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Model Reduction of Multidimensional Linear Shift-Invariant Recursive Systems Using Padé Techniques*

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Abstract. This paper describes a very flexible "general order" multivariate Padé approximation technique for the model reduction of a multidimensional linear shift-invariant recursive system, i.e., a system characterized by a multivariate rational transfer function. The technique presented allows full control of the regions of support in numerator and denominator of the reduced system and also admits a nonbranched continued fraction representation for an easy realization of the model. The method presented here overcomes some of the problems of related approaches to model reduction of multidimensional linear recursive systems. Different rational approximants can be introduced to compute the reduced model, but a drawback is that these approximants are not always readily available in continued fraction form for immediate implementation of the reduced system. Also multibranched continued fractions can be used to approximate the transfer function, but it was pointed out that the regions of support of numerator and denominator blow up rapidly as one considers successive convergents. Both these problems are overcome here.

Key Words: Model reduction, multidimensional systems, multivariate Padé

1. Introduction

Multidimensional systems arise in problems like computer-aided tomography, image processing, image deblurring, seismology, sonar and radar applications, and many other problems. Many operations performed on one-dimensional signals remain valid in the multidimensional case but the mathematics for handling multidimensional systems is less complete than the mathematics for handling one-dimensional systems [1]. Filtering signals, such as in image deblurring, is a discipline born of the computer revolution, that is concerned with the extraction and/or enhancement of information contained in a one-dimensional or multidimensional sequence of measurements. Noises can be filtered from spoken messages or picture images. Systems can transform a message to a form recognizable by a computer. The number of applications is legion. We shall indicate here new techniques for the model reduction of multidimensional linear shift-invariant (LSI) systems with infinite-extent impulse response (IIR). From the one-dimensional theory one knows that the problem of model

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reduction is equivalent to the computation of Padé approximants [2]. It will become clear that this is also true in multidimensional systems theory.

A multidimensional discrete signal is represented by a multidimensional array $x(n_1, \ldots, n_p)$. For simplicity of notation we will restrict ourselves to the case $p = 2$. An important example of discrete signals is the unit impluse $\delta(n_1, n_2)$, defined by $\delta(n_1, n_2) = 1$ for $n_1 = n_2 = 0$ and $\delta(n_1, n_2) = 0$ elsewhere.

When talking about two-dimensional LSI systems we will always refer to recursive systems or systems with infinite-extent impulse response, which transform an input signal $x(n_1, n_2)$ into an output signal $y(n_1, n_2)$ such that $y(n_1, n_2)$ can be described by a difference equation of the form

$$y(n_1, n_2) = \sum_{\substack{(k_1,k_2)\in N \\ N\subset \mathbb{Z}^2}} a(k_1, k_2)x(n_1 - k_1, n_2 - k_2) - \sum_{\substack{(k_1,k_2)\in D^\circ \\ D^\circ\subset \mathbb{Z}^2\setminus\{(0,0)\}}} b(k_1, k_2)y(n_1 - k_1, n_2 - k_2), \tag{1}$$

where $D^\circ \neq \phi$. The sets N and $D = D^\circ \cup \{(0, 0)\}$ are the regions of support of the arrays $a(n_1, n_2)$ and $b(n_1, n_2)$ respectively with $b(0, 0) = 1$. For $x(n_1, n_2) = \delta(n_1, n_2)$ the above difference equation becomes

$$h(n_1, n_2) = a(n_1, n_2) - \sum_{(k_1,k_2)\in D^\circ} b(k_1, k_2)h(n_1 - k_1, n_2 - k_2), \tag{2}$$

and since $D^\circ \neq \phi$ the signal $h(n_1, n_2)$, which is called the impulse response of the system, indeed has infinite extent. Remark that a recursive system is only recursively computable if it allows an ordering by which the output values $y(n_1, n_2)$ can be computed sequentially, given a set of initial conditions. It is clear that this depends on the subset $D^\circ \subset \mathbb{Z}^2$.

