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Multidimensional IIR filters and robust rational interpolation

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Abstract It is well-known that IIR filters can have a much lower order than FIR filters with the same performance. On the downside is that the implementation of an IIR filter is an iterative procedure while that of an FIR filter is a one-shot computation. But in higher dimensions IIR filters are definitely more attractive. We offer a technique where the filter's performance specifications, stability constraints, its convergence speed and a protection against possible adverse effects of perturbations are all included in the design from the start. The technique only needs an off-the-shelf LP solver because the filter is obtained as a Chebyshev center of a convex polytope. The method deals with general non-causal non-separable filters.

Keywords Multidimensional digital filters · Non-causal · IIR · Linear programming · Quadratic programming · Rational approximation

1 Multidimensional recursive systems

There exists a myriad of papers and monographs on the subject of multidimensional IIR filters. The reference that we feel most related to is foremost (Gorinevsky and Boyd 2006). But we also want to point out (Dumitrescu 2005; Jiang and Kwan 2010) on least squares methods for IIR filters, (Wu-Sheng et al. 1998) using quadratic programming, (Piyachaiyakul and Charoenlarpnopparut 2011) on 3D filters and (Lu and Hinamoto 2011) delivering sparse models. For the rest we assume that the interested reader is familiar with the state of the art.

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Let us start with the introduction of the required notation and definitions. Without loss of generality and for ease of notation, we restrict the presentation to the 2D case. For indices we use the notations $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and $\mathbb{N}_0$, $\mathbb{Z}_0$ when the number zero is removed. In addition $-\mathbb{N}_0 = \{-1, -2, \ldots\}$. We consider multidimensional LSI recursive systems or systems with infinite-extent impulse response, which transform an input signal $x(n_1, n_2)$ into an output signal $y(n_1, n_2)$ such that $y(n_1, n_2)$ can be described by a difference equation of the form

$$\begin{aligned} b(0,0)y(n_1,n_2) = &\sum_{\substack{(k_1,k_2)\in N\\ N\subset\mathbb{Z}^2}} a(k_1,k_2)x(n_1-k_1,n_2-k_2) \\ &- \sum_{\substack{(k_1,k_2)\in D_0\\ D_0\subset\mathbb{Z}^2\setminus\{(0,0)\}}} b(k_1,k_2)y(n_1-k_1,n_2-k_2) \end{aligned} \tag{1}$$

where $D_0 \neq \phi$ and $b(0, 0) \neq 0$. The sets N and $D = D_0 \cup \{(0, 0)\}$ are the regions of support of the arrays $a(n_1, n_2)$ and $b(n_1, n_2)$ respectively. The transfer function of a LSI recursive system is the ratio

$$H(z_1,z_2) = \frac{\sum_{\substack{(k_1,k_2)\in N\\ N\subset\mathbb{Z}^2}} a(k_1,k_2)z_1^{-k_1}z_2^{-k_2}}{b(0,0)+\sum_{\substack{(k_1,k_2)\in D_0\\ D_0\subset\mathbb{Z}^2\setminus\{(0,0)\}}} b(k_1,k_2)z_1^{-k_1}z_2^{-k_2}} = \frac{A(z_1,z_2)}{B(z_1,z_2)}. \tag{2}$$

The output signal $y(n_1, n_2)$ of such a system applied to an input which is a complex sinusoid of the form $x(n_1, n_2) = \exp \mathrm{i}(\omega_1 n_1 + \omega_2 n_2)$ with $(\omega_1, \omega_2) \in [-\pi, \pi] \times [-\pi, \pi]$, is characterized by the system's frequency response

$$H\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right) = \frac{\sum_{(k_1,k_2)\in N} a(k_1,k_2)\exp\left(-\mathrm{i}(k_1\omega_1+k_2\omega_2)\right)}{b(0,0)+\sum_{(k_1,k_2)\in D_0} b(k_1,k_2)\exp\left(-\mathrm{i}(k_1\omega_1+k_2\omega_2)\right)}.$$

Often $b(0, 0) = 1$ is assumed in order to normalize the representation of $H\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right)$. However, we propose a different normalization which is discussed at the end of this section.

Usual filtering in one dimension is causal (implying $N, D \subset \mathbb{N}$), but for most multidimensional signals there is no preferred direction for the coordinates because they are often spatial and not time (hence in general $N, D \subset \mathbb{Z}^2$). The most common non-causal filters are zero-phase filters where $A\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right)$ and $B\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right)$ are real and satisfy some symmetry properties. Let $a(n_1, n_2)$ and $b(n_1, n_2)$ be real arrays. When the centro symmetry

$$\begin{aligned} A(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}) &= A(e^{-\mathrm{i}\omega_1}, e^{-\mathrm{i}\omega_2}) \\ B(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}) &= B(e^{-\mathrm{i}\omega_1}, e^{-\mathrm{i}\omega_2}), \end{aligned}$$

holds, the polynomials $A(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2})$ and $B(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2})$ simplify to the real-valued

$$\begin{aligned} A(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}) &= \sum_{\substack{(k_1,k_2)\in N\\ N\subset\mathbb{N}\times\mathbb{N}\cup\mathbb{N}_0\times(-\mathbb{N}_0)}} \tilde{a}(k_1,k_2)\cos(k_1\omega_1+k_2\omega_2), \\ \tilde{a}(0,0) &= a(0,0),\ \tilde{a}(k_1,k_2) = a(k_1,k_2)+a(-k_1,-k_2), \\ B(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}) &= \sum_{\substack{(k_1,k_2)\in D\\ D\subset\mathbb{N}\times\mathbb{N}\cup\mathbb{N}_0\times(-\mathbb{N}_0)}} \tilde{b}(k_1,k_2)\cos(k_1\omega_1+k_2\omega_2), \\ \tilde{b}(0,0) &= b(0,0),\ \tilde{b}(k_1,k_2) = b(k_1,k_2)+b(-k_1,-k_2). \end{aligned} \tag{3}$$

For the additional reflection symmetry conditions

$$A(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}) = A(e^{\mathrm{i}\omega_1}, e^{-\mathrm{i}\omega_2})$$
$$B(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}) = B(e^{\mathrm{i}\omega_1}, e^{-\mathrm{i}\omega_2}),$$

the polynomials $A(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2})$ and $B(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2})$ further simplify to the quadrantally symmetric

$$\begin{aligned} A(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}) &= \sum_{\substack{(k_1,k_2)\in N \\ N\subset\mathbb{N}^2}} \breve{a}(k_1,k_2)\cos(k_1\omega_1)\cos(k_2\omega_2), \\ \breve{a}(k_1,0) &= \tilde{a}(k_1,0),\ \ \breve{a}(k_1,k_2) = \tilde{a}(k_1,k_2) + \tilde{a}(k_1,-k_2), \\ B(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}) &= \sum_{\substack{(k_1,k_2)\in D \\ D\subset\mathbb{N}^2}} \breve{b}(k_1,k_2)\cos(k_1\omega_1)\cos(k_2\omega_2), \\ \breve{b}(k_1,0) &= \tilde{b}(k_1,0),\quad \breve{b}(k_1,k_2) = \tilde{b}(k_1,k_2) + \tilde{b}(k_1,-k_2). \end{aligned} \tag{4}$$

An important issue is stability. If a system is unstable, any input, including computational noise, can cause the output to grow without bound. Thus the condition referred to as bounded-input-bounded-output or BIBO stability, is generally imposed.

Following the reasoning of Gorinevsky and Boyd (2006), we require

$$\left|1 - B\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right)\right| \leq \tau < 1 \tag{5}$$

with $B\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right) > 0$ (the latter is a necessary condition for BIBO stability). Only this last condition needs to be guaranteed, because the obtained polynomials $A(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2})$ and $B(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2})$ can both be multiplied a posteriori with a (nonnegative) normalization constant λ such that (5) is satisfied. The value

$$\tau = \max_{[-\pi,\pi]^2} \left|1 - B\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right)\right|$$

determines the convergence speed of an iterative implementation as in Li and Marks (1989). A lower-bound for the rate of convergence is given in Gorinevsky and Boyd (2006) by

$$\frac{\max\limits_{[-\pi,\pi]^2} B\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right) - \min\limits_{[-\pi,\pi]^2} B\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right)}{\max\limits_{[-\pi,\pi]^2} B\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right) + \min\limits_{[-\pi,\pi]^2} B\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right)} \leq \tau = \max_{[-\pi,\pi]^2} \left|1 - B\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right)\right| \tag{6}$$

Note that the left-hand side of (6) is independent of the chosen normalization constant λ. However, the right-hand side of (6) is not! In order to achieve the fastest possible convergence rate, $H\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right)$ must be normalized a posteriori with

$$\lambda = \frac{2}{\max\limits_{[-\pi,\pi]^2} B\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right) + \min\limits_{[-\pi,\pi]^2} B\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right)}$$

such that (5) is satisfied and equality in (6) is achieved.

