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Sparse Interpolation and Rational Approximation

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Dedicated to Ed Saff's 70th birthday

Abstract. Sparse interpolation or exponential analysis, is widely used and in quite different applications and areas of science and engineering. Therefore researchers are often not aware of similar studies going on in another field. The current text is written as a concise tutorial, from an approximation theorist point of view.

In Section 2 we summarize the mathematics involved in exponential analysis: structured matrices, generalized eigenvalue problems, singular value decomposition. The section is written with the numerical computation of the sparse interpolant in mind.

In Section 3 we outline several connections of sparse interpolation with other mostly non-numeric subjects: computer algebra, number theory, linear recurrences. Some problems are only solved using exact arithmetic.

In Section 4 we connect sparse interpolation to rational approximation theory. One of the major hurdles in sparse interpolation is still the correct detection of the number of components in the model. Here we show how to reliably obtain the number of terms in a numeric and noisy environment.

The new insight allows to improve on existing state-of-the-art algorithms.

1. Motivation

When interpolating data f_j at points x_j with the values f_j coming from a function of the form $f(x) = \alpha_1 + \alpha_2 x^{100}$, classical interpolation needs 101 samples $f_j, j = 0, \ldots, 100$ despite the fact that there are only 4 unknowns involved, namely α_1, α_2 and the knowledge of the two basis functions x^0, x^{100}. Let us look at the challenge to determine these 4 unknown items from only 4 samples.

The problem statement is related to a wide range of topics in the computational sciences and engineering, as reviewed in [**16**]. It is also connected to several mathematical and numerical subjects such as exponential analysis, generalized eigenvalue problems, symbolic computation, orthogonal polynomials, signal processing, moment problems, and last but not least rational approximation theory. In the sequel we assume most of the times that the data f_j are collected at equidistant points $x_j = j\Delta, j = 0, 1, 2, \ldots$

The first sparse interpolation problem was the interpolation of

$$f(x) = \sum_{i=1}^{n} \alpha_i \exp(\phi_i x), \qquad \alpha_i, \phi_i \in \mathbb{C}, \tag{1.1}$$

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formulated in [**5**] and summarized in Section 2.

The multivariate polynomial sparse interpolation of

$$f(x_1,\ldots,x_d) = \sum_{(i_1,\ldots,i_d)\in I} \alpha_{i_1,\ldots,i_d} x_1^{i_1}\cdots x_d^{i_d}, \qquad I \subset \mathbb{N}^d, \#I = n \tag{1.2}$$

was later investigated in [**3**] and is discussed in Section 3.

We build on the connection with Padé approximation theory [**1**] to formulate a way to correctly detect the number of terms in (1.1) and (1.2) in a numerical and hence noisy environment. This new approach is presented and illustrated in Section 4. The proposed method improves on the existing state-of-the-art algorithms.

2. Univariate exponential modeling

Let us consider the nonlinear interpolation problem

$$\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_j) = f_j, \qquad j = 0,\ldots,2n-1, \tag{2.1}$$

where

$$x_j = j\omega, \qquad \omega = 2\pi/M, \qquad |\Im(\phi_i)| < M/2.$$

If we denote

$$\Omega_i = \exp(\phi_i \omega),$$

then it is apparent that the data f_j are structured, namely

$$f_j = \sum_{i=1}^{n} \alpha_i \Omega_i^j, \qquad j = 0,\ldots,2n-1. \tag{2.2}$$

We now want to obtain the values $\Omega_i, i = 1,\ldots,n$ and $\alpha_i, i = 1,\ldots,n$ from the $2n$ samples f_j. From Ω_i the value ϕ_i can easily be deduced because $|\Im(\phi_i)\omega| < \pi$ and hence no periodicity problem arises. In addition to computing Ω_i and α_i we indicate what is known about n. Knowledge of the latter is crucial for the success of the algorithm and is the main subject of Section 4 and this paper. Temporarily we assume that n is known.

