Paper: Multivariate exponential analysis from the minimal number of samples.

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Multivariate exponential analysis from the minimal number of samples

Numerical illustration (part one)

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Script environment

This script does not depend on the random number generator state.

```
clear
close all
```

6. Numerical illustration (part one)

We take $d = 2$, write $u := x_1$, $v := x_2$, $x = (u, v)^t$ and consider the exponential sum

$$f(u, v) = \sum_{j=1}^{4} \alpha_j \exp(\langle \phi_j, x \rangle)$$

with

$$\phi_1 = (-0.5, 1 + \mathrm{i}2\pi \times 0.5), \qquad \alpha_1 = 1.7 \exp(\mathrm{i}2\pi/10),$$

$$\phi_2 = (0.1 + \mathrm{i}2\pi \times 3.4, 1.5 + \mathrm{i}2\pi \times 5.2), \qquad \alpha_2 = 1.1 \exp(\mathrm{i}2\pi/20),$$

$$\phi_3 = (0.1 + \mathrm{i}2\pi \times 3.4, -0.5 + \mathrm{i}2\pi \times 12.6), \qquad \alpha_3 = 0.9,$$

$$\phi_4 = (-2.5 + \mathrm{i}2\pi \times 23.2, -10 + \mathrm{i}2\pi \times 82.3), \qquad \alpha_4 = 9.2 \exp(\mathrm{i}2\pi/2).$$

```
phi = [-0.5, 1+2*pi*1i*0.5;
       0.1+2*pi*1i*3.4, 1.5+2*pi*1i*5.2;
       0.1+2*pi*1i*3.4, -0.5+2*pi*1i*12.6;
       -2.5+2*pi*1i*23.2, -10+2*pi*1i*82.3];
alpha = [1.7*exp(2*pi*1i/10);
         1.1*exp(2*pi*1i/20);
```

```
        0.9;
        9.2*exp(2*pi*1i/2)];

f = @(u,v) sum(alpha.*exp(phi*[u(:),v(:)].'),1);
```

When outputting numerical results for this small scale example, we round all values to 4 significant digits (all relative errors are less than 5×10^{-4}). The numerical effect of the choice of the vectors Δ and δ_i throughout the process, and that of the underlying one-dimensional Prony-like method in use, is beyond the scope of this paper and will be the subject of further investigations.

First we show the simple case described in the Sections 2 and 3, where the number of terms $n = 4$ is known up front and no collisions of the inner products in the samples occur. Of course, the latter is hard to predict in practice. We take $\Delta = (0.01, 0.01)$ and $\delta_1 = (-0.01, 0.01)$. Using 8 equidistant evaluations at $x = s\Delta, s = 0, \ldots, 7$, we obtain from (4) the values of $\exp(\Phi_j)$ and can deduce the $\Phi_j, j = 1, \ldots, 4$ because $|\Im\phi_{ji}| < \pi/|\Delta_i|$:

$$\Phi_1 = \langle \phi_1, \Delta \rangle \approx 0.005000 + 0.03142\mathrm{i},$$

$$\Phi_2 = \langle \phi_2, \Delta \rangle \approx 0.01600 + 0.5404\mathrm{i},$$

$$\Phi_3 = \langle \phi_3, \Delta \rangle \approx -0.004000 + 1.005\mathrm{i},$$

$$\Phi_4 = \langle \phi_4, \Delta \rangle \approx -0.125 + 0.3456\mathrm{i}.$$

```
Delta = 1/4*[0.01,0.01];
delta1 = 1/4*[-0.01,0.01];
F = f((0:7)*Delta(1),(0:7)*Delta(2));
exp_Phi = eig(F(hankel(2:5,5:8)),F(hankel(1:4,4:7)));
Phi_recon = log(exp_Phi)
```

```
Phi_recon =

  -0.0312 + 1.6572i
  -0.0010 + 0.2513i
   0.0040 + 0.1351i
   0.0012 + 0.0079i
```

We obtain the coefficients $\alpha_j, j = 1, \ldots, 4$ from (6):

$$\alpha_1 = 1.700 \exp(\mathrm{i}2\pi \times 0.1000),$$

$$\alpha_2 = 1.100 \exp(\mathrm{i}2\pi \times 0.05000),$$

$$\alpha_3 = 0.9000,$$

$\alpha_4 = 9.200\exp(\mathrm{i}2\pi \times 0.5000)$,

```
V = vandermonde(exp_Phi,4);
alpha_recon = V\F(1:4).'

alpha_recon =

  -9.2000 + 0.0000i
   0.9000 - 0.0000i
   1.0462 + 0.3399i
   1.3753 + 0.9992i
```

From 4 additional evaluations along the identification shift δ_1, we obtain the values of $\exp(\Phi_{11}), \exp(\Phi_{21}), \exp(\Phi_{31}), \exp(\Phi_{41})$ from (8). Their exponents are the projections of the vectors ϕ_j along δ_1:

$\Phi_{11} = \langle \phi_1, \delta_1 \rangle \approx 0.01500 + 0.03142\mathrm{i}$,

$\Phi_{21} = \langle \phi_2, \delta_1 \rangle \approx 0.01400 + 0.1131\mathrm{i}$,

$\Phi_{31} = \langle \phi_3, \delta_1 \rangle \approx -0.006000 + 0.5781\mathrm{i}$,

$\Phi_{41} = \langle \phi_4, \delta_1 \rangle \approx -0.07500 + 3.713\mathrm{i}$,

```
F2 = f((0:3)*Delta(1)+delta1(1),(0:3)*Delta(2)+delta1(2));
Phi2_recon = log((V\F2(:))./alpha_recon)

Phi2_recon =

  -0.0187 + 0.9283i
  -0.0015 + 0.1445i
   0.0035 + 0.0283i
   0.0038 + 0.0079i
```

We finally obtain the values of $\phi_j = (\phi_{j1}, \phi_{j2})^t$ by solving for each $j = 1, \ldots, 4$

$$\begin{pmatrix} \Delta_1 & \Delta_2 \\ \delta_{11} & \delta_{12} \end{pmatrix} \begin{pmatrix} \phi_{j1} \\ \phi_{j2} \end{pmatrix} = \begin{pmatrix} \Phi_j \\ \Phi_{j1} \end{pmatrix}$$

```
phi_recon = zeros(4,2);
for j = 1:4
    phi_recon(j,:) = [Delta;delta1]\[Phi_recon(j);Phi2_recon(j)];
end
phi_recon
```

```
phi_recon =

   1.0e+02 *

  -0.0250 + 1.4577i  -0.1000 + 5.1711i
   0.0010 + 0.2136i  -0.0050 + 0.7917i
   0.0010 + 0.2136i   0.0150 + 0.3267i
  -0.0050 + 0.0000i   0.0100 + 0.0314i
```