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How to get high resolution results from sparse and coarsely sampled data

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ABSTRACT

Sampling a signal below the Shannon–Nyquist rate causes aliasing, meaning different frequencies to become indistinguishable. It is also well-known that recovering spectral information from a signal using a parametric method can be ill-posed or ill-conditioned and therefore should be done with caution.

We present an exponential analysis method to retrieve high-resolution information from coarse-scale measurements, using uniform downsampling. We exploit rather than avoid aliasing. While we loose the unicity of the solution by the downsampling, it allows to recondition the problem statement and increase the resolution.

Our technique can be combined with different existing implementations of multi-exponential analysis (matrix pencil, MUSIC, ESPRIT, APM, generalized overdetermined eigenvalue solver, simultaneous QR factorization, ...) and so is very versatile. It seems to be especially useful in the presence of clusters of frequencies that are difficult to distinguish from one another.

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1. Introduction

Estimating the fine scale spectral information of an exponential sum plays an important role in many signal processing applications. The problem of superresolution [1,2] has therefore recently received considerable attention.

Despite its computational efficiency and wide applicability, the often used Fourier transform (FT) has some well-known limitations, such as its limited resolution and the leakage in the frequency domain. These restrictions complicate the analysis of signals falling exponentially with time. Fourier analysis, which represents a signal as a sum of periodic functions, is not very well suited for the decomposition of aperiodic

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signals, such as exponentially decaying ones. The damping causes a broadening of the spectral peaks, which in its turn leads to the peaks overlapping and masking the smaller amplitude peaks. The latter are important for the fine level signal classification.

Signals that fall exponentially with time appear, for instance, in transient detection, motor fault diagnosis, electrophysiology, magnetic resonance and infrared spectroscopy, vibration and seismic data analysis, music signal processing, corrosion rate and crack initiation modelling, electronic odour recognition, typed keystroke recognition, nuclear science, liquid explosives identification, direction of arrival estimation, and so on.

A different approach to spectral analysis is offered, among others, by parametric methods. However, parametric methods often suffer from ill-posedness and ill-conditioning particularly in the case of clustered frequencies [3–5]. In general, parametric methods also require prior knowledge of the model order. Widely used parametric methods assuming a multi-exponential model include MUSIC [6], ESPRIT [7], the matrix pencil algorithm [8], simultaneous QR factorization [9] or a generalized overdetermined eigenvalue solver [10] and the approximate Prony method APM [11–13].

In general, parametric methods as well as the FT, sample at a rate dictated by the Shannon–Nyquist theorem [14,15]. It states that the sampling rate needs to be at least twice the maximum frequency present in the signal. A coarser time grid causes aliasing, identifying higher frequencies with lower frequencies without being able to distinguish between them. Conventional measurement systems, as used in modern consumer electronics, biomedical monitoring and medical imaging devices, are all based on this theorem.

In the past decade, alternative approaches have proved that signal reconstruction is also possible from sub-Nyquist measurements, if additional information on the structure of the signal is known, such as its sparsity. Many signals are indeed sparse in some domain such as time, frequency or space, meaning that most of the samples of either the signal or its transform in another domain can be regarded as zero. Among others, we refer to compressed sensing [16,17], finite rate of innovation [18], the use of coprime arrays in DOA [19,20].

The ultimate goal is to retrieve fine-scale information directly from coarse-scale measurements acquired at a slower information rate, in function of the sparsity and not the bandwidth of the signal. We offer a technique that allows to overcome the Shannon–Nyquist sampling rate limitation and at the same time may improve the conditioning of the numerical linear algebra problems involved. The technique is exploiting aliasing rather than avoiding it and maintains a regular sampling scheme [21,22]. It relies on the concept of what we call *identification shift* [22,21], which is the additional sampling at locations shifted with respect to the original locations, in order to overcome any ambiguity in the analysis arising from periodicity issues and in order to solve other identification problems occurring in coprime array approaches.

The paper is organized as follows. Exponential analysis following Shannon–Nyquist sampling is repeated in Section 2 and generalized to sub-Nyquist sampling in Section 3. Since a sub-Nyquist rate can cause terms to collide at the time of the sampling, we explain how to unravel collisions in Section 4. Such collisions are very unlikely to happen in practice of course. In Section 5 numerical examples illustrate both the collision-free situation and the case in which the collision of terms happens. The numerical examples at the same time illustrate:

- how the method reconditions a problem statement that is numerically ill-conditioned because of the presence of frequency clusters,
- that it can be combined with an existing implementation of a multi-exponential spectral analysis (we used ESPRIT [7] and `oeig` [10]).

2. The multi-exponential model

In order to proceed we introduce some notations. Let the real parameters $\psi_i, \omega_i, \beta_i$ and γ_i respectively denote the damping, frequency, amplitude and phase in each component of the signal

$$\phi(t) = \sum_{i=1}^{n} \alpha_i \exp(\phi_i t), \qquad \alpha_i = \beta_i \exp(\mathtt{i}\gamma_i), \qquad \phi_i = \psi_i + \mathtt{i}2\pi\omega_i. \tag{1}$$

For the moment, we assume that the frequency content is limited by [14,15]

$$|\Im(\phi_i)|/(2\pi) = |\omega_i| < \Omega/2, \qquad i = 1, \ldots, n,$$

and we sample $\phi(t)$ at the equidistant points $t_j = j\Delta$ for $j = 0, 1, \ldots, 2n-1, \ldots$ with $\Delta \leq 1/\Omega$. In the sequel we denote

$$f_j := \phi(t_j), \qquad j = 0, 1, \ldots, 2n-1, \ldots$$

The aim is to find the model order n, and the parameters $\phi_1, \ldots, \phi_n$ and $\alpha_1, \ldots, \alpha_n$ from the measurements $f_0, \ldots, f_{2n}, \ldots$ We further denote

$$\lambda_i := \exp(\phi_i \Delta), \qquad i = 1, \ldots, n.$$

With

$$H_n^{(k)} := \begin{pmatrix} f_k & \cdots & f_{k+n-1} \\ \vdots & \iddots & \vdots \\ f_{k+n-1} & \cdots & f_{k+2n-2} \end{pmatrix}, \qquad k \geq 0, \qquad n \geq 1,$$

the λ_i are retrieved [8,9,23] as the generalized eigenvalues of the problem

$$H_n^{(1)} v_i = \lambda_i H_n^{(0)} v_i, \qquad i = 1, \ldots, n, \tag{2}$$

where v_i are the generalized right eigenvectors. From the values λ_i, the complex numbers ϕ_i can be retrieved uniquely because of the restriction $|\Im(\phi_i \Delta)| < \pi$.

In the absence of noise, the exact value for n can be deduced from [24, p. 603], because we have for any single value of k that (for a detailed discussion also see [25])

$$\begin{aligned} &\det H_\nu^{(k)} = 0 \text{ only accidentally}, \qquad k \geq 0, \\ &\det H_n^{(k)} \neq 0, \qquad k \geq 0 \\ &\det H_\nu^{(k)} = 0, \qquad \nu > n, \qquad k \geq 0. \end{aligned} \tag{3}$$

The way (3) is checked is usually by computing the numerical rank of a Hankel matrix $H_\nu^{(k)}$ or a rectangular $(\mu - \nu) \times \nu$ version of it with $\mu \geq 2\nu, \nu \geq n$, from its singular value decomposition [23]. In the presence of noise and/or clusters of eigenvalues, this technique may not be reliable though, but then some convergence property can be used instead [26]. Note that hitting a zero value for $\det H_\nu^{(0)}$ accidentally, meaning while $1 \leq \nu < n$, can only happen a finite number of times in a row, namely $n-1$ times (which is extremely unlikely), while the true value of n is confirmed an infinite number of times when overshooting it with any $\nu > n$. Therefore the output of (3) is always probabilistic of nature. In Section 4.2 a similar result is presented in the context of sub-Nyquist sampling where one may loose the mutual distinctiveness of the generalized eigenvalues which is at the basis of (3).