Consider the polynomial

$$\prod_{i=1}^{n}(z-\Omega_i) = z^n + \beta_{n-1}z^{n-1} + \cdots + \beta_1 z + \beta_0 \tag{2.3}$$

with so far unknown coefficients $\beta_i, i = 1,\ldots,n$. Since the Ω_i are its zeroes, we find for $k \geq 0$,

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \alpha_i \Omega_i^k (\Omega_i^n + \beta_{n-1}\Omega_i^{n-1} + \cdots + \beta_0) \\ &= \sum_{i=1}^{n} \alpha_i \Omega_i^{n+k} + \sum_{j=0}^{n-1} \beta_j \left(\sum_{i=1}^{n} \alpha_i \Omega_i^{j+k} \right) \\ &= f_{k+n} + \sum_{j=0}^{n-1} \beta_j f_{k+j}. \end{aligned}$$

In other words, we can conclude that the structured data f_j are linearly generated,

$$\begin{pmatrix} f_0 & \cdots & f_{n-1} \\ \vdots & \iddots & \vdots \\ f_{n-1} & \cdots & f_{2n-2} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{n-1} \end{pmatrix} = - \begin{pmatrix} f_n \\ \vdots \\ f_{2n-1} \end{pmatrix}. \tag{2.4}$$

This linear system allows us to compute the coefficients $\beta_i, i = 0, \ldots, n-1$ and actually compose the polynomial having $\Omega_i, i = 1, \ldots, n$ as its zeroes. Let us now denote by $H_n^{(r)}$ the Hankel matrix

$$H_n^{(r)} = \begin{pmatrix} f_r & \cdots & f_{r+n-1} \\ \vdots & \iddots & \vdots \\ f_{r+n-1} & \cdots & f_{r+2n-2} \end{pmatrix}$$

and by $H_n^{(0)}(z)$ the Hankel polynomial [**13**, p. 625]

$$H_n^{(0)}(z) = \begin{vmatrix} f_0 & \cdots & f_{n-1} & f_n \\ \vdots & \iddots & \vdots & \vdots \\ f_{n-1} & \cdots & f_{2n-2} & f_{2n-1} \\ 1 & \ldots & z^{n-1} & z^n \end{vmatrix}.$$

Then

$$\prod_{i=1}^{n}(z - \Omega_i) = \frac{H_n^{(0)}(z)}{|H_n^{(0)}|},$$

where $|H_n^{(0)}|$ denotes the determinant of $H_n^{(0)}$. From the matrix factorizations

$$H_n^{(0)} = V_n D_\alpha V_n^T,$$

$$H_n^{(1)} = V_n D_\alpha \begin{pmatrix} \Omega_1 & & \\ & \ddots & \\ & & \Omega_n \end{pmatrix} V_n^T,$$

where V_n and D_α respectively denote the Vandermonde matrix

$$V_n = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \Omega_1 & \Omega_2 & \ldots & \Omega_n \\ \vdots & \vdots & & \vdots \\ \Omega_1^{n-1} & \Omega_2^{n-1} & \ldots & \Omega_n^{n-1} \end{pmatrix}$$

and the diagonal matrix

$$D_\alpha = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix},$$

it is easy to see that the polynomial zeroes Ω_i can also be obtained as generalized eigenvalues [**11**, **14**]. So the Ω_i also satisfy

$$\det\left(H_n^{(1)} - \Omega_i H_n^{(0)}\right) = 0, \qquad i = 1, \ldots, n. \tag{2.5}$$

The coefficients α_i in the model (1.1) can be obtained from any set of n interpolation conditions taken from (2.2),

$$\begin{pmatrix} \Omega_1^j & \ldots & \Omega_n^j \\ \vdots & & \vdots \\ \Omega_1^{j+n-1} & \ldots & \Omega_n^{j+n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} f_j \\ \vdots \\ f_{j+n-1} \end{pmatrix}, \qquad 0 \le j \le n. \tag{2.6}$$

With Ω_i computed as above, the remaining equations are linearly dependent.

Now what can be said about n? Merely using some known theorems, its value can be nailed down quite precisely, that is in an exact noisefree context. We read in [**13**] and [**2**] that on the one hand, for $N < n$ and $r \ge 0$, $|H_N^{(r)}|$ is only accidentally zero, depending on the value of ω, while on the other hand, for $N > n$ and $r \ge 0$,

$|H_N^{(r)}|$ is always zero, irrespective of the value of ω. Most importantly, for $N = n, r \geq 0$ and $\Omega_i \neq \Omega_j$ for $i \neq j$, $|H_N^{(r)}| \neq 0$.

