Verified Computation of Packet loss Probabilities in Multiplexer Models using Rational Approximation

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A statistical multiplexer is a basic model used in the design and the dimensioning of communication networks. The multiplexer model consists of a finite buffer, to store incoming packets, served by a single server with constant service time, and a more or less complicated arrival process. The aim is to determine the packet loss probability as a function of the capacity of the buffer. An exact analytic approach is unfeasible, and hence we show how techniques from rational approximation can be applied for the computation of the packet loss. Since the parameters used in such networks may not produce precise probabilities of interest without having to introduce drastic assumptions, we also need to carry out a perturbation analysis with respect to these parameters. This will be done by means of interval arithmetic.

1 Introduction

Fixed length packet switches have been studied extensively in the context of ATM switching models. However, since the Internet is primarily TCP/IP with variable length packets, it is even more important to analyze switching in the new context. Variable bit rate (VBR) communications with real time constraints in general, and video communication services (video phone, video conferencing, television distribution) in particular, are expected to be a major class of services provided by the future Quality of Service (QoS) enabled Internet. The introduction of statistical multiplexing techniques offers the capability to efficiently support VBR connections by taking advantage of the variability of the bandwidth requirements of individual connections.

In order to assess the multiplexing gain, a variety of techniques have been developed in recent years, based on the exact analysis using matrix-analytic methods [3], fluid approximation [4] and simulation [5] to study these multiplexer models. In particular, considerable work has been spent at the development of analytical techniques for evaluating packet loss probabilities, also called cell loss probabilities (CLP).

Recently another and very efficient approach to compute the CLP as a function of the system size has become available, based on the use of rational approximation [6]. The motivation behind this approach is that using the matrix-analytic method, it is computationally feasible to evaluate the CLP as a function of the system size when the system size is small, and moreover it is often possible to obtain information about the asymptotic behavior [7]. In [6], the authors have employed rational approximants to compute the CLP, but only for models where the correlation between the cells was ignored. Considering a high degree of correlation is of major importance when the input consists of more video sources [3]. In [2], a sampling technique in combination with rational data fitting is proposed for models with more correlation between the cells. In rare cases it happens that the poles of the computed rational function disturb the fit of the packet loss probability. In [1], the authors have proposed a method to avoid this problem through an

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a priori optimal placement of poles. The location of the poles is determined, indirectly, from the system parameters.

Since the parameters used in such networks may not produce precise probabilities of interest without having to introduce drastic assumptions, it is interesting and important to carry out a perturbation analysis with respect to these parameters. This will be done here, concurrently with the computation of the rational model, using interval arithmetic.

2   Model Description

In the multiplexer environment, the arrival of packets to the switch happens in discrete time, with discrete service time, which makes the discrete time Markov chain a natural modeling choice. We assume that the arrival of cells which are transmitted by M independent and non-identical information sources to the multiplexer, can be modeled as a discrete time batch Markovian Arrival Process (D-BMAP), the discrete-time version of BMAP. Each information source is controlled by a Markov chain, called the background Markov chain. The basic queueing system which models the multiplexer is a D-BMAP/D/c/N queue with c discrete time servers, where each server can serve at most one cell per time unit. These servers serve a buffer with a capacity of N cells which is fed by M independent information sources. When the server is busy, a maximum of c cells will depart in each slot. Service starts at the beginning of each time slot.

The D-BMAP queueing model is an M/G/1 type queue which is basically a two-dimensional discrete time Markov chain $\{(X_n, Y_n), n \geq 0\}$, where X_n is the number of cells in the buffer and Y_n represents the state of the M sources during the nth time slot. We are interested in the steady state behavior $(X, Y) \equiv \lim_{n \to \infty} (X_n, Y_n)$.

Let $\boldsymbol{D}$ be the transition probability matrix of the process Y and let $\boldsymbol{D_m}$ ($m = 0, 1, \ldots, M$) denote the matrix corresponding to m arrivals during a time slot. These matrices can be calculated from the following system parameters [2]:

1. the number of sources M;

2. the transition probabilities of the background Markov chains:

   - $\boldsymbol{p} = [p_1, p_2, \ldots, p_M]$ and $\boldsymbol{q} = [q_1, q_2, \ldots, q_M]$, if the sources are heterogeneous,
   - p and q, if the sources are homogeneous;

3. the cell generation probability:

   - $\boldsymbol{d} = \begin{pmatrix} d_1(0) & d_1(1) \\ d_2(0) & d_2(1) \\ \vdots & \vdots \\ d_M(0) & d_M(1) \end{pmatrix}$, if the sources are heterogeneous,
   - d, if the sources are homogeneous.