In practice the condition $B\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right) > 0$ is only ensured on a frequency grid and a discussion of the mathematical guarantees offered by this approach is given in Sect. 4.

In Sect. 2 we present our robust interpolation technique for multidimensional recursive filters. It automatically incorporates a safeguard against the uncertainty due to the use of finite precision arithmetic. Our method returns the IIR filter of lowest order with maximal robustness. Several designs computed with this technique are presented in Sect. 3.

2 Rational interpolation of uncertainty intervals

To model some filter frequency response requirements, very often a least squares approach is used. But this nonlinear optimization problem may have many local minima and so the quality of the computed model highly depends on the provided starting value. Our novel approach is to deal with data that represent uncertainty, such as the passband and stopband ripple, the transition band width and finite precision roundings, in another very natural way, namely by means of an uncertainty interval. The technique is very similar to the one in Gorinevsky and Boyd (2006) where the design specifications are also cast as linear inequalities, but in Gorinevsky and Boyd (2006) the method is not obtained from or viewed as a multivariate rational interpolation problem. Our approach provides a mathematical underpinning for the optimization problem that needs to be solved.

We assume that the uncertainty in the independent variables ω_1 and ω_2 (the frequencies) is zero or at least negligible, and that for the frequency response magnitude an uncertainty interval is given which contains the prescribed exact value. We study the problem of approximating these data with a rational function that intersects the given uncertainty intervals. Both the problem statement and the algorithm that we develop can be written down for any number of independent variables. For ease of notation and illustration, we again describe the 2D instead of the general high dimensional case. Let the following set of $n+1$ vertical segments F_i be given at the sample points $(\omega_1^{(i)}, \omega_2^{(i)})$:

$$S_n = \{(\omega_1^{(0)}, \omega_2^{(0)}, F_0), (\omega_1^{(1)}, \omega_2^{(1)}, F_1), \ldots, (\omega_1^{(n)}, \omega_2^{(n)}, F_n)\}.$$

Here $F_i = [\underline{f}_i, \overline{f}_i]$ are real finite intervals with $\underline{f}_i < \overline{f}_i$ and $H(e^{\mathrm{i}\omega_1^{(i)}}, e^{\mathrm{i}\omega_2^{(i)}}) \in F_i$ for $i = 0, \ldots, n$ and none of the points $(\omega_1^{(i)}, \omega_2^{(i)})$ coincide. Let us consider the bivariate generalized polynomials

$$p(\omega_1, \omega_2) = \sum_{(k_1,k_2)\in N} \alpha(k_1, k_2)\phi_{k_1k_2}(\omega_1, \omega_2),$$
$$q(\omega_1, \omega_2) = \sum_{(k_1,k_2)\in D} \beta(k_1, k_2)\phi_{k_1k_2}(\omega_1, \omega_2).$$

The functions $\phi_{k_1k_2}(\omega_1, \omega_2)$ can for instance be $\phi_{k_1k_2}(\omega_1, \omega_2) = \omega_1^{-k_1}\omega_2^{-k_2}$, $\phi_{k_1k_2}(\omega_1, \omega_2) = \cos(k_1\omega_1 + k_2\omega_2)$ or $\phi_{k_1k_2}(\omega_1, \omega_2) = \cos(k_1\omega_1)\cos(k_2\omega_2)$. Further, if the cardinality of the sets N and D is respectively given by $\ell + 1$ and $m + 1$, let us denote

$$\begin{aligned} N &= \left\{(k_1^{(0)}, k_2^{(0)}), \ldots, (k_1^{(\ell)}, k_2^{(\ell)})\right\}, \\ D &= \left\{(k_1^{(0)}, k_2^{(0)}), \ldots, (k_1^{(m)}, k_2^{(m)})\right\}, \ \left(k_1^{(0)}, k_2^{(0)}\right) = (0, 0) \end{aligned} \tag{7}$$

and the irreducible form of the generalized rational function $(p/q)(\omega_1, \omega_2)$ by $r_{\ell,m}(\omega_1, \omega_2)$. Let

$$R_{\ell,m}(S_n) = \left\{r_{\ell,m}(\omega_1, \omega_2) \mid r_{\ell,m}\left(\omega_1^{(i)}, \omega_2^{(i)}\right) \in F_i\,,\ q\left(\omega_1^{(i)}, \omega_2^{(i)}\right) > 0\,,\ 0 \le i \le n\right\}. \tag{8}$$

So the ideal frequency response, providing the data, is denoted by the function $H(e^{\mathrm{i}\omega_1^{(i)}}, e^{\mathrm{i}\omega_2^{(i)}}) = (A/B)(e^{\mathrm{i}\omega_1^{(i)}}, e^{\mathrm{i}\omega_2^{(i)}})$ and its rational approximant is denoted by $r_{\ell,m}(\omega_1, \omega_2) = (p/q)(\omega_1, \omega_2)$. Whereas in traditional rational interpolation one has $\ell + m = n$, here we envisage $\ell + m \ll n$ just as in least squares approximation. For given segments

S_n and given sets N and D of respective cardinality $\ell+1$ and $m+1$, we are concerned with the problem of determining whether $R_{\ell,m}(S_n) \neq \emptyset$.

The interpolation conditions

$$r_{\ell,m}(\omega_1^{(i)}, \omega_2^{(i)}) \in F_i, \qquad i = 0, \ldots, n \tag{9}$$

in (8) amount to

$$\underline{f}_i \leq \frac{p\left(\omega_1^{(i)}, \omega_2^{(i)}\right)}{q\left(\omega_1^{(i)}, \omega_2^{(i)}\right)} \leq \overline{f}_i, \qquad i = 0, \ldots, n.$$

Under the assumption that $q(\omega_1^{(i)}, \omega_2^{(i)}) > 0$ for $i = 0, \ldots, n$, we obtain the following homogeneous system of linear inequalities after linearization

$$\begin{cases} -p\left(\omega_1^{(i)}, \omega_2^{(i)}\right) + \overline{f}_i q\left(\omega_1^{(i)}, \omega_2^{(i)}\right) \geq 0 \\ \;\; p\left(\omega_1^{(i)}, \omega_2^{(i)}\right) - \underline{f}_i q\left(\omega_1^{(i)}, \omega_2^{(i)}\right) \geq 0 \end{cases}, \quad i = 0, \ldots, n. \tag{10}$$

As explained in Salazar Celis et al. (2007), there is no loss of generality in assuming that $q(\omega_1, \omega_2)$ is positive in the interpolation points: the interpolation conditions (9) can be linearized for arbitrary non-zero $q(\omega_1^{(i)}, \omega_2^{(i)})$, positive or negative, without changing the nature of the problem.

For ease of notation, let $\phi_{N,j}^{(i)} = \phi_{k_1^{(j)} k_2^{(j)}}(\omega_1^{(i)}, \omega_2^{(i)})$ for $(k_1^{(j)}, k_2^{(j)})$ in N, $\phi_{D,j}^{(i)} = \phi_{k_1^{(j)} k_2^{(j)}}(\omega_1^{(i)}, \omega_2^{(i)})$ for $(k_1^{(j)}, k_2^{(j)})$ in D, $c = (\alpha_0, \ldots, \alpha_\ell, \beta_0, \ldots, \beta_m)^T$ and $\nu = \ell + m + 2$. We denote by U the $(2n+2) \times \nu$ matrix

$$U = \begin{pmatrix} -\phi_{N,0}^{(0)} & \cdots & -\phi_{N,\ell}^{(0)} & \overline{f}_0 \phi_{D,0}^{(0)} & \cdots & \overline{f}_0 \phi_{D,m}^{(0)} \\ \vdots & & \vdots & \vdots & & \vdots \\ -\phi_{N,0}^{(n)} & \cdots & -\phi_{N,\ell}^{(n)} & \overline{f}_n \phi_{D,0}^{(n)} & \cdots & \overline{f}_n \phi_{D,m}^{(n)} \\ \phi_{N,0}^{(0)} & \cdots & \phi_{N,\ell}^{(0)} & -\underline{f}_0 \phi_{D,0}^{(0)} & \cdots & -\underline{f}_0 \phi_{D,m}^{(0)} \\ \vdots & & \vdots & \vdots & & \vdots \\ \phi_{N,0}^{(n)} & \cdots & \phi_{N,\ell}^{(n)} & -\underline{f}_n \phi_{D,0}^{(n)} & \cdots & -\underline{f}_n \phi_{D,m}^{(n)} \end{pmatrix}.$$

In the sequel, we abbreviate (10) to $Uc \geq 0$ and denote

$$L_{\ell,m}(S_n) = \{c \in \mathbb{R}^\nu \mid Uc \geq 0\}. \tag{11}$$

The linearized version of the problem $R_{\ell,m}(S_n) \neq \emptyset$ now consists in determining whether

$$L_{\ell,m}(S_n) \neq \{0\}. \tag{12}$$

It is shown in Cuyt and Salazar Celis (2012) that if $L_{\ell,m}(S_n)$ contains non-zero elements and if c is normalized such that

$$||c||_\infty \leq 1,$$

then a maximally robust coefficient vector c for the rational interval interpolant $r_{\ell,m}(\omega_1, \omega_2)$ can be found as a Chebyshev center of a convex polytope. This is obtained from the solution of a linear programming (LP) problem. Although a Chebyshev center is not unique, it is always a global optimum, unlike the solution of some nonlinear optimization problems.