In order to inspect $|H_N^{(r)}|$ for $N > n$, additional samples up to f_{r+2N-2} need to be provided, in other words at least the additional sample f_{2n} (in case $r = 0$ and $N = n + 1$). A nice discussion, based on algebraic arguments, is presented in [**17**]. We illustrate the above with an easy noisefree example. Consider the function

$$f(x) = \sum_{i=1}^{4} \alpha_i \exp(\phi_i x)$$

where

$$\begin{aligned} \alpha_1 &= 1 & \phi_1 &= 0 \\ \alpha_2 &= 2.4 & \phi_2 &= -5 + 19.97\mathtt{i} \\ \alpha_3 &= -2.1 & \phi_3 &= 3 + 45\mathtt{i} \\ \alpha_4 &= 0.2 & \phi_4 &= 5.3\mathtt{i} \end{aligned}$$

and take $M = 100$ such that $\max_{1 \leq i \leq 4} |\Im(\phi_i)| < M/2$. Inspecting the singular value decomposition of $H_N^{(0)}$ with $N = 6 > n = 4$ reveals that the numerical rank of $H_N^{(0)}$ equals 4. We give a log-plot of the singular values, in decreasing order of magnitude, in Figure 1: singular values of the order of magnitude of (moderately amplified) numerical round-off errors do not contribute to the numerical rank. The use of the numerical rank for N slightly larger than n is however numerically unreliable in the presence of real-life measurement noise (other than floating-point round-off errors). In Section 4 we indicate how this can be remedied.

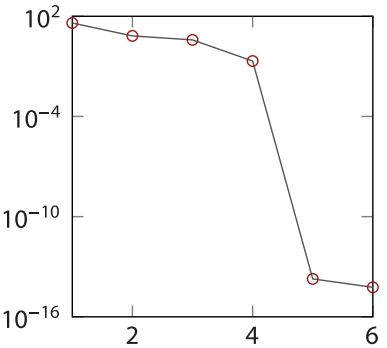

FIGURE 1. $H_N^{(0)}$ with $N = 6 > n = 4$ singular.

The method is not only useful for equidistantly sampled structured data. Other data that obey the same structure can also be dealt with, such as for instance, the successive derivatives of a function of the form

$$f(x) = \alpha_1 \exp(\phi_1 x) + \alpha_2 \exp(\phi_2 x).$$

The values

$$\begin{aligned} f(0) &= \alpha_1 + \alpha_2 \\ f'(0) &= \alpha_1\phi_1 + \alpha_2\phi_2 \\ f''(0) &= \alpha_1\phi_1^2 + \alpha_2\phi_2^2 \\ f'''(0) &= \alpha_1\phi_1^3 + \alpha_2\phi_2^3 \end{aligned}$$

can be used to deduce $\phi_{1,2}$ and $\alpha_{1,2}$ from, in the same way as above, since the sampled values are structured as in (2.2). In addition, it is easy to check that the determinant $|H_3^{(0)}|$ with $f_j = f^{(j)}(0), j = 0, \ldots, 4$ is symbolically zero, indicating that $n = 2$.

3. Multivariate polynomial interpolation

While generalizing to more variables, we restrict ourselves to the monomial basis functions [**3**]. We consider functions f as given in (1.2),

$$f(x_1, \ldots, x_d) = \sum_{(i_1,\ldots,i_d)\in I} \alpha_{i_1,\ldots,i_d} x_1^{i_1} \cdots x_d^{i_d}, \qquad I \subset \mathbb{N}^d, \#I = n.$$

The issue is again to detect from a number of samples of f which monomials are involved in its expression and then compute the coefficients in the interpolant (1.2). To this end we now take our samples equidistantly along the unit circle [**8**].