Taking the z-transform of both sides of (2) results in

$$\begin{aligned} H(z_1, z_2) &= A(z_1, z_2) - \sum_{(k_1,k_2)\in D^\circ} b(k_1, k_2)H(z_1, z_2)z_1^{-k_1}z_2^{-k_2} \\ &= \sum_{(k_1,k_2)\in N} a(k_1, k_2)z_1^{-k_1}z_2^{-k_2} - H(z_1, z_2) \sum_{(k_1,k_2)\in D^\circ} b(k_1, k_2)z_1^{-k_1}z_2^{-k_2}. \end{aligned}$$

Here $H(z_1, z_2)$ is the z-transform of the impulse response $h(n_1, n_2)$ and is called the transfer function of the system. So the transfer function of a recursive system is the ratio of the z-transforms $A(z_1, z_2)$ and $B(z_1, z_2)$ of the coefficient arrays $a(n_1, n_2)$ and $b(n_1, n_2)$, with $b(0, 0) = 1$:

$$H(z_1, z_2) = \frac{\Sigma_{(k_1,k_2)\in N}\, a(k_1, k_2)z_1^{-k_1}z_2^{-k_2}}{1 + \Sigma_{(k_1,k_2)\in D^\circ}\, b(k_1, k_2)z_1^{-k_1}z_2^{-k_2}}$$

$$= \frac{\Sigma_{(k_1,k_2)\in N}\, a(k_1, k_2) z_1^{-k_1} z_2^{-k_2}}{\Sigma_{(k_1,k_2)\in D}\, b(k_1, k_2) z_1^{-k_1} z_2^{-k_2}}$$

$$= \frac{A(z_1, z_2)}{B(z_1, z_2)} \tag{3}$$

2. Model reduction

Without loss of generality [1, pp. 176–180] we restrict ourselves here to systems whose impulse response has support on the first quadrant. We can therefore write for the transfer function of the desired system

$$H(z_1, z_2) = \sum_{n_1=0}^{+\infty} \sum_{n_2=0}^{+\infty} h(n_1, n_2) z_1^{-n_1} z_2^{-n_2},$$

or, equivalently,

$$H\left(\frac{1}{w_1}, \frac{1}{w_2}\right) = \sum_{n_1=0}^{+\infty} \sum_{n_2=0}^{+\infty} h(n_1, n_2) w_1^{n_1} w_2^{n_2}, \tag{4}$$

where the equality sign is only formal. We know from (3) that for a recursive system the transfer function $H(1/w_1, 1/w_2)$ is a bivariate rational function in w_1 and w_2, and hence techniques from bivariate Padé approximation can be used to approximate the series (4). In [3], [4] Canterbury approximants were used to compute the reduced model. A drawback is that these Canterbury approximants are not available in continued fraction form for an easy realization of the model [5], [6]. In [7], [8] multibranched continued fractions are used to approximate the transfer function. However in [9] is pointed out that the index sets or regions of support of numerator and denominator blow up rapidly as one considers successive convergents. In the sequel of this section we present a very flexible multivariate Padé approximation technique that, on one hand, allows full control of the index sets in numerator and denominator of the reduced system and, on the other hand, admits a continued fraction representation for its immediate implementation.

Given a power series expansion

$$\tilde{H}(w_1, w_2) = \sum_{(i,j)\in \mathbb{N}^2} h(i, j) w_1^i w_2^j, \tag{5}$$

we shall compute an approximant $p(w_1, w_2)/q(w_1, w_2)$ to (5) by an accuracy-through-order principle. The polynomials $p(w_1, w_2)$ and $q(w_1, w_2)$ are of the form

$$p(w_1, w_2) = \sum_{(i,j)\in N} a(i, j) w_1^i w_2^j,$$

$$q(w_1, w_2) = \sum_{(i,j)\in D} b(i, j) w_1^i w_2^j,$$

where N (numerator) and D (denominator) are finite subsets of $\mathbb{N}^2$, indicating in a way the degree of $p(w_1, w_2)$ and $q(w_1, w_2)$. Let us denote, with # for the cardinality of a set,