However, the returned Chebyshev center may be very close to the origin and, at the same time, the radius ρ of the largest Euclidean ball with center c contained in $L_{\ell,m}(S_n) \cap [-1, 1]^\nu$ may be terribly small, especially when n is large. Therefore it is sometimes very difficult to conclude whether the returned solution does not coincide with the origin and whether the returned radius ρ is nonzero, in other words whether (12) holds. The set (11) is in essence also the starting point for the LP approach proposed in Gorinevsky and Boyd (2006).

One can avoid these numerical issues by reformulating (12) as a quadratic programming (QP) problem, as we explain now. The set $L_{\ell,m}(S_n)$ in (11) is a convex polyhedral cone with apex in the origin. In Salazar Celis et al. (2007) the following strictly convex quadratic programming (QP) formulation is proposed:

$$\begin{aligned} &\arg\min_{c \in \mathbb{R}^\nu} ||c||_2^2 \\ &\text{subject to } U_j\, c - \delta\, ||U_j||_2 \geq 0, \quad j = 1, \ldots, 2n+2 \end{aligned} \tag{13}$$

where U_j denotes the j-th row of the matrix U, $||\cdot||_2$ denotes the Euclidean norm and $\delta > 0$ is a given constant chosen as indicated below. Problem (13) establishes (under mild conditions) whether $L_{\ell,m}(S_n) \neq \{0\}$ and in such a case returns a coefficient vector in the open interior of (11). Nevertheless, it is clear that there is still an abundance of freedom in choosing elements out of $L_{\ell,m}(S_n)$. Compared to Gorinevsky and Boyd (2006), the current approach has however the following important advantages.

First, if $\min_{i=0,\ldots,n}(\overline{f}_i - \underline{f}_i) > 0$ then it follows from (10) that for c in the open interior of (11), the corresponding denominator polynomial $q(\omega_1, \omega_2)$ is strictly positive at the sampled frequencies $(\omega_1^{(i)}, \omega_2^{(i)})$:

$$q(\omega_1^{(i)}, \omega_2^{(i)}) > 0, \qquad i = 0, \ldots, n.$$

The fact that this follows automatically and hence that $n+1$ additional inequalities need not be imposed (like in Gorinevsky and Boyd 2006) is a substantial gain, especially in the case of more than 2 variables.

So if the width of the intervals F_i is nonzero, then the rational function $r_{\ell,m}(\omega_1, \omega_2)$ passes through the open interior of F_i and the denominator polynomial $q(\omega_1, \omega_2)$ does not become zero in any of the sampled frequencies $(\omega_1^{(i)}, \omega_2^{(i)})$. A multivariate generalization in Sect. 4 of a little known result from rational approximation theory (Pomentale 1968) allows to even guarantee the overall positivity of the denominator $q(\omega_1, \omega_2)$.

Second, finite-word roundoff errors are automatically controlled by the choice of δ. Indeed, if the vector c is a solution of (13) for some $\delta > 0$, then it follows that the perturbed vectors

$$\{c + e \mid e \in \mathbb{R}^\nu : ||e||_2 < \delta\}$$

all lie in the open interior of (11). Moreover, the value δ can be made arbitrary large because $L_{\ell,m}(S_n)$ is unbounded.

So whereas we cannot control the value of ρ in the LP formulation of the problem, we have complete freedom in the choice of δ in the QP formulation. Both play a similar role, namely they express the robustness of the solution vector c because in both the LP and QP formulations a Euclidean ball of radius ρ (in LP) and δ (in QP) and centered at c is guaranteed to be included in the solution set.

By checking the mere existence of a solution for increasing $\ell \geq 0$ and $m \geq 0$, once an order as in (7) of the index tuples is determined, the algorithm can easily return the smallest $\ell+m$ for which an interval interpolant exists. Hence it returns the IIR filter of lowest order, for which $L_{\ell,m}(S_n) \neq \{0\}$. The order one decides upon in (7) can be varied. One can add terms to N and D in a diagonal fashion (increasing the total degree), or along squares (bounding the partial degree in each variable) or in another way. Different illustrations can be found in the Sects. 3 and 4.

3 Designing IIR filters

A given set of frequency response characteristics typically may be met by an IIR filter of considerably lower order than a corresponding FIR design. The design specifications are usually given in terms of the magnitude which possesses certain symmetries, while the phase characteristic is either not known or unimportant. Though the approach is fundamentally focused on zero-phase IIR filters, also more general filter types are possible: the conditions can be written separately for the real and imaginary parts of the transfer function numerator while a zero-phase denominator with positive real frequency response is maintained.

An ideal lowpass filter can for instance be specified by the frequency response

$$H\left(e^{\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}\right) = \begin{cases} 1, & (\omega_1, \omega_2) \in P \subset [-\pi, \pi] \times [-\pi, \pi], \\ 0, & (\omega_1, \omega_2) \notin P, \end{cases} \tag{14}$$

where the domain P can be as simple as a square, disk or diamond (which are all mere 2D balls in the ℓ_∞, ℓ_2 or ℓ_1-norm respectively). We illustrate the technique for the computation of the centro symmetric filter (Reddy et al. 2003) with

$$P = \{\pi/4 \le \omega_1 \le 3\pi/4, -3\pi/4 \le \omega_2 \le -\pi/4\} \\ \cup \{-3\pi/4 \le \omega_1 \le -\pi/4, \pi/4 \le \omega_2 \le 3\pi/4\}$$

and the more difficult 90° quadrantally symmetric fan filter (Harris 1981) where

$$P = \{|\omega_1| \le \omega_2\} \cup \{-|\omega_1| \ge \omega_2\}\ .$$

The practical specifications of such a filter take the form of a tolerance scheme, graphically illustrated in Fig. 1, in which:

– there is a passband wherein the frequency response must approximate 1 within an error of at most $\pm\delta_1$ (indicated with plus signs),

$$r_{\ell,m}(\omega_1, \omega_2) \in [1 - \delta_1, 1 + \delta_1], \qquad (\omega_1, \omega_2) \in P, \tag{15a}$$

– there is a stopband in which the response must approximate zero within an error of at most $\pm\delta_2$ (indicated with crosses),

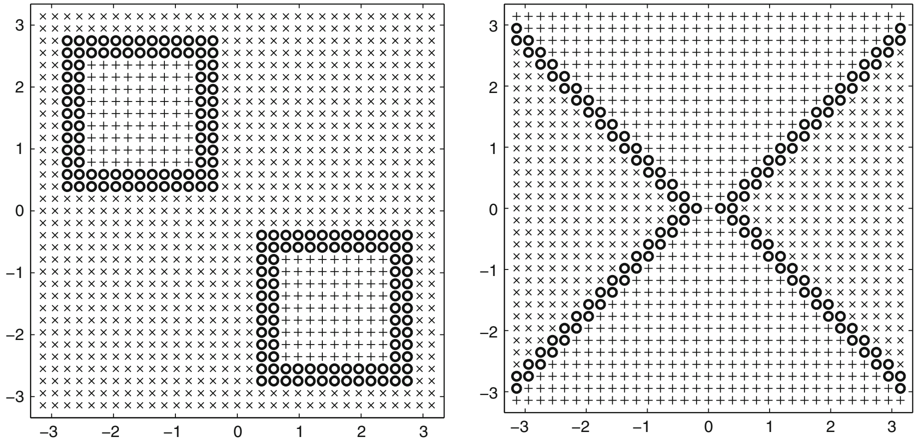
$$r_{\ell,m}(\omega_1, \omega_2) \in [-\delta_2, \delta_2], \qquad (\omega_1, \omega_2) \notin P \cup T, \tag{15b}$$

– there is a transition band of a certain width in which the response drops smoothly from the passband to the stopband (indicated with circles),

$$r_{\ell,m}(\omega_1, \omega_2) \in [-\delta_2, 1 + \delta_1], \qquad (\omega_1, \omega_2) \in T. \tag{15c}$$