We assume we have upperbounds p_k for the partial degree of f in the variable x_k and we take these $p_k, k = 1, \ldots, d$ mutually prime. With p_k we define

$$\omega_k = \exp(2\pi \mathtt{i}/p_k), \qquad k = 1, \ldots, d.$$

We then evaluate $f(x_1, \ldots, x_d)$ in the points

$$(x_{j1}, \ldots, x_{jd}) = (\omega_1^j, \ldots, \omega_d^j), \qquad j = 0, \ldots, 2n-1$$

on the unit circle. Let the n distinct multi-indices in I be numbered $(i_1^{(1)}, \ldots, i_d^{(1)}), \ldots, (i_1^{(n)}, \ldots, i_d^{(n)})$ and denote

$$\Omega_l = \omega_1^{i_1^{(l)}} \cdots \omega_d^{i_d^{(l)}}, \qquad l = 1, \ldots, n.$$

So the $\Omega_1, \ldots, \Omega_n$ reveal the basis functions that appear in (1.2), namely the monomials powered by the multi-indices in I. How can we obtain the values Ω_l? Using the same approach as in Section 2. The coefficients in the polynomial

$$\prod_{i=1}^{n} (z - \Omega_i) = z^n + \beta_{n-1} z^{n-1} + \cdots + \beta_1 z + \beta_0$$

are computed from the linear system (2.4) or the Ω_i are computed from the generalized eigenvalue problem (2.5).

How do we extract the exponents or multi-indices $(i_1^{(l)}, \ldots, i_d^{(l)})$ from Ω_l? We use a reverse of the Chinese remainder theorem, which we can make use of because the p_k are mutually prime [**8**]. Define

$$m = \prod_{k=1}^{d} p_k, \qquad \omega = \exp(2\pi \mathtt{i}/m).$$

Since $\omega_k = \omega^{m/p_k}$, we have

$$\Omega_l = \omega^{i(l)}, \qquad i(l) = i_1^{(l)}\frac{m}{p_1} + \cdots + i_d^{(l)}\frac{m}{p_d}, \qquad l = 1, \ldots, n.$$

With $\gcd(p_k, m/p_k) = 1$ and

$$i(l) \bmod p_k = i_k^{(l)}\frac{m}{p_k} \bmod p_k, \qquad k = 1, \ldots, d, l = 1, \ldots, n$$

we can retrieve the integers $i_k^{(l)} < p_k$. While the coefficients $\alpha_{i_1,\ldots,i_d}$ are computed as in Section 2, for the detection of n usually an alternative is used.

To find n, additional samples are required beyond the $f_0, \ldots, f_{2n-1}$. In exact arithmetic, the value of n is the smallest integer for which the discrepancy δ_j given by the $(n+1)$-term recurrence

$$\delta_j = f_j + \beta_{n-1}f_{j-1} + \cdots + \beta_0 f_{j-n}, \qquad j \geq 2n$$

equals zero [**19**].

We illustrate the algorithm with a simple example. Consider

$$f(x, y) = x^5y + 2.2x^4y^4 - 0.5xy^{11} + 0.1xy^{12}$$

and take $p_1 = 6, p_2 = 13$ and $\omega_1 = \exp(2\pi\mathtt{i}/6), \omega_2 = \exp(2\pi\mathtt{i}/13)$. The sequence $f_0, \ldots, f_7, \ldots$ is linearly generated with $\delta_8 = 0$. Hence $n = 4$. With $m = p_1p_2 = 78$ we find

$$\begin{aligned}
\Omega_1 &= \omega^{i(1)}, & i(1) &= 71 \bmod m = 5 \times 13 + 1 \times 6\\
\Omega_2 &= \omega^{i(2)}, & i(2) &= 76 \bmod m = 4 \times 13 + 4 \times 6\\
\Omega_3 &= \omega^{i(3)}, & i(3) &= 1 \bmod m = 1 \times 13 + 11 \times 6\\
\Omega_4 &= \omega^{i(4)}, & i(4) &= 7 \bmod m = 1 \times 13 + 12 \times 6.
\end{aligned}$$

So $I = \{(5,1), (4,4), (1,11), (1,12)\}$ and the coefficients $\alpha_{5,1}, \alpha_{4,4}, \alpha_{1,11}, \alpha_{1,12}$ in $f(x, y)$ are obtained from the Vandermonde system (2.6).