$$\#N = n + 1, \qquad \#D = m + 1$$

It is now possible to let $p(w_1, w_2)$ and $q(w_1, w_2)$ satisfy

$$(\tilde{H}\, q - p)(w_1, w_2) = \sum_{(i,j)\in\mathbb{N}^2\setminus E} g(i, j) w_1^i w_2^j \tag{6}$$

if, in analogy with the univariate case, the index set E (equations) is such that

$$N \subseteq E, \tag{7a}$$
$$\#(E\setminus N) = m = \#D - 1, \tag{7b}$$
$$E \text{ satisfies the inclusion property.} \tag{7c}$$

The last condition on E means that when a point belongs to the index set E, then the rectangular subset of points emanating from the origin with the given point as its furthermost corner, also lies in E. Condition (7a) enables us to split the system of equations

$$g(i, j) = 0, \qquad (i, j) \in E,$$

in an inhomogeneous part defining the numerator coefficients

$$\sum_{\mu=0}^{i} \sum_{\nu=0}^{j} h(\mu, \nu) b(i - \mu, j - \nu) = a(i, j), \qquad (i, j) \in N, \tag{8a}$$

and a homogeneous part defining the denominator coefficients

$$\sum_{\mu=0}^{i} \sum_{\nu=0}^{j} h(\mu, \nu) b(i - \mu, j - \nu) = 0, \qquad (i, j) \in E\setminus N. \tag{8b}$$

By convention $b(i, j) = 0$ if $(i, j) \notin D$. Condition (7b) guarantees the existence of a nontrivial denominator $q(w_1, w_2)$ because the homogeneous system has one equation less than the number of unknowns and so one unknown coefficient can be chosen freely. Condition (7c) finally takes care of the Padé approximation property, namely

$$\left(\tilde{H} - \frac{p}{q}\right)(w_1, w_2) = \sum_{(i,j)\in\mathbb{N}^2\setminus E} f(i, j) w_1^i w_2^j.$$

For more information we refer to [10]. For the sake of simplicity we assume that the homogeneous system of equations (8b) has maximal rank. However, what follows can easily be extended to the case where this is not true, by adding points to the set $E\setminus N$ until the rank deficiency has disappeared, but at this moment this would only complicate the notation. So for the moment $\#E = n + m + 1$.

Let us now introduce a numbering of the points in $\mathbb{N}^2$. We can for instance enumerate $\mathbb{N}^2$ as follows:

$$(0, 0), (1, 0), (0, 1), (1, 1), (2, 0), (2, 1), (0, 2), (1, 2), (2, 2),$$
$$(3, 0), (3, 1), (3, 2), (0, 3), (1, 3), (2, 3), (3, 3), \ldots .$$

This particular numbering of $\mathbb{N}^2$ is only one possible choice. Any other numbering $r(i, j)$ of $\mathbb{N}^2$ can be chosen as long as it satisfies the following property:

$$i \leq k \text{ and } j \leq l \Rightarrow r(i, j) \leq r(k, l). \tag{9}$$

Using the above enumeration, we can write

$$\begin{aligned} \emptyset &= N_{-1} \subset N_0 \subset N_1 \subset \cdots \subset N_{n-1} \subset N_n = N, \\ \#N_l &= l + 1, \\ N_l \setminus N_{l-1} &= \{(i_l, j_l)\}, \qquad l = 0, \ldots, n, \\ r(i_l, j_l) &= l. \end{aligned}$$