In Fig. 1a,b the transition band is respectively given by

$$\begin{aligned} T = &(\{\pi/8 \le \omega_1 \le 7\pi/8, -7\pi/8 \le \omega_2 \le -\pi/8\} \cup \\ &\{-7\pi/8 \le \omega_1 \le -\pi/8, \pi/8 \le \omega_2 \le 7\pi/8\}) \setminus P, \\ T = &\{-|\omega_2| - \pi/8 \le \omega_1 \le -|\omega_2|\} \cup \\ &\{|\omega_2| \le \omega_1 \le |\omega_2| + \pi/8\}. \end{aligned}$$



(a) Centro symmetric filter **(b)** Fan filter

Fig. 1 Graphical illustration of tolerance schemes

The frequency domain $[-\pi, \pi] \times [-\pi, \pi]$ is covered by a grid of $n + 1 = 1089$ points by sampling in both directions equidistantly with distance $\pi/16$. For the choice $\delta_1 = 0.01$ the passband ripple equals

$$20 \log_{10} \frac{1+\delta_1}{1-\delta_1} = 0.17 \text{ dB}$$

and for $\delta_2 = 0.02$ the stopband attenuation is

$$-20 \log_{10} \delta_2 = 34 \text{ dB}.$$

With these parameters we obtain for the centrosymmetric filter a model with the sets N and D in 3 given by

$$\begin{aligned} N &= D \setminus \{(1, -3)\} \\ D &= \{(k_1, k_2) \in \mathbb{N}^2 \mid 0 \le k_1 + k_2 \le 4\} \cup \\ &\quad \{(k_1, k_2) \in \mathbb{N}_0 \times (-\mathbb{N}_0) \mid 2 \le k_1 - k_2 \le 4\}. \end{aligned}$$

The resulting rational function $r_{\ell,m}(\omega_1, \omega_2) = r_{19,20}(\omega_1, \omega_2)$ can be found in Fig. 2a. As envisaged $\ell + m = 39 \ll n = 1088$.

The model obtained for the fanfilter is of much less complexity than the one presented in Harris (1981) for the same parameters: its numerator and denominator only contain 13 and 15 terms respectively with the sets N and D in (4) given by

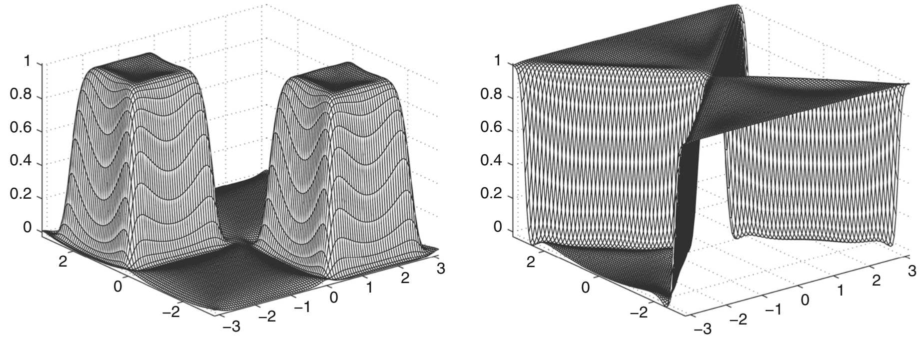
$$\begin{aligned} N &= \{(k_1, k_2) \in \mathbb{N}^2 \mid 0 \le k_1, k_2 \le 3\} \setminus \{(3, 2), (2, 3), (3, 3)\} \\ D &= \{(k_1, k_2) \in \mathbb{N}^2 \mid 0 \le k_1, k_2 \le 3\} \setminus \{(3, 3)\}. \end{aligned}$$

The rational function $r_{12,14}(\omega_1, \omega_2)$ is shown in Fig. 2b. Here again $\ell + m = 26 \ll n = 1088$.

When the parameters are relaxed to $\delta_1 = 0.05$ and $\delta_2 = 0.1$ then, as expected, the obtained model has even lower complexity:

$$N = \{(k_1, k_2) \in \mathbb{N}^2 \mid 0 \le k_1 + k_2 \le 3\} \cup \{(2, 2)\} \quad (16)$$

$$D = \{(k_1, k_2) \in \mathbb{N}^2 \mid 0 \le k_1 + k_2 \le 2\}. \quad (17)$$



(a) $r_{\ell,m}(\omega_1, \omega_2) = r_{19,20}(\omega_1, \omega_2)$ **(b)** $r_{\ell,m}(\omega_1, \omega_2) = r_{12,14}(\omega_1, \omega_2)$

Fig. 2 Rational models for the parameters $\delta_1 = 0.01$ and $\delta_2 = 0.02$

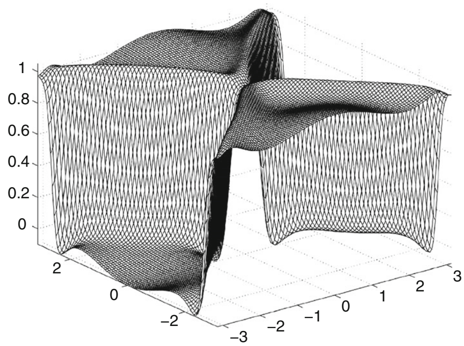


Fig. 3 Rational model $r_{\ell,m}(\omega_1, \omega_2) = r_{10,5}(\omega_1, \omega_2)$ for the relaxed parameters $\delta_1 = 0.05$ and $\delta_2 = 0.1$

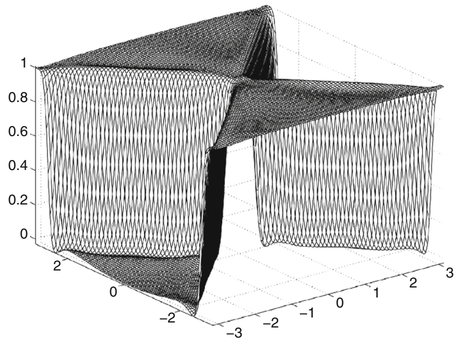


Fig. 4 Rational model $r_{\ell,m}(\omega_1, \omega_2) = r_{12,12}(\omega_1, \omega_2)$ for the data in $[0, \pi] \times [0, \pi]$ and parameters $\delta_1 = 0.01$ and $\delta_2 = 0.02$

The passband ripple becomes 0.87 dB while the stopband attenuation now equals 20 dB. This approximation $r_{\ell,m}(\omega_1, \omega_2) = r_{10,5}(\omega_1, \omega_2)$ is illustrated in Fig. 3.

Instead of relaxing the tolerances δ_1 and δ_2, the dataset in Fig. 1b can be reduced to about one quarter because of the symmetric nature of the basis functions. We have

$$q(\omega_1, \omega_2) = q(-\omega_1, \omega_2) = q(\omega_1, -\omega_2) = q(-\omega_1, -\omega_2)$$

and similarly for $p(\omega_1, \omega_2)$. This may lead to a subtle difference in the neighbourhood of the axes $\omega_1 = 0$ and $\omega_2 = 0$ as can be seen in Fig. 4 compared to Fig. 2b.

Similar experiments were made for the circularly symmetric and diamond-shaped lowpass filters given in Gorinevsky and Boyd (2006).

Note the typical equiripple behaviour around 1 and 0 in the pass- and stopband respectively for the obtained models.

4 Guaranteeing stability in the tolerance scheme

Let us first discuss a result on the stability of the filter $(p/q)(\omega_1, \omega_2)$. We reconsider the denominator $q(\omega_1, \omega_2)$ of the rational approximant. We know that if $\min_{i=0,\ldots,n}(\overline{f}_i - \underline{f}_i) > 0$, then from (10) follows

$$\min_{i=0,\ldots,n} q\left(\omega_1^{(i)}, \omega_2^{(i)}\right) > 0. \tag{18}$$

We ask ourselves whether this gridded positivity constraint may ever imply that $q(\omega_1, \omega_2) > 0$ allover. Surprisingly enough the answer is yes, but the conditions under which it is guaranteed are not feasible in practice.

Take a finite set $S \subset [-\pi, \pi]^2$ of points $(\omega_1^{(k)}, \omega_2^{(k)})$ in the independent variable space, containing at least the $n+1$ given points $(\omega_1^{(i)}, \omega_2^{(i)})$ and the corner points $(\pm\pi, \pm\pi)$. Let

$$d\left(\left(\omega_1^{(i)}, \omega_2^{(i)}\right), \left(\omega_1^{(j)}, \omega_2^{(j)}\right)\right) = \max\left(|\omega_1^{(i)} - \omega_1^{(j)}|, |\omega_2^{(i)} - \omega_2^{(j)}|\right)$$

denote the usual ℓ_∞ distance and let

$$\Delta = \max_S d\left(\left(\omega_1^{(i)}, \omega_2^{(i)}\right), \left(\omega_1^{(j)}, \omega_2^{(j)}\right)\right)$$
$$\mu = \min_S q\left(\omega_1^{(k)}, \omega_2^{(k)}\right).$$

The following more general result than (18) can be proved.