Finally, the α_i are computed from the interpolation conditions

$$\sum_{i=1}^{n} \alpha_i \exp(\phi_i t_j) = f_j, \qquad j = 0, \ldots, 2n-1, \tag{4}$$

either by solving the system in the least squares sense, in the presence of noise, or by solving a subset of n (consecutive) interpolation conditions in case of a noisefree $\phi(t)$. Also, n can everywhere be replaced by $N > n$, in order to model noise on the data by means of some additional $N - n$ noise terms in (1). Note that

$$\exp(\phi_i t_j) = \lambda_i^j$$

and that the coefficient matrix of (4) is therefore a Vandermonde matrix. It is well-known that the conditioning of structured matrices is something that needs to be monitored [27,28].

Without loss of generality, we assume in the sequel that $0 \le \omega_i < \Omega \in \mathbb{N}, i = 1, \ldots, n$ instead of $|\omega_i| < \Omega/2, i = 1, \ldots, n$. Also we assume in Section 3 that n is known or correctly detected as indicated in [26]. In Section 4 we explain how to detect n concurrently with the computation of the ϕ_i and α_i from sub-Nyquist data.

3. Sub-Nyquist multi-exponential analysis

Some basic result is first deduced for $n = 1$. Afterwards this result is made use of for general n. The latter however, demands additional developments.

3.1. Dealing with a single frequency ($n = 1$)

At first we deal with some simple mathematical results, without caring about computational issues. When

$$\phi(t) = \alpha \exp(\psi t + \mathtt{i}2\pi\omega t), \qquad 0 \le \omega < \Omega,$$

and $\phi(t)$ is sampled at $t_j = 0, \Delta, 2\Delta, \ldots$, with for simplicity $\Delta = 1/\Omega$, then ω can uniquely be determined in $[0, \Omega)$ from the samples. No periodicity problem occurs since $\omega\Delta < 1$ in the generalized eigenvalue

$$\lambda = \exp(\psi\Delta)\exp(\mathtt{i}2\pi\omega\Delta).$$

When $\phi(t)$ is sampled at multiples $t_{r_1 j} = 0, r_1\Delta, 2r_1\Delta, \ldots$ with $1 < r_1 \in \mathbb{N}$, then there exist r_1 solutions for ω in $[0, \Omega)$ since $0 \le 2\pi\omega r_1\Delta < 2r_1\pi$. If $\phi(t)$ is also sampled at $t_{r_2 j} = 0, r_2\Delta, 2r_2\Delta, \ldots$ with $0 < r_2 \in \mathbb{N}$, then one obtains another set containing r_2 solutions for ω. Each solution set is extracted from the respective generalized eigenvalues $\exp(\psi r_m\Delta)\exp(\mathtt{i}2\pi\omega r_m\Delta), m = 1, 2$ satisfying (2) where the first generalized eigenvalue problem is set up with the samples $f_{r_1 j} = \phi(0), \phi(r_1\Delta), \phi(2r_1\Delta), \ldots$ and the second generalized eigenvalue problem with the samples $f_{r_2 j} = \phi(0), \phi(r_2\Delta), \phi(2r_2\Delta), \ldots$ In our write-up we have chosen not to add an index r to the notation of the Hankel matrices $H_n^{(0)}$ and $H_n^{(1)}$ when they are filled with samples taken at multiples $t_{rj} = jr\Delta$ in order to not overload the notation. From the context it is always clear which sequence of samples is meant.

It is easy to show that, if in addition $\gcd(r_1, r_2) = 1$, then ω is the unique intersection of the two solution sets.

Lemma 1. *Let $0 \le \omega < \Omega$ and Ω, r_1, r_2 be nonzero positive integers. If $\gcd(r_1, r_2) = 1$ and if $\Delta = 1/\Omega$, then from the values $\exp(\mathtt{i}2\pi\omega r_1\Delta)$ and $\exp(\mathtt{i}2\pi\omega r_2\Delta)$, the frequency ω can uniquely be recovered in $[0, \Omega)$.*

Proof. From the generalized eigenvalue $\lambda^{r_1} = \exp(\psi r_1\Delta)\exp(\mathtt{i}2\pi r_1\Delta)$ we extract r_1 solutions for ω:

$$\omega = \omega^{(1)} + k\frac{\Omega}{r_1}, \qquad 0 \le \omega^{(1)} < \frac{\Omega}{r_1}, \qquad k = 0, \ldots, r_1 - 1. \tag{5}$$

From the value $\lambda^{r_2} = \exp(\psi r_2 \Delta)\exp(\mathtt{i}2\pi r_2\Delta)$ we extract r_2 solutions for ω:

$$\omega = \omega^{(2)} + \ell\frac{\Omega}{r_2}, \qquad 0 \le \omega^{(2)} < \frac{\Omega}{r_2}, \qquad \ell = 0, \dots, r_2 - 1. \tag{6}$$

Note that the frequency ω we are trying to identify satisfies both (5) and (6). Remains to show that the common solution to (5) and (6) is unique. Suppose we have two distinct values for ω that both satisfy (5) and (6). This implies that there exist two distinct $0 \le k_1, k_2 < r_1$ such that

$$\begin{aligned} \omega^{(1)} + k_1\frac{\Omega}{r_1} &= \omega^{(2)} + \ell_1\frac{\Omega}{r_2}, \\ \omega^{(1)} + k_2\frac{\Omega}{r_1} &= \omega^{(2)} + \ell_2\frac{\Omega}{r_2}, \end{aligned} \tag{7}$$

with $0 \le \ell_1, \ell_2 < r_2$ and $\ell_1 \ne \ell_2$ because $k_1 \ne k_2$. From (7) we deduce

$$k_1 - k_2 = \frac{(\ell_1 - \ell_2)r_1}{r_2} \ne 0.$$

Hence r_2 divides $\ell_1 - \ell_2$ because $\gcd(r_1, r_2) = 1$. Since $\ell_1 - \ell_2$ is bounded in absolute value by $r_2 - 1$, this is a contradiction. □

Furthermore, the element ω in the intersection can be obtained from the Euclidean algorithm.

Lemma 2. *Let $0 \le \omega < \Omega$ and Ω, r_1, r_2 be nonzero positive integers. If $\gcd(r_1, r_2) = 1$ and if $\Delta = 1/\Omega$, then from the values $\exp(\mathtt{i}2\pi\omega r_1\Delta)$ and $\exp(\mathtt{i}2\pi\omega r_2\Delta)$, the frequency $\omega \in [0, \Omega)$ is obtained as*

$$\left(p_1\frac{\mathrm{Log}(\exp(\mathtt{i}2\pi\omega r_1\Delta))}{\mathtt{i}2\pi} + p_2\frac{\mathrm{Log}(\exp(\mathtt{i}2\pi\omega r_2\Delta))}{\mathtt{i}2\pi}\right)\Omega = \omega + h\Omega, \quad h \in \mathbb{Z}, \tag{8}$$

where $p_1 r_1 + p_2 r_2 = 1 \mod \Omega$ with $p_1, p_2 \in \mathbb{Z}$ and $\mathrm{Log}(\cdot)$ denotes the principal branch of the complex logarithm.

Proof. We use the same notation as in the proof of Lemma 1. So we have

$$\begin{aligned} \omega &= \omega^{(1)} + k\frac{\Omega}{r_1}, \qquad 0 \le \omega^{(1)} < \frac{\Omega}{r_1}, \\ \omega &= \omega^{(2)} + \ell\frac{\Omega}{r_2}, \qquad 0 \le \omega^{(2)} < \frac{\Omega}{r_2}. \end{aligned}$$

Then

$$\frac{\Omega}{r_1}\frac{\mathrm{Log}\left(\exp(\mathtt{i}2\pi\omega r_1\Delta)\right)}{\mathtt{i}2\pi} = \omega^{(1)}, \qquad \frac{\Omega}{r_2}\frac{\mathrm{Log}\left(\exp(\mathtt{i}2\pi\omega r_2\Delta)\right)}{\mathtt{i}2\pi} = \omega^{(2)},$$

and

$$\begin{aligned} \left(p_1\frac{\mathrm{Log}\left(\exp(\mathtt{i}2\pi\omega r_1\Delta)\right)}{\mathtt{i}2\pi} + p_2\frac{\mathrm{Log}\left(\exp(\mathtt{i}2\pi\omega r_2\Delta)\right)}{\mathtt{i}2\pi}\right)\Omega &= (p_1 r_1 + p_2 r_2)\omega - (p_1 k + p_2\ell)\Omega \\ &= \omega - (p_1 k + p_2\ell)\Omega, \end{aligned}$$

in which $p_1 k + p_2\ell$ is an integer. □

When the integers p_1 and p_2 are small this method is very useful. Otherwise one has to be careful about the numerical stability of (8). One can of course experiment with different r_1 and r_2 values to ensure small p_1 and p_2 values.