The average arrival rate of cells at the multiplexer is given by

$$\lambda = \boldsymbol{\eta} \left(\sum_{m=0}^{M} m \boldsymbol{D_m} \right) \boldsymbol{e}, \tag{1}$$

where $\boldsymbol{e}$ is a column vector of ones and the vector $\boldsymbol{\eta}$ is such that $\boldsymbol{\eta D} = \boldsymbol{\eta}$ with $\boldsymbol{\eta e} = 1$.

Under the condition of ergodicity of the chain (X, Y), i.e. the load $\rho = \lambda/c < 1$, the stationary distribution vector $\boldsymbol{\Pi} := \{\boldsymbol{\pi_0}, \boldsymbol{\pi_1}, \ldots, \boldsymbol{\pi_N}\}$ with $\boldsymbol{\pi_i} \in \mathbb{R}^{2^M}$ satisfies

$$\boldsymbol{\Pi P} = \boldsymbol{\Pi} \quad \text{and} \quad \boldsymbol{\Pi e} = 1, \tag{2}$$

where the transition probability matrix $\boldsymbol{P}$ of the process (X, Y) [2] is a square matrix of size

- $(N+1)2^M \times (N+1)2^M$, if the sources are heterogeneous,

- $(N+1)(M+1) \times (N+1)(M+1)$, if the sources are homogeneous,

and is given by

$$
\boldsymbol{P} = \begin{pmatrix}
\boldsymbol{D_0} & \boldsymbol{D_1} & \ldots & \boldsymbol{D_{N-c}} & \ldots & \boldsymbol{D_{N-1}} & \boldsymbol{B_N} \\
\boldsymbol{D_0} & \boldsymbol{D_1} & \ldots & \boldsymbol{D_{N-c}} & \ldots & \boldsymbol{D_{N-1}} & \boldsymbol{B_N} \\
\vdots & & & & & & \vdots \\
\boldsymbol{D_0} & \boldsymbol{D_1} & \ldots & \boldsymbol{D_{N-c}} & \ldots & \boldsymbol{D_{N-1}} & \boldsymbol{B_N} \\
0 & \boldsymbol{D_0} & \ldots & \boldsymbol{D_{N-c-1}} & \ldots & \boldsymbol{D_{N-2}} & \boldsymbol{B_{N-1}} \\
0 & 0 & \ldots & \boldsymbol{D_{N-c-2}} & \ldots & \boldsymbol{D_{N-3}} & \boldsymbol{B_{N-2}} \\
\vdots & & & & & & \vdots \\
0 & 0 & \ldots & \boldsymbol{D_0} & & \boldsymbol{D_{c-1}} & \boldsymbol{B_c}
\end{pmatrix},
\tag{3}
$$

with $\boldsymbol{B_i} = \sum_{j=i}^{M} \boldsymbol{D_j}$, $i = c, \ldots, N$.

The packet loss probability function, as a function of the buffer size N, is then given by

$$
P_L(N) := \frac{1}{\lambda} \sum_{n=0}^{N} \boldsymbol{\pi_n} \sum_{k=0}^{M} [k + n - (N + \min(N, c))]^+ \boldsymbol{D_k e},
\tag{4}
$$

where $[x]^+ := \max(0, x)$.

It has been proved that for M/G/1-type queues, the $\log P_L(N)$ decays exponentially [7]. That is,

$$
\log P_L(N) \approx \xi N, \text{ as } N \to \infty.
\tag{5}
$$

3 Rational Approximation

Because of the fact that the function $\log P_L(N)$ asymptotically behaves as ξN for large N, polynomial approximation techniques for $\log P_L(N)$ are not suitable. Every polynomial model of degree larger than one, would blow up for large N. However, a rational function $r_n(N)$ of numerator degree $n + 1$ and denominator degree n, has a similar asymptotic behavior as that of $\log P_L(N)$. Remains to compute the coefficients in numerator and denominator of the rational function

$$
r_n(N) = \frac{p_n(N)}{q_n(N)} = \frac{\sum_{i=0}^{n+1} a_i N^i}{\sum_{i=0}^{n} b_i N^i},
\tag{6}
$$

mostly from computed or measured values of $\log P_L(N_j)$ for small buffer sizes N_j, to fit the behavior of $\log P_L(N)$ using (4).

