_{j=0}^{1} f_{1i,0j}B_{ij}(x,y) & \sum_{i=0}^{1}\sum_{j=0}^{1} f_{0i,1j}B_{ij}(x,y) \\ f_{02,01} & f_{12,01} & f_{02,11} \\ f_{01,02} & f_{11,02} & f_{01,12} \end{vmatrix}$$

with

$$\sum_{i=0}^{1}\sum_{j=0}^{1} f_{0i,0j}B_{ij}(x,y)=1+\frac{x}{0\cdot 1+\sqrt{\pi}}+\frac{10}{0\cdot 1+\sqrt{\pi}}x(y+\sqrt{\pi})$$

$$\sum_{i=0}^{1}\sum_{j=0}^{1} f_{1i,0j}B_{ij}(x,y)=\frac{0\cdot 1+2\sqrt{\pi}}{0\cdot 1+\sqrt{\pi}}x+\frac{\sqrt{\pi}}{0\cdot 1(0\cdot 1+\sqrt{\pi})}x(y+\sqrt{\pi})$$

$$\sum_{i=0}^{1}\sum_{j=0}^{1} f_{0i,1j}B_{ij}(x,y)=(y+\sqrt{\pi})+10x(y+\sqrt{\pi})$$

Finally we obtain

$$[N/D]_E(x,y)=\frac{p}{q}(x,y)=\frac{0\cdot 1+\sqrt{\pi}+x-(y+\sqrt{\pi})}{0\cdot 1+\sqrt{\pi}-(y+\sqrt{\pi})}$$

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