Paper: Multivariate exponential analysis from the minimal number of samples.

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Multivariate exponential analysis from the minimal number of samples

Numerical illustration (part two)

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Script environment

This script does not depend on the random number generator state.

```
clear
close all
```

6. Numerical illustration (part two)

We take $d = 2$, write $u := x_1$, $v := x_2$, $x = (u, v)^t$ and consider the exponential sum

$$f(u, v) = \sum_{j=1}^{4} \alpha_j \exp(\langle \phi_j, x \rangle)$$

with

$$\phi_1 = (-0.5, 1 + \mathrm{i}2\pi \times 0.5), \qquad \alpha_1 = 1.7 \exp(\mathrm{i}2\pi/10),$$

$$\phi_2 = (0.1 + \mathrm{i}2\pi \times 3.4, 1.5 + \mathrm{i}2\pi \times 5.2), \qquad \alpha_2 = 1.1 \exp(\mathrm{i}2\pi/20),$$

$$\phi_3 = (0.1 + \mathrm{i}2\pi \times 3.4, -0.5 + \mathrm{i}2\pi \times 12.6), \qquad \alpha_3 = 0.9,$$

$$\phi_4 = (-2.5 + \mathrm{i}2\pi \times 23.2, -10 + \mathrm{i}2\pi \times 82.3), \qquad \alpha_4 = 9.2 \exp(\mathrm{i}2\pi/2).$$

```
phi = [-0.5, 1+2*pi*1i*0.5;
       0.1+2*pi*1i*3.4, 1.5+2*pi*1i*5.2;
       0.1+2*pi*1i*3.4, -0.5+2*pi*1i*12.6;
       -2.5+2*pi*1i*23.2, -10+2*pi*1i*82.3];
alpha = [1.7*exp(2*pi*1i/10);
         1.1*exp(2*pi*1i/20);
```

```
          0.9;
          9.2*exp(2*pi*1i/2)];

f = @(u,v) sum(alpha.*exp(phi*[u(:),v(:)].'),1);
```

Take $\Delta = (0.03, 0)$ and $\delta_1 = (0, 0.01)$. The projections of ϕ_2 and ϕ_3 along Δ clearly coincide. After 7 evaluations at $x = s\Delta, s = 0, \ldots, 6$ we found that $\nu_0 = 3$ and we obtain from (11) that (without actually knowing the values of the h_j which we list only to help the reader follow the example):

$\Phi_{h_1} = \langle \phi_1, \Delta \rangle \approx -0.01500,$

$\Phi_{h_2} = \langle \phi_3, \Delta \rangle \approx 0.003000 + 0.6409\mathrm{i},$

$\Phi_{h_3} = \langle \phi_4, \Delta \rangle \approx -0.07500 + 4.373\mathrm{i},$

```
Delta = 1/4*[0.03,0];
delta1 = 1/4*[0,0.01];
F = f((0:6)*Delta(1),(0:6)*Delta(2));
nu0 = rank(F(hankel(1:4,4:7)))
exp_Phi = eig(F(hankel(2:4,4:6)),F(hankel(1:3,3:5)));
Phi_tmp = log(exp_Phi)
```

```
nu0 =

     3


Phi_tmp =

  -0.0187 + 1.0933i
   0.0008 + 0.1602i
  -0.0037 + 0.0000i
```

We proceed without knowing n and without knowing whether and where some collisions have occurred. But we know, since $d = 2$, that after adding an independent shift vector δ_1, all terms will have revealed themselves.

So we add evaluations $F_{s\ell 1} = f(\kappa_{\ell 1}\Delta + s\delta_1)$ with $\ell = 1, 2, 3$ and $s = 1, 2, \ldots$. For simplicity we choose $\kappa_{\ell 1} = \ell - 1$. With $\ell = 1$ and $s = 1, 2$ we find that the matrix

$$\begin{pmatrix} A_1 & A_{111} \\ A_{111} & A_{211} \end{pmatrix},$$

where the A_{sji} are computed from (14), has rank 1, and so $h_1 = 1 = g_1$.