In other words, for each $l = 0, \ldots, n$ we add to N_{l-1} the point (i_l, j_l) which is next in line in $N \cap \mathbb{N}^2$. Analogously we write

$$\emptyset = D_{-1} \subset D_0 \subset D_1 \subset \cdots \subset D_{m-1} \subset D_m = D$$

with

$$\begin{aligned} \#D_l &= l + 1, \\ D_l \setminus D_{l-1} &= \{(d_l, e_l)\}, \qquad l = 0, \ldots, m, \end{aligned}$$

and

$$E = \bigcup_{l=0}^{n+m} E_l = E_{n+m},$$

with

$$\begin{aligned} E_l &= N_l, \qquad l = 0, \ldots, n, \\ E_{n+l} \setminus E_{n+l-1} &= \{(i_{n+l}, j_{n+l})\}, \qquad l = 1, \ldots, m, \\ r(i_{n+l}, j_{n+l}) &= n + l. \end{aligned}$$

Note that because of condition (9) the subsets E_l ($l = 0, \ldots, n + m$) consisting of the first $l + 1$ elements of E, individually satisfy the inclusion property.

It was shown in [10] that a determinant representation for

$$p_l(w_1, w_2) = \sum_{(i,j)\in N_l} a(i, j) w_1^i w_2^j, \qquad 0 \le l \le n,$$

and

$$q_r(w_1, w_2) = \sum_{(i,j)\in D_r} b(i, j) w_1^i w_2^j, \qquad 0 \le r \le m,$$

satisfying

$$(\tilde{H}q_r - p_l)(w_1, w_2) = \sum_{(i,j)\in \mathbb{N}^2 \setminus E_{l+r}} g(i, j) w_1^i w_2^j$$

is given by

$$p_l(w_1, w_2) = \begin{vmatrix} \sum_{(i,j)\in N_l} h(i - d_0, j - e_0) w_1^i w_2^j & \dots & \sum_{(i,j)\in N_l} h(i - d_r, j - e_r) w_1^i w_2^j \\ h(i_{l+1} - d_0, j_{l+1} - e_0) & \dots & h(i_{l+1} - d_r, j_{l+1} - e_r) \\ \vdots & & \vdots \\ h(i_{l+r} - d_0, j_{l+r} - e_0) & \dots & h(i_{l+r} - d_r, j_{l+r} - e_r) \end{vmatrix}$$

and

$$q_r(w_1, w_2) = \begin{vmatrix} w_1^{d_0} w_2^{e_0} & \dots & w_1^{d_r} w_2^{e_r} \\ h(i_{l+1} - d_0, j_{l+1} - e_0) & \dots & h(i_{l+1} - d_r, j_{l+1} - e_r) \\ \vdots & & \vdots \\ h(i_{l+r} - d_0, j_{l+r} - e_0) & \dots & h(i_{l+r} - d_r, j_{l+r} - e_r) \end{vmatrix}$$

where $h(i, j) = 0$ if $i < 0$ or $j < 0$.

A solution of the original problem (6) is then given by $p_n(w_1, w_2)/q_m(w_1, w_2)$ since $N_n = N$, $D_m = D$ and $E_{n+m} = E$. However, the rational functions $p_l(w_1, w_2)/q_r(w_1, w_2)$ are themselves Padé approximants. They satisfy only part of the approximation conditions (6) (determined by E_{l+r}) and have N_l and D_r as "degree" of numerator and denominator. If one does not need explicit knowledge of the numerator and denominator coefficients $a(i, j)$ and $b(i, j)$, it is shown in [11] that the value of the Padé approximants $p_l(w_1, w_2)/q_r(w_1, w_2)$ can easily be computed recursively for $l + r \le n + m$.

We shall now show how a nonbranched continued fraction representation can be constructed for the Padé approximants p_l/q_r. However, we first want to point out that the approximants introduced in [12] differ from the Padé approximants described here, in the sense that in [12] the index set E must have a well-specified form which does not satisfy the inclusion property. Therefore, the recursive computation scheme of [12] is different from the one discussed in [11]. Also, no continued fraction representation has been given for the approximants in [12].