Sometimes $f(x_1, \ldots, x_d)$ is sparse only after performing a shift such as in

$$f(x, y) = (x-3)^5(y+5) + 2.2(x-3)^4(y+5)^4 - 0.5(x-3)(y+5)^{11} + 0.1(x-3)(y+5)^{12}.$$

The computation of such a shift is again carried out in exact arithmetic [**7**]. The sparse interpolation algorithm is then performed on $\overline{f}(u, v) = f(x, y)$ with $u = x - 3$ and $v = y + 5$. So when evaluating $\overline{f}$ at $u = \omega_1^j$ and $v = \omega_2^j$, we're actually evaluating f at $x = \omega_1^j + 3$ and $y = \omega_2^j - 5$.

4. Connection with Padé approximation

With $f_j, j = 0, 1, 2, \ldots$ we now define the noisefree

$$F(z) = \sum_{j=0}^{\infty} f_j z^j. \tag{4.1}$$

The Padé approximant $r_{m,n}(z)$ of degree m in the numerator and n in the denominator is defined as the irreducible form of the rational function $p(z)/q(z)$ satisfying

$$F(z)q(z) - p(z) = \sum_{j \geq m+n+1} c_j z^j, \qquad \partial p \leq m, \partial q \leq n.$$

With

$$p(z) = \sum_{i=0}^{m} a_i z^i,$$
$$q(z) = \sum_{i=0}^{n} b_i z^i,$$

we find that $a_0, \ldots, a_m$ and $b_0, \ldots, b_n$ need to satisfy

$$\begin{cases} f_0 b_0 = a_0 \\ f_1 b_0 + f_0 b_1 = a_1 \\ \quad \vdots \\ f_m b_0 + \cdots + f_{m-n} b_n = a_m \end{cases}$$
$$\begin{cases} f_{m+1} b_0 + \cdots + f_{m-n+1} b_n = 0 \\ \quad \vdots \\ f_{m+n} b_0 + \cdots + f_m b_n = 0 \end{cases}$$

where we tacidly assume that $f_j = 0$ for $j < 0$. Choosing $b_0 = 1$ in the homogeneous system (if allowed) results in

$$H_n^{(m+1-n)} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = - \begin{pmatrix} f_{m+1} \\ \vdots \\ f_{m+n} \end{pmatrix}.$$

Since

$$f_j = \sum_{i=1}^{n} \alpha_i \exp(j\phi_i \omega) = \sum_{i=1}^{n} \alpha_i \Omega_i^j,$$

we can rewrite

$$F(z) = \sum_{i=1}^{n} \frac{\alpha_i}{1 - z\Omega_i}. \tag{4.2}$$

So we see that $F(z)$ is itself a rational function of degree $n-1$ in the numerator and n in the denominator, with poles $1/\Omega_i$. Hence, from Padé approximation theory we know (as to be expected) that $r_{n-1,n}(z)$ reconstructs $F(z)$, in other words

$$r_{n-1,n}(z) = F(z)$$

with denominator

$$q(z) = \prod_{i=1}^{n} (1 - z\Omega_i) = \beta_0 z^n + \beta_1 z^{n-1} + \cdots + \beta_{n-1} z + 1.$$

The partial fraction decomposition (4.2) is related to both the Laplace transform and the z-transform of the exponential model (1.1), which explains why this approach is known as the Padé-Laplace method. The connection was originally noticed in [**24**], but in the meantime a lot of new insight can be added.

Let us now add a white circular Gaussian noise term ϵ_j to each sample f_j. In the sequel we denote the noisy series by

$$F(z) + \epsilon(z) = \sum_{j=0}^{\infty} (f_j + \epsilon_j) z^j.$$

A number of very nice approximation and convergence results exist for Padé approximants. They express what one would expect intuitively from such approximants:

they are especially useful if the approximated function is meromorphic in some substantial region of the complex plane.

The first result we mention is the convergence theorem of de Montessus de Ballore for Padé approximants of a meromorphic function [**4**]. It states that if $F(z)+\epsilon(z)$ is a meromorphic function in the complex region $\{z : 0 \le |z| < R\}$ with poles of total multiplicity n in that open disc, then the sequence of Padé approximants $\{r_{m,n}(z)\}_{m\in\mathbb{N}}$ converges to $F(z)+\epsilon(z)$ uniformly on compact sets excluding the poles. What's more, the poles of $F(z)+\epsilon(z)$ attract poles of $r_{m,n}(z)$ according to their multiplicity. Although this result is the best one can expect for $r_{m,n}(z)$, it does not leave much room to model the noise: the denominator has a fixed degree and so the only way to model the noise $\epsilon(z)$ separately from the underlying signal $F(z)$ is through the numerator. Numerical experiments have taught us that this does not work well.