3.2. Dealing with several terms ($n > 1$)

When $\phi(t)$ contains several terms, then we obtain n solution sets for the $\omega_i, i = 1, \ldots, n$ from the first batch of evaluations at multiples of $r_1\Delta$ and another n solution sets for these frequencies from the second batch of samples at multiples of $r_2\Delta$. But now we are facing the problem of correctly matching the solution set from the first batch to the solution set from the second batch that refer to the same ω_i. Of course, we want to avoid such combinatorial steps in our algorithm. To solve this problem we are going to choose the second batch of sampling points in a smarter way.

Before we proceed we assume that we don't have $\exp(\phi_k r\Delta) = \exp(\phi_\ell r\Delta)$ for distinct k and ℓ with $1 \le k, \ell \le n$. In Section 4 we explain how to deal with the collision of terms, which we exclude in the sequel of this section.

The sampling strategy that we propose is the following. Sampling at $t_{rj} = jr\Delta$ with fixed $1 < r \in \mathbb{N}$, gives us only aliased values for ω_i, obtained from $\mathrm{Log}(\exp(\mathtt{i}2\pi\omega r\Delta))$. This aliasing can be fixed at the expense of the following additional samples. In what follows n can also everywhere be replaced by $N > n$ when using $N - n$ additional terms in (1) to model the noise.

To fix the aliasing, we add n samples to the already collected $f_0, f_r, \ldots, f_{(2n-1)r}$, namely at the shifted points

$$t_{rj+\rho} = jr\Delta + \rho\Delta, \qquad r, \rho \text{ fixed},$$
$$j = h, \ldots, h + n - 1, \qquad 0 \le h \le n.$$

An easy choice for ρ is a number mutually prime with r. For the most general choice allowed, we refer to [29]. An easy practical generalization is when r and ρ are rational numbers r/s and ρ/σ respectively with $r, s, \sigma \in \mathbb{N}$ and $\rho \in \mathbb{Z}$. In that case the condition $\gcd(r, \rho) = 1$ is replaced by $\gcd(\overline{r}, \overline{\rho}) = 1$ where $r/s = \overline{r}/\tau, \rho/\sigma = \overline{\rho}/\tau$ with $\tau = \mathrm{lcm}(s, \sigma)$. Also, the indices of the shifted points need not be consecutive, but for ease of notation we assume this for now.

From the samples $f_0, f_r, \ldots, f_{(2n-1)r}$ we first compute the generalized eigenvalues $\lambda_i^r = \exp(\phi_i r\Delta)$ and the coefficients α_i going with λ_i^r in the model

$$\begin{aligned} \phi(jr\Delta) &= \sum_{i=1}^{n} \alpha_i \exp(\phi_i jr\Delta) \\ &= \sum_{i=1}^{n} \alpha_i \lambda_i^{jr}, \quad j = 0, \ldots, 2n - 1. \end{aligned} \tag{9}$$

So we know which coefficient α_i goes with which generalized eigenvalue λ_i^r, but we just cannot identify the correct $\Im(\phi_i)$ from λ_i^r. The samples $f_{jr+\rho}$ at the additional points $t_{rj+\rho}$ satisfy

$$\begin{aligned} \phi(jr\Delta + \rho\Delta) &= \sum_{i=1}^{n} \alpha_i \exp(\phi_i(jr + \rho)\Delta) \\ &= \sum_{i=1}^{n} (\alpha_i \lambda_i^{\rho}) \lambda_i^{jr}, \\ &\qquad j = h, \ldots, h + n - 1, \qquad 0 \le h \le n, \end{aligned} \tag{10}$$

which can be interpreted as a linear system with the same coefficient matrix entries as (9), but now with a new left hand side and unknowns $\alpha_1\lambda_1^\rho, \ldots, \alpha_n\lambda_n^\rho$ instead of $\alpha_1, \ldots, \alpha_n$. And again we can associate each computed coefficient $\alpha_i\lambda_i^\rho$ with the proper generalized eigenvalue λ_i^r. Then by dividing the $\alpha_i\lambda_i^\rho$ computed from (10) by the α_i computed from (9), for $i = 1, \ldots, n$, we obtain from λ_i^ρ a second set of ρ plausible values for ω_i. Because of the fact that we choose ρ and r relatively prime, the two sets of plausible values for ω_i have only one value in their intersection, as explicited in Lemma 1 and 2. Thus the aliasing problem is solved.

4. When aliasing causes terms to collide

When $\exp(\phi_k r\Delta) = \exp(\phi_\ell r\Delta)$ with $k \neq \ell$, then different exponential terms in (9) collide into one term as a consequence of the undersampling and the aliasing effect. Note that then for the moduli of the exponential terms holds that $\exp(\psi_k r\Delta) = \exp(\psi_\ell r\Delta)$ and consequently $\psi_k = \psi_\ell$. As long as $\psi_k \neq \psi_\ell$, exponential terms can be distinguished on the basis of their modulus. So our focus is on the situation where

$$\phi_k = \psi_k + \mathtt{i}2\pi\omega_k \neq \phi_\ell = \psi_\ell + \mathtt{i}2\pi\omega_\ell, \qquad \psi_k = \psi_\ell, \quad r\omega_k = r\omega_\ell + h\Omega, \quad h \in \mathbb{Z}.$$

Since terms can collide when subsampling, their correct number n may not be revealed when sampling at multiples of $r\Delta$, in other words, when sampling at the rate Ω/r instead of Ω. Let us assume that (3), or its practical implementation in [26] on $N \times N$ Hankel matrices with $N > n$, reveals a total of n_0 terms after the first batch of evaluations at $t_{rj} = jr\Delta$ with fixed r. We call $\lambda_i^{(0)}$ the n_0 generalized eigenvalues of (2) computed from the f_{jr} as in Section 3. Since some of the terms in (9) may have collided, we have

$$\phi(t_{rj}) = \sum_{i=1}^{n_0} \alpha_i^{(0)} \exp(\phi_i^{(0)} t_{rj}) \tag{11}$$

with

$$\lambda_i^{(0)} = \exp(\phi_i^{(0)} r\Delta), \qquad i = 1, \ldots, n_0,$$

and some of the $\alpha_i^{(0)}$ being sums of the α_i from (9). In Section 4.1 we assume that all $\alpha_i^{(0)}$ are nonzero. The case where some of the collisions have disappeared because of cancellations in the coefficients, meaning that some of the $\alpha_i^{(0)}, i = 1, \ldots, n_0$ are zero, is dealt with in Section 4.2.

It should be clear that the value of n_0 depends on r, as can be seen in the following simple example (nevertheless we do not want to burden the notation n_0 with this evidence). Consider the function $\phi(t)$ given by

$$\phi(t) = e^{\mathtt{i}2\pi t} - e^{\mathtt{i}2\pi 21t} + e^{\mathtt{i}2\pi 41t} - e^{\mathtt{i}2\pi 61t} + e^{\mathtt{i}2\pi 11t} - e^{\mathtt{i}2\pi 31t} + e^{\mathtt{i}2\pi 51t}.$$

With $\Delta = 1/100$ and $r = 5$ we find that in the evaluations $\phi(jr\Delta)$ the first 4 terms cancel each other and the last 3 terms collide into

$$f_{5j} = e^{\mathtt{i}2\pi 55j/100}, \qquad n_0 = 2. \tag{12}$$

With $\Delta = 1/100$ and $r = 12$ the fourth and the fifth term cancel each other and the first and the last term collide, giving

$$f_{12j} = 2e^{\mathtt{i}2\pi 12j/100} - e^{\mathtt{i}2\pi 52j/100} + e^{\mathtt{i}2\pi 92j/100} - e^{\mathtt{i}2\pi 72j/100}, \qquad n_0 = 4.$$