From now on we shall write for the lower-order solutions of the Padé approximation problem (6)

$$\frac{p_l}{q_r}(w_1, w_2) = [N_l/D_r]_{E_{l+r}}.$$

With these solutions we can fill up a table of Padé approximants where $[N_n/D_m]_{E_{n+m}} = [N/D]_E$:

$$\begin{array}{lll} [N_0/D_0]_{E_0} & [N_0/D_1]_{E_1} & [N_0/D_2]_{E_2} \ \cdots \\ [N_1/D_0]_{E_1} & [N_1/D_1]_{E_2} & [N_1/D_2]_{E_3} \ \cdots \\ [N_2/D_0]_{E_2} & [N_2/D_1]_{E_3} & \ddots \\ \vdots & \vdots & \end{array}$$

In this table we consider descending staircases of multivariate Padé approximants, like

$$\begin{array}{llll} [N_s/D_0]_{E_s} & & & \\ [N_{s+1}/D_0]_{E_{s+1}} & [N_{s+1}/D_1]_{E_{s+2}} & & \\ & [N_{s+2}/D_1]_{E_{s+3}} & [N_{s+2}/D_2]_{E_{s+4}} & \\ & & \vdots & \cdots \end{array}$$

and construct continued fractions of which the l^{th} convergent equals the $(l+1)^{\text{th}}$ entry on the staircase. We restrict ourselves to the case where every three successive elements on the staircase are different. In [13] it is proved that, given this staircase, a continued fraction representation of the form

$$[N_s/D_0]_{E_s} + \frac{[N_{s+1}/D_0]_{E_{s+1}} - [N_s/D_0]_{E_s} \big|}{\big| 1} + \frac{-q_1^{(s+1)} \big|}{\big| 1 + q_1^{(s+1)}} + \frac{-e_1^{(s+1)} \big|}{\big| 1 + e_1^{(s+1)}}$$
$$+ \frac{-q_2^{(s+1)} \big|}{\big| 1 + q_2^{(s+1)}} + \frac{-e_2^{(s+1)} \big|}{\big| 1 + e_2^{(s+1)}} + \cdots$$

exists of which the successive convergents equal the successive Padé approximants on the staircase. We now resume the formulas necessary for the computation of $q_l^{(s+1)}$ and $e_l^{(s+1)}$ in our case. A proof of these formulas can be constructed in a completely analogous way as was done in [13] for a more general problem.

Input of the algorithm is

$$\tilde{H}(w_1, w_2) = \sum_{(i,j)\in \mathbb{N}^2} h(i, j) w_1^i w_2^j.$$

With the unit impulse response $h(i, j)$ and with the sets

$$E_l = \{(i_0, j_0), \ldots, (i_l, j_l)\},$$
$$D_r = \{(d_0, e_0), \ldots, (d_r, e_r)\}$$

we construct for $l = 0, \ldots, n + m$ and $r = 0, \ldots, m$,

$$t_r(l) = \sum_{(i,j)\in E_l} h(i - d_r, j - e_r) w_1^{i-d_r} w_2^{j-e_r},$$

and for $l = 0, \ldots, n + m$ and $r = 1, \ldots, m$,

$$g_{0,r}^{(l)} = t_r(l) - t_{r-1}(l),$$
$$g_{r,s}^{(l)} = \frac{g_{r-1,s}^{(l)} g_{r-1,r}^{(l+1)} - g_{r-1,s}^{(l+1)} g_{r-1,r}^{(l)}}{g_{r-1,r}^{(l+1)} - g_{r-1,r}^{(l)}}, \qquad s = r + 1, r + 2, \ldots.$$

The $q_l^{(s+1)}$ and $e_l^{(s+1)}$ are stored as in table (10):

$$\begin{array}{ccccc}
q_1^{(1)} & & & & \\
 & e_1^{(1)} & & & \\
q_1^{(2)} & & q_2^{(1)} & & \\
 & e_1^{(2)} & & e_2^{(1)} & \ddots \\
q_1^{(3)} & & q_2^{(2)} & & \\
 & e_1^{(3)} & & e_2^{(2)} & \ddots \\
q_1^{(4)} & & q_2^{(3)} & & \\
\vdots & e_1^{(4)} & \vdots & e_2^{(3)} & \\
 & \vdots & & \vdots &
\end{array}$$