It is numerically preferable to build on another famous convergence result for Padé approximants, which is the theorem of Nuttall, later generalized by Pommerenke. It states that if $F(z)+\epsilon(z)$ is analytic throughout the complex plane except for a countable number of poles [**20**] and essential singularities [**21**], then the paradiagonal sequence $\{r_{m-1,m}(z)\}_{m\in\mathbb{N}}$ converges to $F(z)+\epsilon(z)$ in measure on compact sets. So here no assertion is made about pointwise or uniform convergence. Instead, the result states that for every sufficiently large m, the measure of the set where the convergence is disrupted, so where $|F(z)+\epsilon(z)-r_{m-1,m}(z)| \ge \tau$ for some given threshold τ, tends to zero as m tends to infinity. When applying this result to our case, $F(z)$ being a rational function of degree $n-1$ in the numerator and n in the denominator, the theorem leaves ample room to model the noise $\epsilon(t)$ in the remaining $m-n$ zeroes and poles of $r_{m-1,m}(z)$.

As a matter of fact, the pointwise convergence is disrupted by the $m-n$ unwanted pole-zero combinations of the Padé approximants, also called Froissart doublets [**6**, **9**, **10**], that are added to the n true poles and $n-1$ true zeroes of $F(z)$. But these Froissart doublets offer a way to filter the noise $\epsilon(z)$ from the underlying signal $F(z)$. And because of the Padé convergence theorem, the true (physical) poles can be identified as stable poles in successive $r_{m-1,m}(z)$, while the spurious (nonphysical, noisy) poles are distinguished by their instability.

Because of their ability to model the noise, Froissart doublets should not be avoided in the computation, as in [**12**] and [**15**], but should be filtered out at a later stage in the computation. We now describe how to apply this idea to exponential analysis. When inspecting the numerical rank of $H_N^{(0)}$, not only should $N > n$, but also the Padé approximant $r_{N-1,N}(z)$ should have converged beyond the noise limit ϵ. This can be observed from the convergence of the n poles $1/\Omega_i$ in (4.2). In practice the $\Omega_i, i = 1,\ldots,N$ are computed from the generalized eigenvalue problem

$$H_N^{(1)} v_i = \Omega_i H_N^{(0)} v_i, \qquad i = 1,\ldots,N$$

and the residues α_i are obtained from the interpolation problem

$$\sum_{i=1}^{n} \alpha_i \Omega_i^j = f_j, \qquad j = 0,\ldots,2N-1,$$

where the $2N$ interpolation conditions are now considered in the least squares sense.

We again illustrate the above with an example. Consider

$$f(x) = \sum_{i=1}^{4} \alpha_i \exp(\phi_i x)$$

where

$$\begin{aligned} \alpha_1 &= 1 & \phi_1 &= 0 \\ \alpha_2 &= 2 & \phi_2 &= -0.2 + 39.5\mathtt{i} \\ \alpha_3 &= 4 & \phi_3 &= -0.5 + 40\mathtt{i} \\ \alpha_4 &= 8 & \phi_4 &= -1, \end{aligned}$$

with $M = 100$ to satisfy $|\Im(\phi_i)| < M/2, i = 1, \ldots, 4$. The noise terms ϵ_j are scaled such that $\max_j |\epsilon_j| = 10^{-2}$. When inspecting the singular value decomposition of $H_N^{(0)}$ for $N = 10 > n = 4$, shown on a log-plot in Figure 2, one cannot easily deduce the correct numerical rank $n = 4$. But from the singular value decomposition of $H_N^{(0)}$ with $N = 50$, shown in Figure 3, it is clear, thanks to the convergence of the Padé approximation technique: 46 singular values are of the order of magnitude of $\max_j |\epsilon_j|$ (slightly or moderately amplified).