Theorem 1 *If S exists with $\Delta M < \mu$ where*

$$M = \max\left(\max_{[-\pi,\pi]^2}\left|\frac{\partial q(\omega_1, \omega_2)}{\partial \omega_1}\right|, \max_{[-\pi,\pi]^2}\left|\frac{\partial q(\omega_1, \omega_2)}{\partial \omega_2}\right|\right),$$

then the rational approximant $r_{\ell,m}(\omega_1, \omega_2)$ satisfying the tolerance scheme (15) has no poles in $[-\pi, \pi]^2$.

Proof Assume that the denominator has a zero $q(\tilde{\omega_1}, \tilde{\omega_2}) = 0$ with $(\tilde{\omega_1}, \tilde{\omega_2}) \in [-\pi, \pi]^2$ (such a zero is not isolated). Then find its nearest neighbour $(\omega_1^{(k)}, \omega_2^{(k)})$ among the points in S, according to the ℓ_∞ distance (this may be non-unique). To this end $(\tilde{\omega_1}, \tilde{\omega_2})$ is taken as the center of an ℓ_∞-ball emanating from it. Hence for the nearest neighbour we find $|\omega_1^{(k)} - \tilde{\omega_1}| \le \Delta/2$ and $|\omega_2^{(k)} - \tilde{\omega_2}| \le \Delta/2$. Using Taylor's theorem we have

$$q(\omega_1^{(k)}, \omega_2^{(k)}) = q(\tilde{\omega_1}, \tilde{\omega_2}) + \frac{\partial q(\omega_1, \omega_2)}{\partial \omega_1}(\hat{\omega_1}, \hat{\omega_2})\left(\omega_1^{(k)} - \tilde{\omega_1}\right)$$
$$+ \frac{\partial q(\omega_1, \omega_2)}{\partial \omega_2}(\hat{\omega_1}, \hat{\omega_2})\left(\omega_2^{(k)} - \tilde{\omega_2}\right)$$

with $(\hat{\omega_1}, \hat{\omega_2})$ somewhere on the segment connecting $(\omega_1^{(k)}, \omega_2^{(k)})$ and $(\tilde{\omega_1}, \tilde{\omega_2})$. Hence

$$\left|q(\omega_1^{(k)}, \omega_2^{(k)})\right| \le M\Delta$$

with $M < \infty$. Since $q(\omega_1^{(k)}, \omega_2^{(k)}) \geq \mu$, we have a contradiction. □

Note that the values of M and μ are both determined up to the same multiplicative constant which depends on the normalization of $p(\omega_1, \omega_2)$ and $q(\omega_1, \omega_2)$ which is free. In practice this result is not very useful though, because it may imply a rather large set S due to the need for a sufficiently small Δ. Fortunately, the gridded positivity condition for $q(\omega_1, \omega_2)$ usually delivers an overall positive denominator for quite some larger Δ.

In order to guarantee the stability of the filter $(p/q)(\omega_1, \omega_2)$ a priori, when the basis functions are cosines, we can complement the system of homogeneous inequalities (10) with

$$\begin{cases} \beta(0,0) \geq 0 \\ \beta(0,0) + \sum_{(k_1,k_2)\in D_0} \pm\beta(k_1,k_2) \geq 0 \end{cases}$$

where each possible sign combination in the sum attributes an inequality. These conditions guarantee that the denominator $q(\omega_1, \omega_2)$ does not change sign. Since the quadratic programming technique delivers a solution for the $\beta(k_1, k_2)$ in the interior of the set, automatically the strict inequalities

$$\begin{cases} \beta(0,0) > 0 \\ \beta(0,0) + \sum_{(k_1,k_2)\in D_0} \pm\beta(k_1,k_2) > 0 \end{cases}$$

are satisfied. Of course it is to be expected that adding constraints to guarantee stability a priori will tighten the conditions on the coefficients $\alpha(k_1, k_2)$ and $\beta(k_1, k_2)$ and hence increase the complexity of the rational model that satisfies all inequalities. When adding the stability constraint to the so-called relaxed centrosymmetric filter and fan filter ($\delta_1 = 0.05$, $\delta_2 = 0.1$), then the algorithm respectively returns the models $r_{41,16}(\omega_1, \omega_2)$ and $r_{45,3}(\omega_1, \omega_2)$ displayed in Fig. 5a,b. Here respectively

$$\begin{aligned} N &= \{(k_1,k_2) \in \mathbb{N}^2 \mid 0 \leq k_1 + k_2 \leq 6\} \cup \\ &\quad \{(k_1,k_2) \in \mathbb{N}_0 \times (-\mathbb{N}_0) \mid 2 \leq k_1 - k_2 \leq 6\} \setminus \{(1,-5)\} \\ D &= \{(k_1,k_2) \in \mathbb{N}^2 \mid 0 \leq k_1 + k_2 \leq 4\} \cup \\ &\quad \{(k_1,k_2) \in \mathbb{N}_0 \times (-\mathbb{N}_0) \mid 2 \leq k_1 - k_2 \leq 3\} \end{aligned}$$

and

$$\begin{aligned} N &= \{(k_1,k_2) \in \mathbb{N}^2 \mid 0 \leq k_1, k_2 \leq 6\} \setminus \{(6,5),(5,6),(6,6)\} \\ D &= \{(k_1,k_2) \in \mathbb{N}^2 \mid 0 \leq k_1, k_2 \leq 1\}. \end{aligned}$$

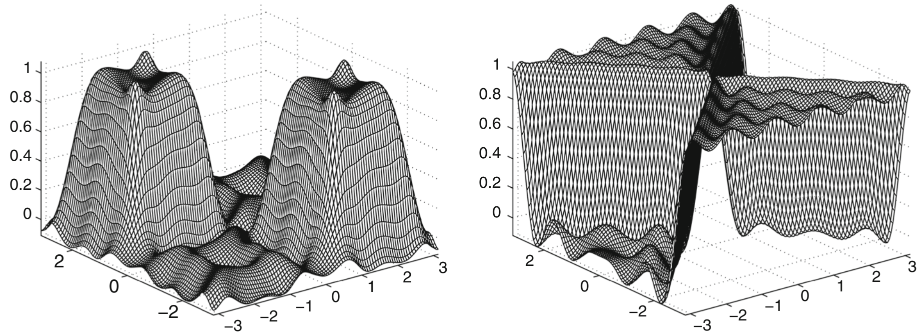
Again the filter displays a typical equiripple behaviour.

The number of additional inequalities to guarantee stability a priori increases exponentially with the denominator degree though, since it is given by $1 + 2^m$ with $m = \#D_0$. This drawback for higher degree denominators can be overcome by computing a denominator of the form $q(\omega_1, \omega_2) = (q_1 q_2)(\omega_1, \omega_2)$ where each factor is of sufficiently low degree and where for separately obtained q_1, for instance

$$\frac{\tilde{p}}{q_1} \in \left[f - \tilde{\delta}_i, f + \tilde{\delta}_i\right], \qquad \tilde{\delta}_1, \tilde{\delta}_2 > 0,$$

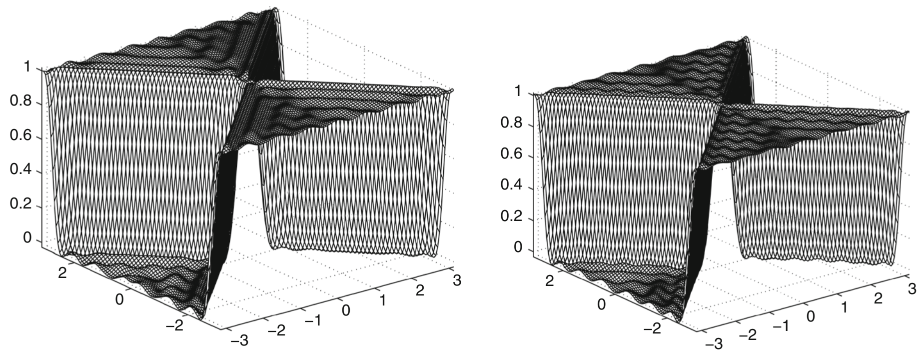
it holds that

$$\frac{p}{q_2} \in [(f - \delta_i)q_1, (f + \delta_i)q_1]$$



(a) $r_{\ell,m}(\omega_1, \omega_2) = r_{41,16}(\omega_1,\omega_2)$ **(b)** $r_{\ell,m}(\omega_1, \omega_2) = r_{45,3}(\omega_1,\omega_2)$

Fig. 5 Relaxed filters with stability conditions



(a) **(b)**

Fig. 6 Fan filters with stability conditions added to a seperable denominator

in pass- and stopband (and similarly for the transition band). This idea can be repeated to obtain more factors in the denominator polynomial. For the stricter fan filter ($\delta_1 = 0.01$, $\delta_2 = 0.02$) we obtain with $q(\omega_1, \omega_2) = (q_1 q_2 q_3)(\omega_1, \omega_2)$ the rational model displayed in Fig. 6a. Here we have chosen each factor q_j of total degree 3 with for $j = 1, 2, 3$ in the pass- and stopband $\tilde{\delta}_1 = 2^{3-j}\delta_1$, $\tilde{\delta}_2 = 2^{3-j}\delta_2$ respectively. The numerator polynomial $p(\omega_1, \omega_2)$ is of total degree 15.