4.1. Collision without cancellation

We remark that $n_0 \le n$ and that the $\phi_i^{(0)}$ are definitely among the n parameters ϕ_i in (9). Without loss of generality we assume that the colliding terms are successive,

$$\begin{pmatrix} \alpha_1^{(0)} \\ \vdots \\ \alpha_{n_0}^{(0)} \end{pmatrix} = \begin{pmatrix} \alpha_{h_1} + \cdots + \alpha_{h_2-1} \\ \vdots \\ \alpha_{h_{n_0}} + \cdots + \alpha_{h_{n_0+1}-1} \end{pmatrix}, \qquad h_1 = 1, \qquad h_i \le h_{i+1}, \qquad 1 \le i \le n_0, \qquad h_{n_0+1} = n+1.$$

In brief, when collisions occur, the computations return the results

$$\begin{aligned} \alpha_i^{(0)} &= \sum_{\ell=h_i}^{h_{i+1}-1} \alpha_\ell, \qquad i = 1, \ldots, n_0 \\ \lambda_i^{(0)} &= \lambda_{h_i}^r = \ldots = \lambda_{h_{i+1}-1}^r, \qquad i = 1, \ldots, n_0. \end{aligned} \tag{13}$$

Note that only the nonzero $\alpha_i^{(0)}$ and the distinct $\lambda_i^{(0)}$ are revealed, without any knowledge about the $h_i, 1 \le i \le n_0$. In Section 4.2 we explain how to deal with the additional problem where some of the α_i cancel each other and therefore some of the $\lambda_i^{(0)}$ have gone missing in the samples $\phi(t_{rj})$.

For the sake of completeness we explicit the linear system that delivered the $\alpha_i^{(0)}$, namely

$$\begin{pmatrix} 1 & \ldots & 1 \\ \lambda_1^{(0)} & \ldots & \lambda_{n_0}^{(0)} \\ \vdots & & \vdots \\ (\lambda_1^{(0)})^{n_0-1} & \ldots & (\lambda_{n_0}^{(0)})^{n_0-1} \end{pmatrix} \begin{pmatrix} \alpha_1^{(0)} \\ \vdots \\ \alpha_{n_0}^{(0)} \end{pmatrix} = \begin{pmatrix} f_0 \\ f_r \\ \vdots \\ f_{(n_0-1)r} \end{pmatrix} \tag{14}$$

or, as is most often the case, an overdetermined version of it. We now explain how to disentangle the collisions, again making use of some additional samples at shifted locations. Let r and ρ be fixed as before with $\gcd(r, \rho) = 1$. If $\gcd(r, \rho) > 1$ for some reason or because of a practical constraint, then the procedure may be an iterative one, as we indicate further below.

Let us sample $\phi(t)$ at the shifted locations $t_{rj+\rho k} = (jr + k\rho)\Delta, j = 0, \ldots, n_0 - 1, k \ge 1$. These sample values equal

$$f_{jr+k\rho} := \sum_{i=1}^{n_0} \left(\sum_{\ell=h_i}^{h_{i+1}-1} \alpha_\ell \exp(\phi_\ell k\rho\Delta) \right) \exp(\phi_i^{(0)} jr\Delta). \tag{15}$$

In (15) we abbreviate

$$\alpha_i^{(1)}(k) := \sum_{\ell=h_i}^{h_{i+1}-1} \alpha_\ell \exp(\phi_\ell k\rho\Delta), \qquad i = 1, \ldots, n_0. \tag{16}$$

For $k = 0$ we have $\alpha_i^{(1)}(0) = \alpha_i^{(0)}, i = 1, \ldots, n_0$. For fixed $k > 0$ the values $\alpha_i^{(1)}(k), i = 1, \ldots, n_0$ are obtained from (15) and

$$\begin{pmatrix} 1 & \ldots & 1 \\ \lambda_1^{(0)} & \ldots & \lambda_{n_0}^{(0)} \\ \vdots & & \vdots \\ (\lambda_1^{(0)})^{n_0-1} & \ldots & (\lambda_{n_0}^{(0)})^{n_0-1} \end{pmatrix} \begin{pmatrix} \alpha_1^{(1)}(k) \\ \vdots \\ \alpha_{n_0}^{(1)}(k) \end{pmatrix} = \begin{pmatrix} f_{k\rho} \\ f_{r+k\rho} \\ \vdots \\ f_{(n_0-1)r+k\rho} \end{pmatrix} \tag{17}$$

or its least squares version. The Vandermonde coefficient matrix of (17) is the same as the one used to compute $\alpha_i^{(0)}, i = 1, \ldots, n_0$ in (14) from the samples f_{jr}, which is the case $k = 0$. So the Vandermonde matrix is reused as it is independent of the index k appearing in the right hand side and in the vector of unknowns.

When collecting in this way, for each $1 \le i \le n_0$, the values $\alpha_i^{(1)}(0)$, $\alpha_i^{(1)}(1)$, $\alpha_i^{(1)}(2)$, ... we have a separate exponential analysis problem per i, namely to identify the number of terms in $\alpha_i^{(1)}(k)$ in (16). Note that the sampling rate used to collect the $\alpha_i^{(1)}(k)$ is Ω/ρ.

Now we fix $1 \le i \le n_0$ and proceed. When the samples $\alpha_i^{(1)}(k)$ take the place of the values f_k in (2) and (4) and $h_{i+1} - h_i$ that of n, then:

- the generalized eigenvalue problem (2) delivers the components $\lambda_\ell^{(1)} = \exp(\phi_\ell \rho \Delta)$ in (16),
- and the respective Vandermonde system (4) delivers the α_ℓ for $\ell = h_i, \ldots, h_{i+1} - 1$.

Both can again be set up in a least squares sense, in a similar way as for the determination of the $\lambda_i^{(0)}$ and $\alpha_i^{(0)}$. As shown in Lemma 1, the exponential sums $\alpha_i^{(1)}(k)$ are fully disentangled and all n terms in (1) are identified when $\gcd(r, \rho) = 1$, which is what we try to achieve in practice.

With

$$\lambda_\ell^{(1)} = \exp(\phi_\ell \rho \Delta) = \lambda_\ell^{\rho}, \qquad \ell = h_i, \ldots, h_{i+1} - 1, \quad i = 1, \ldots, n_0, \tag{18}$$

and

$$\lambda_i^{(0)} = \exp(\phi_{h_i} r \Delta) = \lambda_\ell^{r}, \qquad \ell = h_i, \ldots, h_{i+1} - 1, \quad i = 1, \ldots, n_0,$$

we have what we need in order to identify the $\phi_i, i = 1, \ldots, n$ using Lemma 2, since

$$n = \sum_{i=1}^{n_0} (h_{i+1} - h_i).$$

An illustration of the procedure above is presented in Section 5.2.

When for one or other reason $\gcd(r, \rho) = s \neq 1$ then the above procedure needs to be repeated with r replaced by s and ρ replaced by a suitable σ. Then again additional samples are collected at shifted locations $t_{sj+\sigma k} = (js + k\sigma)\Delta$, namely

$$f_{js+k\sigma} := \phi\left((js + k\sigma)\Delta\right),$$

and the procedure is repeated from (15) on. When $\gcd(r, \rho, \sigma) = 1$ the procedure ends, otherwise it continues as described.

4.2. Collision with cancellation

To complete the method, we discuss the special situation where some of the terms $\alpha_i \exp(\phi_i j r \Delta)$ cancel each other when evaluating (1) at the t_{rj}, a situation which is illustrated in Section 5.3.

So at the first batch of evaluations f_{jr}, in addition to collision, one encounters cancellation for one or more indices $i, 1 \le i \le n_0$, meaning that one or more $\alpha_i^{(0)} = \alpha_i^{(1)}(0) = 0$. The fundamental question is whether the $\alpha_i^{(1)}(k)$ can continue to evaluate to zero for all k in the second shifted batch of evaluations $f_{jr+k\rho}$ when $\gcd(r, \rho) = 1$? The answer is no, not even when the ϕ_ℓ in (16) have the same decay rate, as becomes clear from the Lemmas 3 and 4 below.