The rational model is fully specified when we know its numerator and denominator coefficients $b_1, \ldots, b_n$ and $a_0, \ldots, a_{n+1}$, a total of $2n + 2$ coefficients (the constant term b_0 in the denominator is only a normalization constant for the rational function [10] and is therefore assigned the value 1, whenever possible). These coefficients are determined from sampling $\log P_L(N)$ at chosen N_j for $j = 0, \ldots, 2n$ while one value is determined from the asymptotic behavior

$$
\lim_{N \to \infty} \log P_L(N) \approx \xi N = \frac{a_{n+1}}{b_n} N.
\tag{7}
$$

In rare cases it happens that the poles of the computed rational function disturb the fit of the packet loss probability. To circumvent this problem, in [1], the authors have used a multipoint Padé-type approximation technique.

4 Interval Arithmetic

In this paper we study the effect of uncertainties in the parameters $\boldsymbol{p}$ (or p), $\boldsymbol{q}$ (or q) and $\boldsymbol{d}$ (or d) on one hand and/or uncertainties in the values of $\log P_L(N_j)$, when coming from measurements or verified computations performed for minimal and maximal load of ρ of the networks under investigation. To automatically incorporate this perturbation analysis in the computation of the rational fitting technique, we need to use tools from interval arithmetic. The verified computation of $r_n(N)$ requires enclosures for or the verified computation of:

- $\boldsymbol{p}$ (or p), $\boldsymbol{q}$ (or q) and $\boldsymbol{d}$ (or d) which are involved in the matrices $\boldsymbol{D_m}$;

- the samples $\log P_L(N_j)$;

- the asymptotic slope ξ [9].

One of the important finds is that $P_L(N)$ is rather sensitive with respect to small changes in the parameters. This of course has its effect on the data fitting problem. When using a rational model with prescribed poles to fit the interval data, as in the multipoint Pade-type approach, the verified computation of $r_n(N)$ can be based on the results developed in [8].

References

[1] A. Cuyt, R.B. Lenin, *Computing packet loss probabilities in multiplexer models using adaptive rational interpolation with optimal pole placement.* submitted.

[2] A. Cuyt, R.B. Lenin, G. Willems, C. Blondia, and P. Rousseeuw, *Rational approximation technique to compute cell loss probabilities in multiplexer models*, IEEE Trans. on Computers, **52**, pp. 1-12 (2003).

[3] C. Blondia and O. Casals, *Performance analysis of statistical multiplexing of VBR sources: A matrix-analytical approach*, Performance Evaluation, **16**, pp. 5-20 (1992).

[4] A.I. Elwalid and D. Mitra, *Analysis, approximations and admission control of a multi-service multiplexing system with priorities*, Proc. INFOCOM '95, pp. 463-472 (1995).

[5] C.S. Chang, P. Heidelberger, S. Juneja and P. Shahabuddin, *Effective bandwidth and fast simulation of ATM intree networks*, Proc. Performance '93, Rome (Italy, October 1993).

[6] W.B. Gong and H. Yang, *On Global Rational Approximants for Stochastic Discrete Event Systems*, International Journal of Discrete Event Dynamic Systems, **7**, pp. 93-116 (1997).

[7] E. Falkenberg, *On the asymptotic behavior of the stationary distribution of Markov chains of M/G/1-type*, Commun. Statist. Stochastic Models, **10**, 1994, pp. 75-97.

[8] S. Markov, E. Popova, U. Schneider, J. Schulze, *On linear interpolation under interval data*, Math. Comput. Simulation, **42**, pp. 35-45 (1996).

[9] S.M. Rump, *Computational error bounds for multiple or nearly multiple eigenvalues*, Linear Algebra and Appl., **324**, pp. 209-226 (2001).

[10] D. Warner, *Hermite interpolation with rational functions* (Ph.D. dissertation, University of California, 1974).