while the values $g_{r,s}^{(l)}$ are stored as in table (11).
The first column of (10) is given by

$$q_1^{(s+1)} = \frac{t_0(s + 2) - t_0(s + 1)}{t_0(s + 1) - t_0(s)} \frac{g_{0,1}^{(s+1)}}{g_{0,1}^{(s+1)} - g_{0,1}^{(s+2)}}$$
$$= \frac{h(i_{s+2} - d_0, j_{s+2} - e_0) w_1^{i_{s+2}-d_0} w_2^{j_{s+2}-e_0}}{h(i_{s+1} - d_0, j_{s+1} - e_0) w_1^{i_{s+1}-d_0} w_2^{j_{s+1}-e_0}} \frac{g_{0,1}^{(s+1)}}{g_{0,1}^{(s+1)} - g_{0,1}^{(s+2)}}.$$

Subsequent e-columns are computed from

$$e_l^{(s+1)} + 1 = \frac{g_{l-1,l}^{(s+l)} - g_{l-1,l}^{(s+l+1)}}{g_{l-1,l}^{(s+l)}} (q_l^{(s+2)} + 1),$$

$$
\begin{array}{c|cc|c|cccc|c|ccc}
g_{0,1}^{(0)} & g_{0,2}^{(0)} & & & g_{0,r}^{(0)} & & & & & g_{0,m}^{(0)} & & \\
 & & g_{1,2}^{(0)} & & & g_{1,r}^{(0)} & & & & & \ddots & \\
g_{0,1}^{(1)} & g_{0,2}^{(1)} & & & g_{0,r}^{(1)} & & \ddots & & & g_{0,m}^{(1)} & & \\
 & & g_{1,2}^{(1)} & & & g_{1,r}^{(1)} & & g_{r-1,r}^{(0)} & & \vdots & & g_{m-1,m}^{(0)} \\
g_{0,1}^{(2)} & g_{0,2}^{(2)} & \vdots & \cdots & g_{0,r}^{(2)} & \vdots & & \vdots & \cdots & & & \vdots \\
\vdots & \vdots & & & \vdots & & & g_{r-1,r}^{(n+m-r+1)} & & & & g_{m-1,m}^{(n+1)} \\
 & & & & & & \iddots & & & & & \\
 & & g_{1,2}^{(n+m-1)} & & & g_{1,r}^{(n+m-1)} & & & & & \iddots & \\
g_{0,1}^{(n+m)} & g_{0,2}^{(n+m)} & & & g_{0,r}^{(n+m)} & & & & & g_{0,m}^{(n+m)} & &
\end{array}
\qquad (11)
$$

and subsequent q-columns by

$$q_l^{(s+1)} = \frac{e_{l-1}^{(s+2)} q_{l-1}^{(s+2)}}{e_{l-1}^{(s+1)}} \frac{g_{l-2,l-1}^{(s+l-1)} - g_{l-2,l-1}^{(s+l)}}{g_{l-2,l-1}^{(s+l-1)}} \frac{g_{l-1,l}^{(s+l)}}{g_{l-1,l}^{(s+l)} - g_{l-1,l}^{(s+l+1)}}.$$

In case not all three successive approximants on the descending staircase are different, singular rules must be used for the computation of the $q_l^{(s+1)}$- and $e_l^{(s+1)}$-values. These singular rules can be found in [14].

The staircases considered above were all lying below the main diagonal of the table of multivariate Padé approximants. If a Padé approximant $(p_l/q_r)(w_1, w_2)$ with $l < r$ has to be computed, then the following reciprocal covariance property could be used [15]: calculate the Padé approximant $(q_r/p_l)(w_1, w_2)$ to the series $1/\tilde{H}$ and invert it.