Lemma 3. *Let for $\phi_k \neq \phi_\ell$ and $r \neq 0$ hold that $\exp(\phi_k r\Delta) = \exp(\phi_\ell r\Delta)$. If $\gcd(r, \rho) = 1$ then*

$$\exp(\phi_k \rho\Delta) \neq \exp(\phi_\ell \rho\Delta).$$

Proof. As pointed out it is sufficient to deal with the imaginary parts of ϕ_k and ϕ_ℓ. We use a similar notation as in Lemma 1. The proof is by contraposition. From $\exp(\phi_k r\Delta) = \exp(\phi_\ell r\Delta)$ and $\exp(\phi_k \rho\Delta) = \exp(\phi_\ell \rho\Delta)$ we find that there exist integers p_k, p_ℓ, q_k, q_ℓ such that

$$\begin{aligned}
\omega_k &= \omega^{(1)} + p_k \frac{\Omega}{r}, && 0 \le p_k \le r-1 \\
\omega_\ell &= \omega^{(1)} + p_\ell \frac{\Omega}{r}, && 0 \le p_\ell \le r-1 \\
\omega_k &= \omega^{(2)} + q_k \frac{\Omega}{\rho}, && 0 \le p_k \le \rho-1 \\
\omega_\ell &= \omega^{(2)} + q_\ell \frac{\Omega}{\rho}, && 0 \le p_k \le \rho-1.
\end{aligned}$$

Then

$$\omega_k - \omega_\ell = (p_k - p_\ell)\frac{\Omega}{r} = (q_k - q_\ell)\frac{\Omega}{\rho}$$

or

$$p_k - p_\ell = \frac{q_k - q_\ell}{\rho} r,$$

which is a contradiction since the left hand side is an integer and $q_k - q_\ell$ in the right hand side is in absolute value bounded by $\rho - 1$. □

Lemma 4. *Let $\alpha_i^{(1)}(k)$ be given by (16). Then $\forall\, 1 \le i \le n_0, \exists\, 0 \le k \le h_{i+1} - h_i : \alpha_i^{(1)}(k) \neq 0$.*

Proof. We consider the following square Vandermonde system which is obtained from (16) for fixed i and with k increased from 0 to $h_{i+1} - h_i$,

$$\begin{pmatrix} 1 & \dots & 1 \\ \exp(\phi_{h_i}\rho\Delta) & \dots & \exp(\phi_{h_{i+1}-1}\rho\Delta) \\ \vdots & & \vdots \\ \exp(\phi_{h_i}(h_{i+1}-h_i)\rho\Delta) & \dots & \exp(\phi_{h_{i+1}-1}(h_{i+1}-h_i)\rho\Delta) \end{pmatrix} \begin{pmatrix} \alpha_{h_i} \\ \vdots \\ \alpha_{h_{i+1}-1} \end{pmatrix} = \begin{pmatrix} \alpha_i^{(1)}(0) \\ \vdots \\ \alpha_i^{(1)}(h_{i+1}-h_i) \end{pmatrix}. \quad (19)$$

From Lemma 3 we know that this $(h_{i+1} - h_i) \times (h_{i+1} - h_i)$ Vandermonde matrix is regular. If the right hand side of this small linear system consists of all zeroes, then we must therefore conclude incorrectly that

$$\alpha_{h_i} = \cdots = \alpha_{h_{i+1}-1} = 0.$$

So from this we know that the evaluation of $\alpha_i^{(1)}(k)$ to zero cannot persist up to and including $k = h_{i+1} - h_i$. □

The important conclusion here is that in a finite number of steps the true value of n_0, which represents the number of distinct generalized eigenvalues existing at the sampling rate $r\Delta$, is always revealed. The evaluations at the shifted sample points $(jr + k\rho)\Delta$ with $k \neq 0$ serve the purpose to provide a different view on the coefficients, namely the values $\alpha_i^{(1)}(k)$ for $k \neq 0$. These additional evaluations do not alter or

touch the generalized eigenvalues $\lambda_i^{(0)}$. That is why a shift is so helpful. And Lemma 4 confirms, that even if initially some $\alpha_i^{(1)}(0)$ are zero, eventually all $\alpha_i^{(1)}(k)$ must become visible. This fact is entirely similar to the conclusion in (3), but with the function $\phi(t)$ replaced by $\alpha_i^{(1)}(k)$ for some fixed i and with the matrix $H_\nu^{(0)}$ replaced by the matrix

$$\begin{pmatrix} \alpha_i^{(1)}(0) & \dots & \alpha_i^{(1)}(\kappa-1) \\ \vdots & & \vdots \\ \alpha_i^{(1)}(\kappa-1) & \dots & \alpha_i^{(1)}(2\kappa-2) \end{pmatrix}$$

of increasing size $\kappa \times \kappa$.

To illustrate this we return to (12). While only one of the $n_0 = 2$ terms is visible when evaluating at $jr\Delta$ when $r = 5$, the evaluations f_{5j+12k} with $\rho = 12$, give us

$$\begin{aligned} f_{5j+12k} &= \left(e^{\mathrm{i}2\pi 12k/100} - e^{\mathrm{i}2\pi 52k/100} + e^{\mathrm{i}2\pi 92k/100} - e^{\mathrm{i}2\pi 32k/100}\right) e^{\mathrm{i}2\pi 5j/100} + \\ &\qquad \left(e^{\mathrm{i}2\pi 32k/100} - e^{\mathrm{i}2\pi 72k/100} + e^{\mathrm{i}2\pi 12k/100}\right) e^{\mathrm{i}2\pi 55j/100} \\ &= \alpha_{h_1}^{(1)}(k) e^{\mathrm{i}2\pi 5j/100} + \alpha_{h_2}^{(1)}(k) e^{\mathrm{i}2\pi 55j/100}. \end{aligned}$$

For $k \geq 1$ and $\nu \geq 2$ we find that the rank of

$$H_\nu^{(12k)} \begin{pmatrix} f_{12k} & f_{r+12k} & \cdots & f_{(\nu-1)r+12k} \\ f_{r+12k} & & & \\ \vdots & & & \vdots \\ f_{(\nu-1)r+12k} & \cdots & & \end{pmatrix}$$

equals $n_0 = 2$.

5. Numerical illustration

We illustrate the working of (9) and (10) from Section 3 and that of (14) and (17) from Section 4 on two examples in the respective Sections 5.1 and 5.2. In the former numerical example the undersampling will not cause collisions, while in the latter illustration it will. In addition, in Section 5.3, we show the detection of terms that have not only collided but entirely vanished in the first sampling at multiples of $r\Delta$. We conclude in Section 5.4 with pseudocode for the full-blown algorithm, which is most easy to understand after going through the examples. The pseudocode deals with all possible combinations of situations and is therefore even more general than the example in Section 5.3.

5.1. Collision-free example

For our first example the α_i and ϕ_i are given in Table 1. We take $\Omega = 1000$ and $\Delta = 1/\Omega$. The $n = 20$ frequencies ω_i form 5 clusters, as is apparent from the FT, computed from 1000 samples and shown in Fig. 1. For completeness we graph the signal in Fig. 2. In Fig. 3 we show the generalized eigenvalues $\lambda_i = \exp(\phi_i\Delta), i = 1, \dots, 20$ computed from the noisefree samples, to illustrate the ill-conditioning of the problem as a result of the clustering of the frequencies.

To (1) we add white Gaussian noise with SNR = 32 dB. For comparison with our new method, we show in Fig. 4 the (ω_i, β_i) results computed by means of ESPRIT using 240 samples, namely $f_0, \dots, f_{239}$. A signal space of dimension 20 and a noise space of dimension 40, so a total dimension $N = 60$, produced a typical ESPRIT result, from a size 180×60 problem. The true (ω_i, β_i) couples from Table 1 are indicated using

Table 1
Collision-free example.