When taking denominator factors of partial degree 2 in each of the variables, a simpler model can be found. With $q(\omega_1, \omega_2) = (q_1 q_2)(\omega_1, \omega_2)$ and $\tilde{\delta}_1 = 2\delta_1$, $\tilde{\delta}_2 = 2\delta_2$ the degree of the numerator is indexed by

$$N = \{(k_1, k_2) \in \mathbb{N}^2 \mid 0 \leq k_1, k_2 \leq 9\} \setminus \{(9, 9)\}$$

and the rational function is illustrated in Fig. 6b.

The coefficients of all rational models used in the paper can be found in the appendix. Note the expected symmetry patterns among related coefficients.

5 Conclusion

The reformulation of the rational approximation problem using uncertainty intervals leads to a linear programming, or preferably a quadratic programming problem with a unique global minimum. The problem statement is independent of the number of variables and hence truly multidimensional. Its solution is the rational interpolant of minimal order that passes through the uncertainty intervals as closely to their midpoint as possible, hence guaranteeing robustness in stability. Because of the use of uncertainty intervals, this rational function automatically and strictly obeys prescribed error tolerances. On the whole the technique offers several advantages compared to least squares techniques. We have illustrated its use for the development of stable symmetric multidimensional recursive filters, a technology which is required in many diverse areas, including image processing, video signal filtering, tomography and different grid-based methods in scientific computing.

A Coefficients of the obtained models

The coefficients of the rational models in this paper are obtained using MATLAB and the MOSEK[1] optimization toolbox for solving the quadratic programming problem (Salazar Celis et al. 2007).

See Tables 1, 2, 3, 4, 5, 6, 7 and 8.

Table 1 Coefficients belonging to Fig. 2a

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(0, 0)	1.651324158846769e+01	(0, 0)	4.639787058234575e+02
(1, 0)	6.365957184641155e−12	(1, 0)	−5.078040292326658e−10
(0, 1)	1.911024862632245e−12	(0, 1)	5.126174903356194e−10
(2, 0)	−1.783814442760839e+01	(2, 0)	−3.881786452646074e+02
(1, 1)	3.480534414387040e+00	(1, 1)	−5.755363975681557e+02
(0, 2)	−1.769115478011714e+01	(0, 2)	−3.907891835918771e+02
(1,-1)	−3.479180079768970e+00	(1,-1)	5.755326813847495e+02
(3, 0)	−3.367033546455229e−12	(3, 0)	2.423417572427720e−10
(2, 1)	−1.615303121247842e−11	(2, 1)	2.075139860673522e−10
(1, 2)	4.967415890641323e−12	(1, 2)	−2.153692315623192e−10
(0, 3)	−6.463240642452745e−12	(0, 3)	−2.512248804280807e−10
(2,-1)	1.979918484643810e−11	(2,-1)	−5.168997053202553e−10
(1,-2)	−7.003432084187437e−12	(1,-2)	5.283329717227284e−10
(4, 0)	−2.409709667326543e−01	(4, 0)	1.651351153356798e+01
(3, 1)	−1.729764820649716e+00	(3, 1)	8.059003157291744e+01
(2, 2)	9.904853514226945e+00	(2, 2)	1.430283826244234e+02
(1, 3)	−1.098097422716069e−03	(1, 3)	8.436372891206345e+01
(0, 4)	−5.086598397343498e−01	(0, 4)	1.773365756981746e+01
(3,-1)	1.729848499168839e+00	(3,-1)	−8.057721964717290e+01
(2,-2)	9.908133830363790e+00	(2,-2)	1.430175975626173e+02
		(1,-3)	−8.435582886925751e+01

[1] http://www.mosek.com

Table 2 Coefficients belonging to Fig. 2b

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(0,0)	6.381754129963678e+02	(0,0)	8.531810862137290e+02
(1,0)	9.074944502138662e+02	(1,0)	5.662255233620328e+02
(0,1)	−9.074944501745391e+02	(0,1)	−5.662255233046639e+02
(1,1)	−1.243241928578952e+03	(1,1)	−1.736533781887765e+03
(2,0)	2.968051980733823e+02	(2,0)	4.358851995707559e+02
(0,2)	2.968051980184051e+02	(0,2)	4.358851995223264e+02
(2,1)	−3.842954042670931e+02	(2,1)	−1.747670039033167e+02
(1,2)	3.842954042094776e+02	(1,2)	1.747670038089761e+02
(2,2)	7.496497839352641e+01	(2,2)	1.393070974039180e+02
(3,0)	3.633830202013743e+01	(3,0)	4.942849837850705e+00
(0,3)	−3.633830200011637e+01	(0,3)	−4.942849799739452e+00
(3,1)	−3.076422417863555e+01	(3,1)	−6.287214209483753e+01
(1,3)	−3.076422416295730e+01	(1,3)	−6.287214207053407e+01
		(3,2)	−1.383212008481558e+00
		(2,3)	1.383212009072552e+00

Table 3 Coefficients belonging to Fig. 3

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(0,0)	1.199990098544420e+02	(0,0)	1.931244585945764e+02
(0,1)	−1.682548600651715e+02	(0,1)	−6.052774624890729e+01
(1,0)	1.682548600651715e+02	(1,0)	6.052774624890731e+01
(1,1)	−1.963929648926201e+02	(1,1)	−3.423309403923468e+02
(2,0)	4.061172916517052e+01	(2,0)	7.664103839770986e+01
(0,2)	4.061172916517052e+01	(0,2)	7.664103839770998e+01
(0,3)	−6.107104253940785e+00		
(3,0)	6.107104253940819e+00		
(1,2)	5.235905237126963e+01		
(2,1)	−5.235905237126880e+01		
(2,2)	−8.678812623055843e−01		

Table 4 Coefficients belonging to Fig. 4

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(0,0)	6.628329186491410e+02	(0,0)	8.671760373832705e+02
(1,0)	9.423231759616856e+02	(1,0)	6.182004066718139e+02
(0,1)	−9.429926972237510e+02	(0,1)	−6.193976685341664e+02
(1,1)	−1.294693625069721e+03	(1,1)	−1.759347467574567e+03
(2,0)	3.097348037055071e+02	(2,0)	4.399859686978245e+02
(0,2)	3.106853507544964e+02	(0,2)	4.406185244684142e+02
(2,1)	−4.010433231086274e+02	(2,1)	−2.047678144395121e+02
(1,2)	4.020468397991291e+02	(1,2)	2.068239089136406e+02
(2,2)	7.940271803571203e+01	(2,2)	1.364546424841361e+02
(3,0)	3.825999707317656e+01	(3,0)	8.866022605658740e+00
(0,3)	−3.861827671733029e+01	(0,3)	−9.748898995373230e+00
(3,1)	−3.284462993253570e+01	(3,1)	−6.138752402438668e+01
(1,3)	−3.313056926800742e+01	(1,3)	−6.151335686129232e+01