We shall now illustrate the above technique with an example. Consider the LSI system given by the transfer function

$$H(z_1, z_2) = \frac{1}{B(z_1, z_2)}, \tag{12a}$$

where

$$\begin{aligned} B(z_1, z_2) &= B(w_1^{-1}, w_2^{-1}) \\ &= (1 - 0.1w_1 - 0.1w_2 - 0.1w_1w_2)(1 - 0.15w_1 - 0.15w_2 - 0.2w_1w_2) \\ &\qquad (1 - 0.2w_1 - 0.2w_2 - 0.4w_1w_2). \end{aligned} \tag{12b}$$

The Taylor series expansion of $\tilde{H}(w_1, w_2) = H(w_1^{-1}, w_2^{-1})$ is given by

$$\tilde{H}(w_1, w_2) = 1 + 0.45w_1 + 0.45w_2 + 0.1375w_1^2 + 0.975w_1w_2 + 0.1375w_2^2 + \cdots.$$

We choose the sets N, D, and E for the general-order Padé approximant to $\tilde{H}(w_1, w_2)$ as follows:

$$\begin{aligned} N &= \{(0, 0), (1, 1)\}, \\ D &= \{(0, 0), (1, 0), (0, 1)\}, \\ E &= N \cup \{(1, 0), (0, 1)\}. \end{aligned}$$

Note that the sets N, D, and E satisfy condition (7) and that, besides E, also D satisfies the inclusion property. The sets N and D are the index sets of the polynomials $p(w_1, w_2)$ and $q(w_1, w_2)$; i.e.,

$$\begin{aligned} p(w_1, w_2) &= a(0, 0) + a(1, 1)w_1w_2, \\ q(w_1, w_2) &= b(0, 0) + b(1, 0)w_1 + b(0, 1)w_2, \end{aligned}$$

where the coefficients $a(i, j)$ and $b(i, j)$ must satisfy (8). In this example, we can immediately solve the system of equations (8) and find

$$p(w_1, w_2) = 1 + 0.495w_1w_2,$$
$$q(w_1, w_2) = 1 - 0.45w_1 - 0.45w_2.$$

In other words, the transfer function of the reduced system obtained through the Padé approximation technique described here is given by

$$\bar{H}(z_1, z_2) = \frac{1 + 0.495z_1^{-1}z_2^{-1}}{1 - 0.45z_1^{-1} - 0.45z_2^{-1}}.$$

Whereas the number of terms in the numerator and denominator of the original transfer function equals 16, for $\bar{H}(z_1, z_2)$ this number amounts to 4.

For model reduction techniques the issue of stability of the reduced system is an important one. A system is called bounded-input bounded-output (BIBO) stable if the output signal is bounded whenever the input signal is bounded. As in the one-dimensional case, this definition of stability can be reformulated in terms of the transfer function of the system. The following theorem summarizes some of the existing results.

THEOREM [1]: Let T be a two-dimensional first-quadrant LSI system with a rational transfer function given by (3) and having no nonessential singularities of the second kind on the unit bicircle. Then the system is stable if and only if

(i) $B(z_1, z_2) \neq 0$ for $|z_1| \geq 1, |z_2| \geq 1$
if and only if
(ii) (a) $B(z_1, z_2) \neq 0$ for $|z_1| \geq 1, |z_2| = 1$
(b) $B(z_1, z_2) \neq 0$ for $|z_1| = 1, |z_2| \geq 1$
if and only if
(iii) (a) $B(z_1, z_2) \neq 0$ for $|z_1| = 1, |z_2| = 1$
(b) $B(a, z_2) \neq 0$ for $|z_2| \geq 1$ for any a such that $|a| = 1$
(c) $B(z_1, b) \neq 0$ for $|z_1| \geq 1$ for any b such that $|b| = 1$.

It is clear from the above theorem that the stability of multidimensional LSI systems is essentially related to the zero set of the denominator polynomial. In order to be able to discuss the stability of the reduced system, we are therefore further investigating the polar sets of the multivariate Padé approximants which model the reduced system. Partial results have been obtained so far and can be found in [16], [17].