i	α_i	ϕ_i
1	$6.5\exp(0.15\mathtt{i})$	$-0.19-\mathtt{i}2\pi 453.1$
2	6.8	$-0.132-\mathtt{i}2\pi 452.19$
3	$6.8\exp(0.3\mathtt{i})$	$-0.183-\mathtt{i}2\pi 451.02$
4	$6.4\exp(0.9\mathtt{i})$	$-0.11-\mathtt{i}2\pi 450.21$
5	$7.1\exp(0.7\mathtt{i})$	$-0.21-\mathtt{i}2\pi 448.39$
6	$4.71\exp(0.12\mathtt{i})$	$-0.106-\mathtt{i}2\pi 132.5$
7	$3.9\exp(0.1\mathtt{i})$	$-0.129-\mathtt{i}2\pi 131.4$
8	$7.2\exp(-0.234\mathtt{i})$	$-0.203-\mathtt{i}2\pi 130.01$
9	$7.43\exp(0.2\mathtt{i})$	$-0.16-\mathtt{i}2\pi 129.17$
10	$4.4\exp(-0.52\mathtt{i})$	$-0.19-\mathtt{i}2\pi 128.39$
11	$3\exp(0.21\mathtt{i})$	$-0.101+\mathtt{i}2\pi 9.1$
12	$3\exp(-0.8\mathtt{i})$	$-0.127+\mathtt{i}2\pi 11.81$
13	$7.2\exp(-0.106\mathtt{i})$	$-0.21+\mathtt{i}2\pi 126.01$
14	$6.53\exp(0.2\mathtt{i})$	$-0.15+\mathtt{i}2\pi 127.62$
15	$6.7\exp(-0.3\mathtt{i})$	$-0.173+\mathtt{i}2\pi 128.98$
16	$6.8\exp(-0.15\mathtt{i})$	$-0.11+\mathtt{i}2\pi 334.01$
17	$6\exp(0.26\mathtt{i})$	$-0.12+\mathtt{i}2\pi 335.18$
18	$7.1\exp(-0.2\mathtt{i})$	$-0.157+\mathtt{i}2\pi 336.01$
19	7.1	$-0.120+\mathtt{i}2\pi 337.91$
20	$6\exp(-0.1\mathtt{i})$	$-0.18+\mathtt{i}2\pi 339.61$

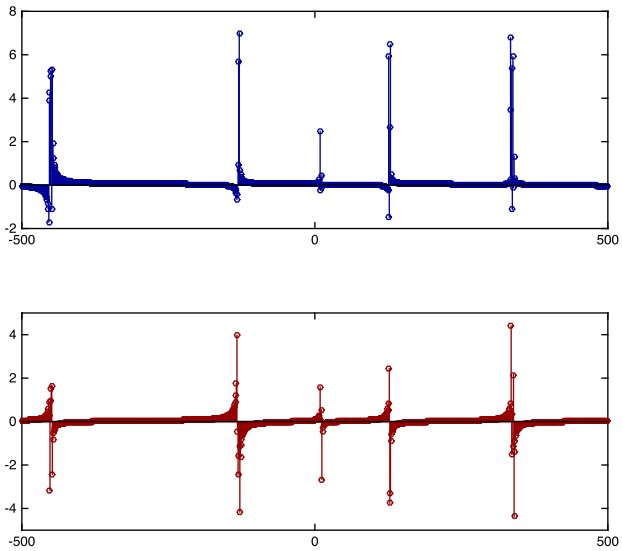

Fig. 1. Real (top) and imaginary (bottom) part of the FT.

circles. The ESPRIT output is indicated using bullets. So ideally every circle should be hit by a bullet. The ill-conditioning has clearly created a serious problem in identifying the individual input frequencies and amplitudes.

Next we choose $r = 11$ and $\rho = 5$. The originally clustered eigenvalues are now much better separated. To illustrate this we show in Fig. 5 the noisefree generalized eigenvalues $\lambda_i^r, i = 1, \ldots, 20$ of the r-fold undersampled exponential analysis problem.

With the noisy samples, we again take $N = 60 > n = 20$ and set up a 120×60 generalized eigenvalue problem (2) with the samples $f_{jr}, j = 0, \ldots, 179$ and the 120×60 Vandermonde system (9) that respectively deliver the λ_i^r and the α_i for $i = 1, \ldots, N$. With the samples $f_{jr+\rho}, j = 0, \ldots, 59$ we set up the 60×60 linear system (10) from which we compute the $\alpha_i \lambda_i^\rho, i = 1, \ldots, 60$ and subsequently the λ_i^ρ. This brings our total number of samples used also to 240, comparable to the ESPRIT procedure. An advantage for ESPRIT is that the signal has less decayed in the first 240 samples, compared to the 240 samples used here. Using

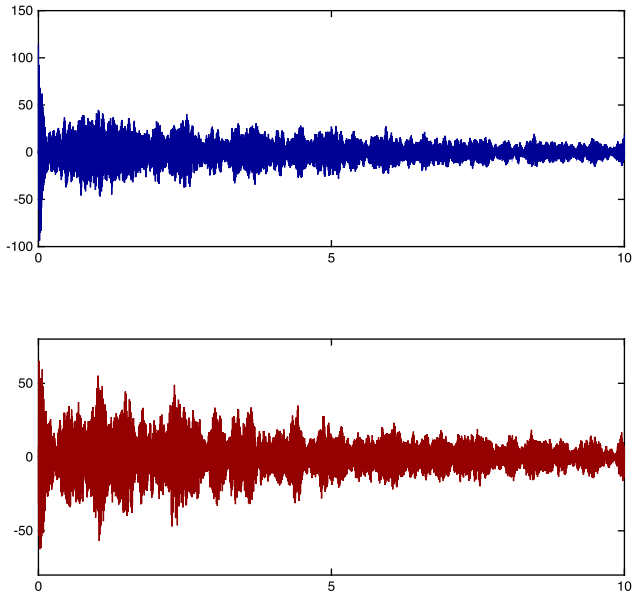

Fig. 2. Real (top) and imaginary (bottom) part of the signal.

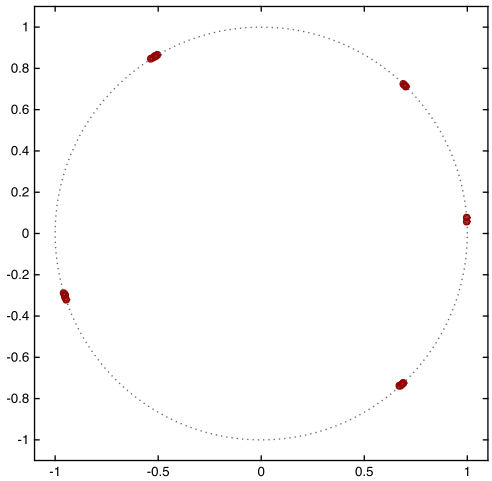

Fig. 3. Generalized eigenvalues λ_i for (1) with data from Table 1.

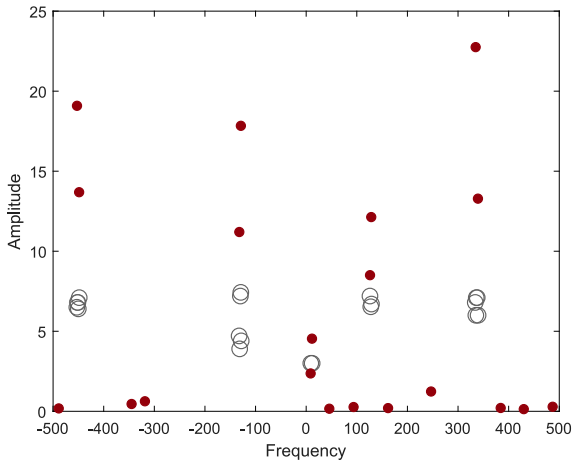

Fig. 4. ESPRIT output $(\omega_i, \beta_i), i = 1, \ldots, 20$ computed from $f_0, \ldots, f_{239}$.

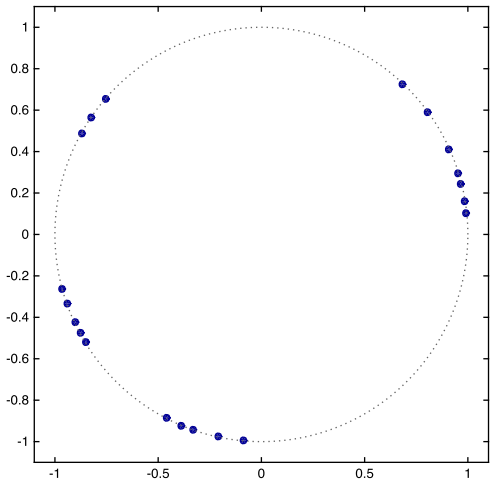

Fig. 5. Generalized eigenvalues λ_i^r for (1) with data from Table 1 and $r = 11$.