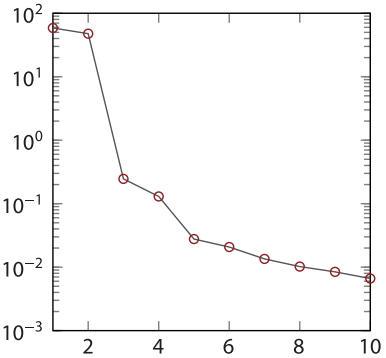

FIGURE 2. Singular values of $H_N^{(0)}$ with $N = 10 > n = 4$.

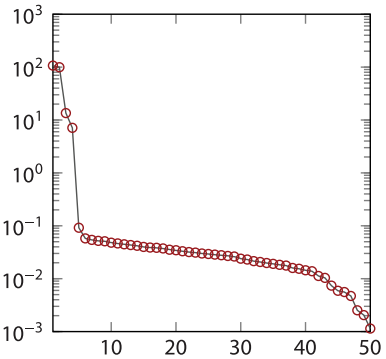

FIGURE 3. Singular values of $H_N^{(0)}$ with $N = 50$.

Let us compare these results to the ones obtained using on the one hand the ESPRIT algorithm [**22**] and on the other hand the robust Padé approximation algorithm [**12**]. We start with the latter.

Providing all 100 samples $f_j + \epsilon_j, j = 0, \ldots, 99$ to the code computing the robust Padé approximant, leads to different estimates for n depending on the error tolerance E that the user enters simultaneously with the data. For $E = 10^{-2}$ and $E = \sum_{j=0}^{99} |\epsilon_j|/100 = 4.1 \times 10^{-3}$ the algorithm returns $n = 2$. Up to $E = 2.25885437964 \times 10^{-4}$, the returned estimate for n is at most 3. Shrinking it more, by just 10^{-15}, to $E = 2.25885437963 \times 10^{-4}$, results in the estimate $n = 5$. For $E = 10^{-4}$ the estimate is $n = 33$. Altogether, getting n correct is very tricky.

Running ESPRIT with only 20 datapoints (as in Figure 2) results in the singular value decomposition shown in Figure 4 for the 10×10 covariance matrix produced from a 10×20 Hankel matrix. Running it with all 100 datapoints (as in Figure 3) produces Figure 5. Although the order of magnitude of the third and fourth largest singular value is the same as that of the noise, one could deduce from Figure 5 that the number of terms in $f(x)$ is $n = 4$. The remaining singular values are of the order of $\max_j^2 |\epsilon_j|$.

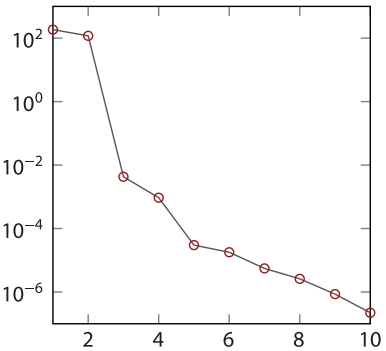

FIGURE 4. Rank estimation using ESPRIT with $N = 10$ and 20 datapoints.

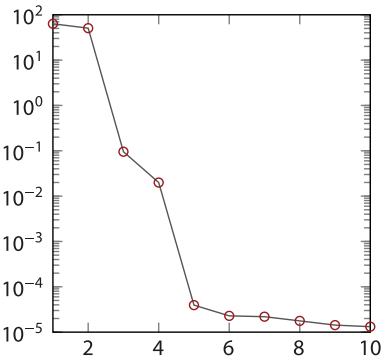

FIGURE 5. Rank estimation using ESPRIT with $N = 10$ and 100 datapoints.

That the connection with Padé approximation theory is really powerful, is illustrated in the next example, where we dig up one term of $f(x)$ that is almost

burried in the noise. Consider

$$\begin{aligned} \alpha_1 &= 10^{-3} & \phi_1 &= 1.5\mathtt{i} \\ \alpha_2 &= 2 & \phi_2 &= 12.7\mathtt{i} \\ \alpha_3 &= 4 & \phi_3 &= -0.1 + 40\mathtt{i} \\ \alpha_4 &= 8 & \phi_4 &= -0.3 + 25.2\mathtt{i} \end{aligned}$$

with $M = 100$ as before and the noise scaled to satisfy $\max_j |\epsilon_j| = 2 \times 10^{-3}$. The singular value decomposition of $H_N^{(0)}$ with $N = 10$ doesn't provide the correct information, but the one with $N = 100$ clearly does: 96 singular values are of the order of magnitude of $\max_j |\epsilon_j|$ (slightly or moderately amplified). So n appears to equal 4. The log-plots are respectively found in the Figures 6 and 7.