Table 5 Coefficients belonging to Fig. 5a

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(0, 0)	3.316059063225431e+01	(0, 0)	1.560193405541942e+02
(1, 0)	2.827194670879147e−10	(1, 0)	−3.950593364232784e−11
(0, 1)	4.473554818689851e−10	(0, 1)	−2.911437746055384e−09
(2, 0)	−3.250751814159970e+01	(2, 0)	−5.852598525306075e+00
(1, 1)	4.615059338707415e+01	(1, 1)	−2.898042987123809e+01
(0, 2)	−2.793438982441199e+01	(0, 2)	4.598118358115006e−08
(1, -1)	−3.974602636496758e+01	(1,-1)	1.332740059830540e−01
(3, 0)	4.292401466031529e−11	(3, 0)	−7.469760286762462e−10
(2, 1)	9.505976272137319e−10	(2, 1)	−3.916347729681059e−10
(1, 2)	1.383433836835346e−09	(1, 2)	−1.999565838270551e−09
(0, 3)	5.311658340722555e−10	(0, 3)	8.592250734420980e−10
(2, -1)	−4.855918785221173e−10	(2,-1)	2.364115766477167e−09
(1, -2)	−7.323251802884898e−10	(1,-2)	−2.459657055672917e−09
(4, 0)	−5.074137979137787e+00	(4, 0)	1.709813038161260e+01
(3, 1)	−1.254890409285706e+01	(3, 1)	−4.673325394918729e+01
(2, 2)	2.071098719146597e+01	(2, 2)	−7.201225944823976e+00
(1, 3)	−8.192397209485527e+00	(1, 3)	−4.951287561086198e+01
(0, 4)	−8.111019716278742e+00		
(3, -1)	4.739576072802652e+00		
(2, -2)	1.017511309770704e+01		
(1, -3)	4.721523728273553e−01		
(5, 0)	−9.910733066859427e−10		
(4, 1)	4.513832639434310e−10		
(3, 2)	−4.736353222011735e−10		
(2, 3)	−4.938015411677864e−10		
(1, 4)	−1.486729261634895e−10		
(0, 5)	−5.159018472858196e−10		
(4, -1)	3.761570675729092e−10		
(3, -2)	1.303723156592340e−09		
(2, -3)	5.248176894263027e−10		
(1, -4)	5.339549915733015e−10		
(6, 0)	−3.897989926191942e+00		
(5, 1)	4.475869914455196e+00		
(4, 2)	−2.390084839259809e+00		
(3, 3)	5.807375988971365e+00		
(2, 4)	6.925200032700233e−01		
(1, 5)	9.071724380453116e−01		
(0, 6)	−4.899701671071803e+00		
(5, -1)	−1.779900362940602e+00		
(4, -2)	4.474973810897193e+00		
(3, -3)	5.127146019612504e+00		
(2, -4)	7.490351866543907e+00		

Table 6 Coefficients belonging to Fig. 5b

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(0,0)	3.736172378317067e+02	(0,0)	6.769106064365624e+02
(1,0)	3.624062487101197e+02	(1,0)	4.678457435219729e+00
(0,1)	−3.624062487100840e+02	(0,1)	−4.678457435178398e+00
(1,1)	−3.019332904638772e+02	(1,1)	−6.674682228721139e+02
(2,0)	−3.237360519774465e+01		
(0,2)	−3.237360519773400e+01		
(2,1)	3.534798389365433e+01		
(1,2)	−3.534798389368139e+01		
(2,2)	6.194057429900786e+01		
(3,0)	−2.049574557837724e+01		
(0,3)	2.049574557841158e+01		
(3,1)	−2.961208096033837e+01		
(1,3)	−2.961208096040617e+01		
(3,2)	3.516579076762741e+01		
(2,3)	−3.516579076759066e+01		
(3,3)	4.384190967691313e+01		
(4,0)	−1.837951020015194e+00		
(0,4)	−1.837951019980902e+00		
(4,1)	−1.554503066613432e+01		
(1,4)	1.554503066606798e+01		
(4,2)	−2.466092332634530e+01		
(2,4)	−2.466092332629933e+01		
(4,3)	2.790408351793911e+01		
(3,4)	−2.790408351796146e+01		
(4,4)	6.221743966979296e+01		
(5,0)	1.841617337979359e+00		
(0,5)	−1.841617337952468e+00		
(5,1)	5.922286011926203e+00		
(1,5)	5.922286011872261e+00		
(5,2)	−1.332835540758634e+00		
(2,5)	1.332835540806822e+00		
(5,3)	−2.749470488344321e+01		
(3,5)	−2.749470488348308e+01		
(5,4)	−1.760753985100208e+01		
(4,5)	1.760753985102446e+01		
(5,5)	3.910888886682526e+01		
(6,0)	−1.580716282043982e+00		
(0,6)	−1.580716282033275e+00		
(6,1)	−3.994295900046221e+00		
(1,6)	3.994295900023601e+00		
(6,2)	−3.585113750558630e+00		
(2,6)	−3.585113750536241e+00		

Table 6 Continued

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(6,3)	−9.230597079571752e−01		
(3,6)	9.230597079362570e−01		
(6,4)	−1.953794000524621e+01		
(4,6)	−1.953794000523334e+01		

Table 7 Coefficients belonging to Fig. 6a

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(0,0)	8.469089732089978e+01	(0,0)	1.796421047633581e+03
(0,1)	−1.157236346712771e+02	(0,1)	−3.341052216475141e−11
(1,0)	1.157236343034689e+02	(1,0)	3.240804091459014e−11
(1,1)	−1.651570828454372e+02	(1,1)	−1.599598270945978e+03
(2,0)	3.611293659225733e+01	(2,0)	−5.913716346350414e−10
(0,2)	3.611293747084370e+01	(0,2)	−5.904631433344395e−10
(0,3)	−7.734132813657423e+00	(0,3)	−9.796145204653948e+01
(3,0)	7.734132399353046e+00	(3,0)	9.796145204654452e+01
(1,2)	6.326319108389590e+01	(1,2)	−9.074101787750276e−10
(2,1)	−6.326319025104787e+01	(2,1)	9.068632195841379e−10
(2,2)	3.990283772952442e+01	(k_1, k_2)	$\beta_2(k_1, k_2)$
(4,0)	3.939479140806489e+00	(0,0)	5.258307392056687e−02
(0,4)	3.939479325265189e+00	(0,1)	5.610744066876251e−12
(3,1)	−1.905923364178958e+01	(1,0)	−3.141193456941425e−12
(1,3)	−1.905923448609409e+01	(1,1)	−4.062315648109036e−02
(0,5)	−2.461600973036756e−01	(2,0)	5.970624366143740e−03
(5,0)	2.461601943243279e−01	(0,2)	5.970625587468837e−03
(1,4)	4.075554216768530e+00	(0,3)	−1.276513862702011e−11
(4,1)	−4.075554265304498e+00	(3,0)	2.679755078330446e−12
(2,3)	−1.119584516125997e+01	(1,2)	−1.029622623054946e−11
(3,2)	1.119584507381882e+01	(2,1)	−1.504763841166207e−12
(3,3)	2.022977708720793e+00	(k_1, k_2)	$\beta_3(k_1, k_2)$
(6,0)	1.741067211845264e−01	(0,0)	1.075749702052532e+00
(0,6)	1.741068074499769e−01	(0,1)	−1.732633215890307e−10
(5,1)	−3.706821398285720e−01	(1,0)	3.971525133838591e−10
(1,5)	−3.706822991634831e−01	(1,1)	−1.068682310185149e+00
(4,2)	−1.465379012835172e+00	(2,0)	3.478501316182089e−03
(2,4)	−1.465378853562072e+00	(0,2)	3.478492121308063e−03
(0,7)	3.217508252961519e−01	(0,3)	4.210323184373265e−10
(7,0)	−3.217507662870586e−01	(3,0)	−4.309299030093476e−10
(1,6)	−8.244920908264222e−01	(1,2)	−1.372786488974350e−08
(6,1)	8.244920284389139e−01	(2,1)	1.229627363148403e−08

Table 7 Continued

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(2 , 5)	8.895849926973295e−01		
(5 , 2)	−8.895849937155715e−01		
(3 , 4)	−1.256263647150319e+00		
(4 , 3)	1.256263650251759e+00		
(4 , 4)	3.860162999401985e+00		
(8 , 0)	4.603445869053960e−02		
(0 , 8)	4.603451445953313e−02		
(7 , 1)	−3.936111254124656e−01		
(1 , 7)	−3.936112510981863e−01		
(6 , 2)	1.212472514407823e+00		
(2 , 6)	1.212472556297577e+00		
(5 , 3)	−2.851711207822016e+00		
(3 , 5)	−2.851711183582002e+00		
(0 , 9)	−1.886892003723736e−01		
(9 , 0)	1.886892189057709e−01		
(1 , 8)	−2.066012432553423e−01		
(8 , 1)	2.066011407324324e−01		
(2 , 7)	6.529325106179837e−01		
(7 , 2)	−6.529324494677279e−01		
(3 , 6)	−7.369928270417756e−01		
(6 , 3)	7.369928439365954e−01		
(4 , 5)	6.098045550500720e−01		
(5 , 4)	−6.098045761602933e−01		
(5 , 5)	2.323414176710885e+00		
(10 , 0)	−1.250390977177981e−01		
(0 ,10)	−1.250391587623128e−01		
(9 , 1)	6.994898656485085e−02		
(1 , 9)	6.994898011127475e−02		
(8 , 2)	1.267946670628435e−01		
(2 , 8)	1.267947545395793e−01		
(7 , 3)	2.716633727090701e−01		
(3 , 7)	2.716633244763975e−01		
(6 , 4)	−1.497877636853258e+00		
(4 , 6)	−1.497877640699451e+00		
(0 ,11)	−3.206726857875420e−01		
(11 , 0)	3.206726195863052e−01		
(1 ,10)	6.183208597602813e−01		
(10 , 1)	−6.183207963198536e−01		
(2 , 9)	−2.160040628556896e−01		
(9 , 2)	2.160040976823973e−01		
(3 , 8)	−1.592235323155309e−01		
(8 , 3)	1.592234866686290e−01		