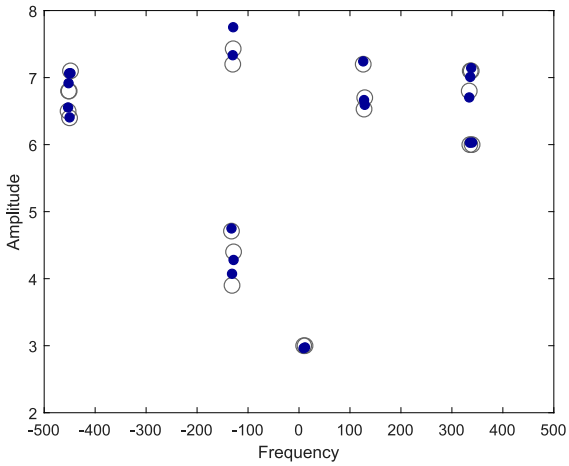

Fig. 6. Output $(\omega_i, \beta_i), i = 1, \ldots, 20$ computed from 240 samples $f_{jr+\rho}, r = 11, \rho = 5$.

the Euclidean algorithm, as explicited in Lemma 2, we recover from λ_i^r and λ_i^ρ the true frequencies ω_i with $p_1 = 1, p_2 = -2$ and $p_1 r + p_2 \rho = 1$. With the new method we find the (ω_i, β_i) couples shown as bullets in Fig. 6. In Fig. 6 the reader can even clearly count the number of frequencies retrieved in each cluster, which is the correct number when comparing to the input values in Table 1. Clearly Fig. 6 is a tremendous improvement over Fig. 4.

5.2. Example where collisions occur without cancellation

In Table 2 we list the α_i and ϕ_i of an exponential model, chosen in such a way that the aliasing causes terms to collide. This enables us to illustrate the workings of the technique explained in Section 4.

The bandwidth is again $\Omega = 1000$ and we take $\Delta = 1/\Omega$ and $r = 100$. We add white Gaussian noise to the samples with SNR = 20 dB and start our computations. When subsampling, the 6 terms collide into 3, as indicated in Fig. 7 by the singular value decomposition of $H_N^{(0)}$ with $N = 30$, which reveals its numerical rank. Actually

$$\phi(t_{rj}) = (\alpha_1 + \alpha_2 + \alpha_3)\exp(\phi_1 jr\Delta) + \alpha_4 \exp(\phi_4 jr\Delta) + (\alpha_5 + \alpha_6)\exp(\phi_5 jr\Delta).$$

We recall that $H_N^{(0)}$ is filled with the samples $f_{jr}, j = 0, \ldots, 59$ and not with the samples $f_j, j = 0, \ldots, 59$.

Table 2
Example where collisions occur.

i	α_i	ϕ_i
1	18	$\mathtt{i}2\pi 191.9$
2	-20	$\mathtt{i}2\pi 291.9$
3	20	$\mathtt{i}2\pi 391.9$
4	5	$\mathtt{i}2\pi 526.2$
5	5	$\mathtt{i}2\pi 858.1$
6	11	$\mathtt{i}2\pi 958.1$

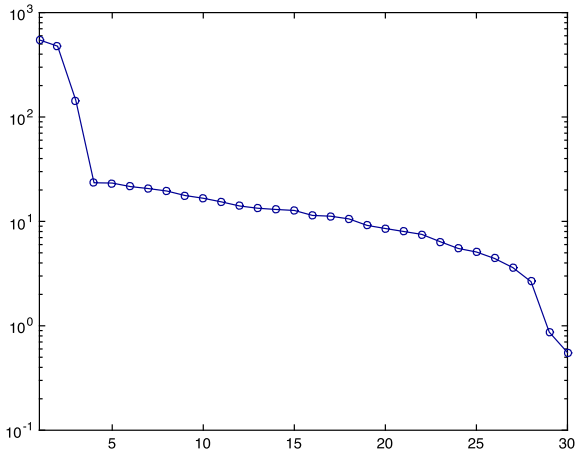

Fig. 7. SVD of $H_{30}^{(0)}$ for (1) with data from Table 2 and $r = 100$.

We set up the 30×30 generalized eigenvalue problem (2) with the samples $f_{jr}, j = 0, \ldots, 59$ which we solve using `oeig`, and the 60×30 Vandermonde system (9) that respectively deliver the λ_i^r and the α_i for $i = 1, \ldots, N$. When retaining the components with largest $|\alpha_i|$, we find

$$\lambda_1^{(0)} \approx 0.36845 + 0.93042\mathtt{i}$$
$$\lambda_2^{(0)} \approx 0.36745 - 0.92977\mathtt{i}$$
$$\lambda_3^{(0)} \approx -0.72761 - 0.68801\mathtt{i}$$

and

$$\begin{aligned} \alpha_1^{(0)} = \alpha_1 + \alpha_2 + \alpha_3 &\approx 17.718 + 0.25273\mathtt{i} \\ \alpha_2^{(0)} = \alpha_4 &\approx 16.126 + 0.057118\mathtt{i} \\ \alpha_3^{(0)} = \alpha_5 + \alpha_6 &\approx 4.5732 - 0.53331\mathtt{i} \end{aligned} \tag{20}$$

At this point we have not yet been able to recover the correct λ_i and α_i for the signal defined by the parameters in Table 2 (we have unearthed only 3 terms instead of 6) because of two reasons. First, the subsampling creates an aliasing effect and second the aliasing causes frequencies to collide. As explained in Section 4, we can disentangle the information in the collisions from more values $\alpha_i^{(1)}(k), k = 1, 2, \ldots$, where $\alpha_i^{(1)}(0) = \alpha_i^{(0)}$, simply because the $\alpha_i^{(1)}(k)$ are themselves linear combinations of exponentials. To not complicate matters too much yet, the example is cancellation free: so the correct value $n_0 = 3$ is immediately discovered from the sampling at the multiples of $r\Delta$, as we see in (20).

For the disentanglement, we choose $\rho = 133$ and we set up the Vandermonde systems (17),

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1^{(0)} & \lambda_2^{(0)} & \lambda_3^{(0)} \\ \vdots & \vdots & \vdots \\ (\lambda_1^{(0)})^9 & (\lambda_2^{(0)})^9 & (\lambda_3^{(0)})^9 \end{pmatrix} \begin{pmatrix} \alpha_1^{(1)}(k) \\ \alpha_2^{(1)}(k) \\ \alpha_3^{(1)}(k) \end{pmatrix} = \begin{pmatrix} f_{k\rho} \\ f_{r+k\rho} \\ \vdots \\ f_{9r+k\rho} \end{pmatrix}, \qquad k = 1, \ldots, 11.$$

In total so far 170 samples are used. A singular value analysis of the Hankel matrices

$$\begin{pmatrix} \alpha_i^{(1)}(0) & \alpha_i^{(1)}(1) & \ldots & \alpha_i^{(1)}(5) \\ \alpha_i^{(1)}(1) & \alpha_i^{(1)}(2) & \ldots & \alpha_i^{(1)}(6) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_i^{(1)}(5) & \alpha_i^{(1)}(6) & \ldots & \alpha_i^{(1)}(10) \end{pmatrix}, \qquad i = 1, 2, 3$$

reveals the number of components that one can distinguish in and consequently extract from the $\alpha_i^{(1)}(k)$. The numbers are respectively 3, 2, 1 for $i = 1, 2, 3$ and so $h_1 = 1, h_2 = 4, h_3 = 6, h_4 = 7$. The size of these Hankel matrices filled with values $\alpha_i^{(1)}(k)$, is chosen somewhat larger than necessary so that the correctness of their rank is confirmed a number of times. We can also conclude that

$$n = \sum_{i=1}^{n_0} (h_{i+1} - h_i) = 6.$$

For $i = 1, 2, 3$ the generalized eigenvalue problems

$$\begin{pmatrix} \alpha_i^{(1)}(1) & \ldots & \alpha_i^{(1)}(6) \\ \vdots & \ddots & \vdots \\ \alpha_i^{(1)}(6) & \ldots & \alpha_i^{(1)}(11) \end{pmatrix} v_\ell = \lambda_\ell^{(1)} \begin{pmatrix} \alpha_i^{(1)}(0) & \ldots & \alpha_i^{(1)}(5) \\ \vdots & \ddots & \vdots \\ \alpha_i^{(1)}(5) & \ldots & \alpha_i^{(1)}(10) \end{pmatrix} v_\ell$$