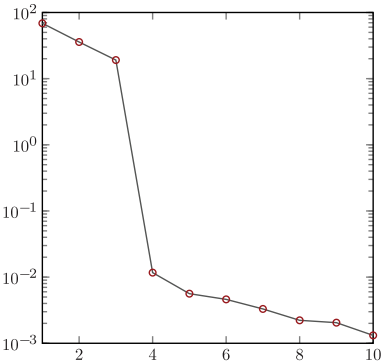

FIGURE 6. Singular values of $H_N^{(0)}$ with $N = 10 > n = 4$.

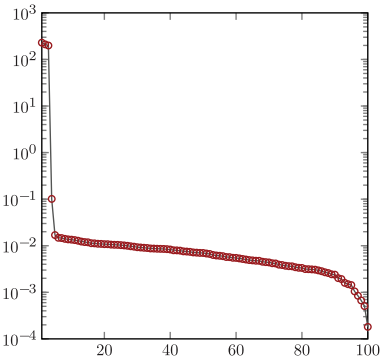

FIGURE 7. Singular values of $H_N^{(0)}$ with $N = 100$.

A numerical comparison with [**22**] and [**12**] leads to similar conclusions as in the previous example.

The robust Padé approximation algorithm with $E = 2{\times}10^{-3}$ or $E = \sum_{j=0}^{199} |\epsilon_j|/200 = 6.8 \times 10^{-4}$ delivers $n = 3$. Running it with the smaller $E = 2.4 \times 10^{-5}$ (trial and error) returns the correct $n = 4$.

The results for ESPRIT are shown in the Figures 8 and 9. Using only 20 datapoints the estimate for n is clearly erroneous. Using all 200 datapoints it is still unclear. In both figures we took $N = 10$ (as in the Figures 4 and 5).

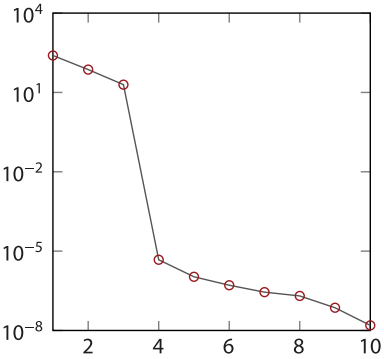

FIGURE 8. Rank estimation using ESPRIT with $N = 10$ and 20 datapoints.

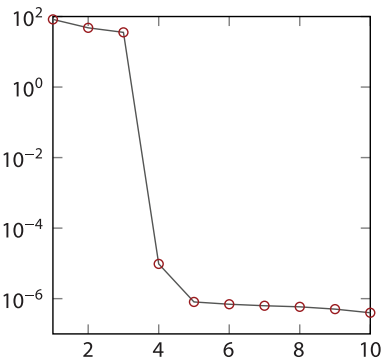

FIGURE 9. Rank estimation using ESPRIT with $N = 10$ and 200 datapoints.

5. Additional remarks

The assumption that the generalized eigenvalues Ω_i are distinct guarantees the existence of a solution to the exponential interpolation problem (2.1). It also entails that the zeroes of (2.3) and the poles of (4.2) are simple. The more general case, where confluence of the Ω_i is allowed, is treated in [**23**] and [**18**].

In the former, the basis $\{\exp(\phi_i x)\}_{i\geq 1}$ for the generalized polynomial is enlarged to $\{x^j \exp(\phi_i x)\}_{i\geq 1, j\geq 0}$. The solution to the exponential interpolation problem then takes the form $\sum_{i=1}^{\nu} p_i(x) \exp(\phi_i x)$, where the degree of $p_i(x)$ is one less than the multiplicity of Ω_i.

In the latter, the matrix pencil method is generalized to the confluent case: in the case of poles of higher multiplicity in the Padé approximant, the generalized eigenvalues are adjusted by a combinatorial factor. At convergence, however, they coincide with the poles, counting multiplicities.

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