Table 7 Continued

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(4,7)	$2.364556897109658e{-}01$		
(7,4)	$-2.364557099099172e{-}01$		
(5,6)	$-8.325674795976534e{-}02$		
(6,5)	$8.325677480319484e{-}02$		
(6,6)	$2.002126116233022e{+}00$		
(12,0)	$-1.196545817776470e{-}01$		
(0,12)	$-1.196546440740202e{-}01$		
(11,1)	$2.887924697895161e{-}01$		
(1,11)	$2.887926109213027e{-}01$		
(10,2)	$-8.431336015783888e{-}02$		
(2,10)	$-8.431342882210725e{-}02$		
(9,3)	$-2.826631160430215e{-}01$		
(3,9)	$-2.826631349508720e{-}01$		
(8,4)	$7.378567926751568e{-}01$		
(4,8)	$7.378568321809232e{-}01$		
(7,5)	$-1.512828008521664e{+}00$		
(5,7)	$-1.512828024753956e{+}00$		
(0,13)	$-1.120607617221355e{-}01$		
(13,0)	$1.120607299831924e{-}01$		
(1,12)	$4.224441301149029e{-}01$		
(12,1)	$-4.224440314662920e{-}01$		
(2,11)	$-4.517070519858217e{-}01$		
(11,2)	$4.517069820434019e{-}01$		
(3,10)	$2.505568056772829e{-}01$		
(10,3)	$-2.505568122039620e{-}01$		
(4,9)	$-1.090657523448959e{-}02$		
(9,4)	$1.090664481083639e{-}02$		
(5,8)	$-1.144278395443393e{-}01$		
(8,5)	$1.144277672355987e{-}01$		
(6,7)	$2.565481673829056e{-}01$		
(7,6)	$-2.565481386314332e{-}01$		
(7,7)	$1.566209577149418e{+}00$		
(14,0)	$6.795691029354550e{-}03$		
(0,14)	$6.795682762832452e{-}03$		
(13,1)	$9.142077830259862e{-}02$		
(1,13)	$9.142082273321496e{-}02$		
(12,2)	$-1.709245431222977e{-}01$		
(2,12)	$-1.709245947972156e{-}01$		
(11,3)	$2.490489920602853e{-}02$		
(3,11)	$2.490493336988972e{-}02$		
(0,15)	$-2.395094710223234e{-}03$		
(15,0)	$2.395093653010287e{-}03$		

Table 7 Continued

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(1,14)	5.507299539982150e−02		
(14,1)	−5.507298138007446e−02		
(2,13)	−1.343011529208195e−01		
(13,2)	1.343011281732314e−01		
(3,12)	1.372859601782090e−01		
(12,3)	−1.372859437146590e−01		
(4,11)	−9.900076816414249e−02		
(11,4)	9.900077757630149e−02		
(5,10)	3.356439071687140e−01		
(10,5)	−3.356439512537950e−01		
(6,9)	−1.493136044754118e+00		
(9,6)	1.493136109695435e+00		
(7,8)	3.025396315841480e+00		
(8,7)	−3.025396349309753e+00		

Table 8 Coefficients belonging to Fig. 6b

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(0,0)	1.944877908203831e+01	(0,0)	2.633764931003522e+03
(1,0)	2.218142685675664e+01	(1,0)	−2.937924079770487e−11
(0,1)	−2.218142685647408e+01	(0,1)	3.857707531357830e−11
(1,1)	−2.396222308035551e+01	(1,1)	−2.321158025019969e+03
(2,0)	2.850717669983482e+00	(2,0)	−5.448820861430599e−10
(0,2)	2.850717669667875e+00	(0,2)	−4.262423134163868e−10
(2,1)	−2.659389898322998e+00	(2,1)	6.321873998446662e+01
(1,2)	2.659389898055924e+00	(1,2)	−6.321873998458624e+01
(2,2)	7.443562139261313e−01	(2,2)	−1.853601529952025e+02
(3,0)	2.461284976915254e−02	(k_1, k_2)	$\beta_2(k_1, k_2)$
(0,3)	−2.461284975795432e−02	(0,0)	1.210009072671117e−02
(3,1)	−1.010984953698465e+00	(1,0)	2.576576617320667e−13
(1,3)	−1.010984953666688e+00	(0,1)	9.515906546998465e−15
(3,2)	−1.506586511766102e−02	(1,1)	−7.966848526971463e−03
(2,3)	1.506586512553965e−02	(2,0)	1.338856963063041e−03
(3,3)	2.406201800272446e+00	(0,2)	1.338856962978238e−03
(4,0)	−4.371950603635969e−02	(2,1)	−2.324421881135792e−14
(0,4)	−4.371950602966369e−02	(1,2)	1.755951318920714e−15
(4,1)	−3.089214481619190e−01	(2,2)	−1.449104637649834e−03
(1,4)	3.089214481560866e−01		
(4,2)	−8.281510203475669e−01		
(2,4)	−8.281510203716866e−01		

Table 8 Continued

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(4,3)	5.024773738467837e−01		
(3,4)	−5.024773738183894e−01		
(4,4)	1.391073109778080e+00		
(5,0)	−1.003944577994073e−01		
(0,5)	1.003944577957896e−01		
(5,1)	−3.089526840322641e−02		
(1,5)	−3.089526838991598e−02		
(5,2)	−1.028204883858354e−01		
(2,5)	1.028204883811935e−01		
(5,3)	−5.992329820576513e−01		
(3,5)	−5.992329820744852e−01		
(5,4)	2.800848399824792e−02		
(4,5)	−2.800848398521759e−02		
(5,5)	1.015801103632452e+00		
(6,0)	3.812408513398009e−02		
(0,6)	3.812408513456082e−02		
(6,1)	−1.721681990904223e−02		
(1,6)	1.721681989520925e−02		
(6,2)	−8.253168698706596e−02		
(2,6)	−8.253168697696288e−02		
(6,3)	1.863375271397620e−02		
(3,6)	−1.863375274582161e−02		
(6,4)	−4.909624809633435e−01		
(4,6)	−4.909624809593242e−01		
(6,5)	−2.197979017261854e−01		
(5,6)	2.197979017337340e−01		
(6,6)	6.315082020415009e−01		
(7,0)	−4.070156187235936e−04		
(0,7)	4.070156330570882e−04		
(7,1)	−1.550551121044997e−02		
(1,7)	−1.550551120287156e−02		
(7,2)	−3.082259758398134e−02		
(2,7)	3.082259756993801e−02		
(7,3)	−4.438551862203208e−02		
(3,7)	−4.438551864003468e−02		
(7,4)	8.007249824733581e−02		
(4,7)	−8.007249823360361e−02		
(7,5)	−2.644033581987892e−01		
(5,7)	−2.644033581817832e−01		
(7,6)	−3.270661436517789e−02		
(6,7)	3.270661436628375e−02		

Table 8 Continued

(k_1, k_2)	$\alpha(k_1, k_2)$	(k_1, k_2)	$\beta_1(k_1, k_2)$
(7,7)	5.891786029136298e−01		
(8,0)	4.956507336916979e−02		
(0,8)	4.956507335673031e−02		
(8,1)	8.521516521264528e−04		
(1,8)	−8.521516412734176e−04		
(8,2)	8.658693185713486e−02		
(2,8)	8.658693189308481e−02		
(8,3)	−2.926347649480631e−02		
(3,8)	2.926347650498035e−02		
(8,4)	−4.310150421156841e−02		
(4,8)	−4.310150424323096e−02		
(8,5)	−2.064871635556232e−01		
(5,8)	2.064871635271779e−01		
(8,6)	−2.950254776587476e−01		
(6,8)	−2.950254776429926e−01		
(8,7)	−5.706821336192940e−01		
(7,8)	5.706821336451177e−01		
(8,8)	8.339800104668572e−02		
(9,0)	2.514014649964506e−03		
(0,9)	−2.514014654462727e−03		
(9,1)	−5.493035937279511e−02		
(1,9)	−5.493035937847541e−02		
(9,2)	−1.774711220533731e−03		
(2,9)	1.774711197937876e−03		
(9,3)	−1.487860639332973e−02		
(3,9)	−1.487860639484312e−02		
(9,4)	1.095214493221580e−01		
(4,9)	−1.095214493114009e−01		
(9,5)	1.629969690103796e−04		
(5,9)	1.629969834326749e−04		
(9,6)	2.562477273846552e−01		
(6,9)	−2.562477273968600e−01		
(9,7)	−3.634747153259665e−02		
(7,9)	−3.634747153876455e−02		
(9,8)	−4.079110503749314e−01		
(8,9)	4.079110503747073e−01		

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Author Biographies

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