With $\ell = 2$ and $s = 1, 2, 3, 4$ we find that the matrix

$$\begin{pmatrix} A2 & A_{121} & A_{221} \\ A_{121} & A_{221} & A_{321} \\ A_{221} & A_{321} & A_{421} \end{pmatrix}$$

has rank 2. This indicates with high probability that there are 2 terms coinciding at Φ_{h_2} (hence h_2 = 3 and g_2 = 2, g_3 = 3). Remember that in order to obtain $A_{s21}, 1 \leq s \leq 4$, we need to solve (13) which involves the samples $F_{sj1}, 1 \leq j \leq 3$. Hence continuing the sampling for $\ell = 2$ drags along $\ell = 1, 3$ at the same time. In other words, we are now spending 3×4 sampels for $\ell = 1, 2, 3$ rather than only 4 samples for $\ell = 2$.

We now reveal $\langle \phi_2, \delta_1 \rangle$ and $\langle \phi_3, \delta_1 \rangle$ by solving the generalized eigenvalue problem

$$\begin{pmatrix} A_{121} & A_{221} \\ A_{221} & A_{321} \end{pmatrix} v = \lambda \begin{pmatrix} A_2 & A_{121} \\ A_{121} & A_{221} \end{pmatrix} v.$$

With $\ell = 3$ and $s = 1, 2$ we find the same conclusion as with $\ell = 1$, now for

$$\begin{pmatrix} A_3 & A_{131} \\ A_{131} & A_{231} \end{pmatrix},$$

and so $\nu_1 = 4$ with $h_3 = 4$, $g_4 = 4$.

At the expense of a total of $(2 \times 3 + 1) + 3 \times 4 = 19$ evaluations, we find that $n = 4$ and we can identify all ϕ_{ji} and α_j for $j = 1, \ldots, 4$ and $i = 1, 2$.

```
V = vandermonde(exp_Phi,nu0);
A = zeros(5,nu0,1);
A(1,:,:) = V\F(1:3).';
for s = 1:4
    Fs=f((0:2)*Delta(1)+s*delta1(1),(0:2)*Delta(2)+s*delta1(2));
    A(s+1,:,:) = V\Fs.';
end

alpha_recon = zeros(4,1);
Phi_recon = zeros(4,1);
Phi2_recon = zeros(4,1);

h1 = rank(reshape(A(hankel(1:3,3:5),1,1),[3,3]),1e-10)
alpha_recon(1) = A(1,1,1);
Phi_recon(1) = Phi_tmp(1);
Phi2_recon(1) = log(A(2,1,1)/A(1,1,1));

h2 = h1 + rank(reshape(A(hankel(1:3,3:5),2,1),[3,3]),1e-10)
```

```
Phi_recon(2:3) = repmat(Phi_tmp(2),2,1);
Phi2_recon(2:3)=log(eig([A(2,2,1),A(3,2,1);A(3,2,1),A(4,2,1)],...
                        [A(1,2,1),A(2,2,1);A(2,2,1),A(3,2,1)]));
alpha_recon(2:3) = vandermonde(exp(Phi2_recon(2:3)),2)\...
                   [A(1,2,1);A(2,2,1)];

h3 = h2 + rank(reshape(A(hankel(1:3,3:5),3,1),[3,3]),1e-10)
alpha_recon(4) = A(1,3,1);
Phi_recon(4) = Phi_tmp(3);
Phi2_recon(4) = log(A(2,3,1)/A(1,3,1));

phi_recon = zeros(4,2);
for j = 1:4
    phi_recon(j,:) = [Delta;delta1]\[Phi_recon(j);Phi2_recon(j)];
end
phi_recon
alpha_recon

h1 =

     1


h2 =

     3


h3 =

     4


phi_recon =

   1.0e+02 *

  -0.0250 + 1.4577i  -0.1000 + 5.1711i
   0.0010 + 0.2136i  -0.0050 + 0.7917i
   0.0010 + 0.2136i   0.0150 + 0.3267i
  -0.0050 + 0.0000i   0.0100 + 0.0314i


alpha_recon =

  -9.2000 + 0.0000i
```

```
0.9000 + 0.0000i
1.0462 + 0.3399i
1.3753 + 0.9992i
```