In our example, the original system given by (12) is stable. This can be verified by looking at the root map of $B(z_1, z_2)$ [1]. The root map of $B(z_1, z_2)$ consists of two root images:

one root image shows the loci of the roots of $B(z_1, z_2)$ as z_1 traverses the unit circle $z_1 = e^{i\phi}$ for $-\pi \leq \phi \leq \pi$. The other root image shows the loci of the roots of $B(z_1, z_2)$ as z_2 traverses the unit circle. We remark that part (ii) of the theorem states that the system is stable if both root images of $B(z_1, z_2)$ lie inside the unit circle. Since $B(z_1, z_2)$ given by (12) is symmetric in z_1 and z_2, the two root images of $B(z_1, z_2)$ coincide and are given by

$$z_1 = z_2 = \frac{a_k + c_k e^{-i\phi}}{1 - b_k e^{-i\phi}}, \qquad -\pi \leq \phi \leq \pi, \quad k = 1, 2, 3,$$

where

$$\begin{array}{lll} a_1 = 0.1 & b_1 = 0.1 & c_1 = 0.1 \\ a_2 = 0.15 & b_2 = 0.15 & c_2 = 0.2 \\ a_3 = 0.2 & b_3 = 0.2 & c_3 = 0.4 \end{array}$$

One can easily verify that with those values of a_k, b_k and c_k, $k = 1, 2, 3$, the root images of $B(z_1, z_2)$ lie inside the unit circle and hence the original system is stable. If we look at the root images of the reduced system, we find

$$z_1 = z_2 = \frac{0.45}{1 - 0.45e^{-i\phi}}, \qquad -\pi \leq \phi \leq \pi.$$

Again it can easily be verified that these root images lie entirely inside the unit circle and therefore the reduced system, obtained by the Padé approximation technique, is also stable in this example.

3. Summary

The technique for model reduction of a multidimensional linear shift-invariant recursive system

$$H(z_1, z_2) = \frac{\Sigma_{(i,j)\in N}\, a(i, j) z_1^{-i} z_2^{-j}}{1 + \Sigma_{(i,j)\in D^\circ}\, b(i, j) z_1^{-i} z_2^{-j}}$$

which is known through its unit impulse response

$$\tilde{H}(w_1, w_2) = \sum_{(i,j)\in \mathbb{N}^2} h(i, j) w_1^i w_2^j$$

can be summarized as follows. Choose subsets $N_l \subset N$, $D_r \subset D = D^\circ \cup \{(0, 0)\}$ indexing multivariate polynomials $p_l(w_1, w_2)$ and $q_r(w_1, w_2)$ respectively and impose the following conditions on $p_l(w_1, w_2)$ and $q_r(w_1, w_2)$:

$$(\tilde{H}q_r - p_l)(w_1, w_2) = \sum_{(i,j)\in \mathbb{N}^2 \setminus E_{l+r}} g(i, j) w_1^i w_2^j,$$

where N_l, D_r, and E_{l+r} must satisfy conditions (7) and the subscripts l, r, and $l + r$ come from a particular enumeration of $\mathbb{N}^2$ which satisfies (9). The multivariate Padé approximant $(p_l/q_r)(w_1, w_2)$ can be computed either through an explicit determinant formula or in continued fraction form for an easy realization of the reduced model. To this end, put $s = l - r$ and write

$$\frac{p_l(w_1, w_2)}{q_r(w_1, w_2)} = \sum_{(i,j)\in N_s} h(i, j) w_1^i w_2^j + \frac{h(i_{s+1}, j_{s+1}) w_1^{i_{s+1}} w_2^{j_{s+1}} \,\big|}{\big|\, 1} + \sum_{i=1}^{r-1} \left(\frac{-q_i^{(s+1)} \,\big|}{\big|\, 1 + q_i^{(s+1)}} + \frac{-e_i^{(s+1)} \,\big|}{\big|\, 1 + e_i^{(s+1)}} \right) + \frac{-q_r^{(s+1)} \,\big|}{\big|\, 1 + q_r^{(s+1)}}.$$

Notes

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