reveal the $\lambda_\ell^{(1)} = \exp(\phi_\ell \rho \Delta), \ell = h_i, \ldots, h_{i+1} - 1, i = 1, \ldots, n_0$. Note that we chose a notation where the $\lambda_\ell^{(1)}$ are not indexed by a double index $(\ell, i), \ell = 1, \ldots, h_{i+1} - h_i, i = 1, \ldots, n_0$ but are indexed consecutively from $\ell = h_1 = 1$ to $\ell = h_{n_0+1} - 1 = n$. This matches the indexing of the $\lambda_\ell^{(0)}$ of which some are coalescent, namely $\lambda_{h_i}^{(0)} = \cdots = \lambda_{h_{i+1}-1}^{(0)}, i = 1, \ldots, n_0$. The respective Vandermonde systems with unknowns $\alpha_{h_i}, \ldots, \alpha_{h_{i+1}-1}$ and right hand sides $\alpha_i^{(1)}(0)$, …, $\alpha_i^{(1)}(11)$ reveal the $\alpha_\ell, \ell = h_i, \ldots, h_{i+1} - 1$ in (16). Again we retain only the $h_{i+1} - h_i$ components with largest $|\alpha_\ell|$. From the $\lambda_\ell^{(0)} = \exp(\phi_\ell r \Delta), \ell = h_i, \ldots, h_{i+1} - 1, i = 1, \ldots, n_0$ and $\lambda_\ell^{(1)} = \exp(\phi_\ell \rho \Delta), \ell = 1, \ldots, n$ the imaginary part of ϕ_i can be recovered as indicated in Lemma 2: with $p_1 = 4$ and $p_2 = -3$ we have $p_1 r + p_2 \rho = 1$ and so

$$\Im(\phi_\ell) = 4 \operatorname{Arg}(\lambda_\ell^{(0)})\Omega - 3 \operatorname{Arg}(\lambda_\ell^{(1)})\Omega + 2\pi h \Omega, \qquad 1 \le \ell \le 6, \qquad h \in \mathbb{Z},$$

where h is taken such that $0 \le \Im(\phi_\ell) < 2\pi\Omega$. Eventually we unearth the following 6 ϕ_i and α_i:

$$\begin{aligned} \phi_1 &\approx -0.021600 + \mathtt{i}2\pi 192.29, \\ \phi_2 &\approx -0.0085122 + \mathtt{i}2\pi 289.87, \\ \phi_3 &\approx -0.025728 + \mathtt{i}2\pi 386.69, \\ \phi_4 &\approx -0.066292 + \mathtt{i}2\pi 538.18, \\ \phi_5 &\approx -0.043745 + \mathtt{i}2\pi 858.70, \\ \phi_6 &\approx 0.0026126 + \mathtt{i}2\pi 956.23 \end{aligned}$$

and

$$\begin{aligned}
\alpha_1 &\approx 19.011 + \mathtt{i}0.53818,\\
\alpha_2 &\approx -20.481 + \mathtt{i}0.89352,\\
\alpha_3 &\approx 21.445 - \mathtt{i}1.5790,\\
\alpha_4 &\approx 5.8439 - \mathtt{i}0.035907,\\
\alpha_5 &\approx 5.0770 + \mathtt{i}0.037562,\\
\alpha_6 &\approx 10.758 - \mathtt{i}0.40878.
\end{aligned}$$

5.3. Example with cancellations in the collisions

The cancellation strategy is most clearly illustrated by means of a noisefree example, where exact cancellations are observed. The actual occurrence of this situation in case of real-life data is extremely small, but we primarily want to show that the proposed sub-Nyquist method is capable of recovering from it.

Let

$$\begin{aligned}\phi(t) = \exp(2\pi\mathtt{i}t) - \exp(2\pi\mathtt{i}21t) + \exp(2\pi\mathtt{i}41t) - \exp(2\pi\mathtt{i}61t) + \\ e^{\mathtt{i}2\pi 72/100}\exp(2\pi\mathtt{i}11t) - e^{\mathtt{i}2\pi 32/100}\exp(2\pi\mathtt{i}31t) + \exp(2\pi\mathtt{i}9t).\end{aligned} \tag{21}$$

We take $\Omega = 100, \Delta = 0.01$ and sample $f_j = \phi(j\Delta)$ for particular values of j. With $r = 5$ the first four terms cancel each other and the fifth and sixth term collide:

$$f_{5j} = 0\exp(2\pi\mathtt{i}5j/100) + \left(e^{\mathtt{i}2\pi 72/100} - e^{\mathtt{i}2\pi 32/100}\right)\exp(2\pi\mathtt{i}55j/100) + \exp(2\pi\mathtt{i}45j/100). \tag{22}$$

So from the samples $f_0, f_5, f_{10}, \ldots$ only two terms can be retrieved:

$$\operatorname{rank} H_3^{(0)} = \operatorname{rank}\begin{pmatrix} f_0 & f_5 & f_{10}\\ f_5 & f_{10} & f_{15}\\ f_{10} & f_{15} & f_{20}\end{pmatrix} = 2$$

and rank $H_N^{(0)} = 2, N \geq 2$. The 2 eigenvalues that we can already compute, are $\lambda_5^{(0)} = \exp(2\pi\mathtt{i}55/100)$ and $\lambda_7^{(0)} = \exp(2\pi\mathtt{i}45/100)$, satisfying

$$\begin{pmatrix} f_5 & f_{10}\\ f_{10} & f_{15}\end{pmatrix} v = \lambda \begin{pmatrix} f_0 & f_5\\ f_5 & f_{10}\end{pmatrix} v.$$

From the Vandermonde system

$$\begin{pmatrix} 1 & 1\\ \lambda_5^{(0)} & \lambda_7^{(0)}\end{pmatrix}\begin{pmatrix} \alpha_5^{(0)}\\ \alpha_7^{(0)}\end{pmatrix} = \begin{pmatrix} f_0\\ f_5\end{pmatrix}$$

we find $\alpha_5^{(0)} = e^{\mathtt{i}2\pi 72/100} - e^{\mathtt{i}2\pi 32/100}$ and $\alpha_7^{(0)} = 1$. We now need to ask ourselves whether n_0 truly equals 2 or whether some cancellation of terms has happened. With $\rho = 12$ we find that

$$\begin{aligned}f_{5j+12k} = \left(e^{\mathtt{i}2\pi 12k/100} - e^{\mathtt{i}2\pi 52k/100} + e^{\mathtt{i}2\pi 92k/100} - e^{\mathtt{i}2\pi 32k/100}\right)\exp(2\pi\mathtt{i}5j/100) + \\ \left(e^{\mathtt{i}2\pi 72/100}e^{\mathtt{i}2\pi 32k/100} - e^{\mathtt{i}2\pi 32/100}e^{\mathtt{i}2\pi 72k/100}\right)\exp(2\pi\mathtt{i}55j/100) + e^{\mathtt{i}2\pi 8k/100}\exp(2\pi\mathtt{i}45j/100).\end{aligned} \tag{23}$$

For $k = 1$ we hit an accidental zero for the coefficient of $\exp(2\pi \mathtt{i} 55j/100)$ and therefore

$$\text{rank } H_3^{(12)} = \text{rank} \begin{pmatrix} f_{12} & f_{17} & f_{22} \\ f_{17} & f_{22} & f_{27} \\ f_{22} & f_{27} & f_{32} \end{pmatrix} = 2$$

again, with rank $H_N^{(12)} = 2, N \geq 2$. The generalized eigenvalues satisfying

$$\begin{pmatrix} f_{17} & f_{22} \\ f_{22} & f_{27} \end{pmatrix} v = \lambda \begin{pmatrix} f_{12} & f_{17} \\ f_{17} & f_{22} \end{pmatrix} v,$$

namely $\lambda_1^{(0)} = \exp(2\pi \mathtt{i} 5/100)$ and $\lambda_7^{(0)} = \exp(2\pi \mathtt{i} 45/100)$, also belong to the n_0 eigenvalues that are identifiable from the evaluations at the multiples of $r\Delta$, since a shift does not change the generalized eigenvalues, only their coefficients. From the Vandermonde system

$$\begin{pmatrix} 1 & 1 \\ \lambda_1^{(0)} & \lambda_7^{(0)} \end{pmatrix} \begin{pmatrix} \alpha_1^{(1)}(1) \\ \alpha_7^{(1)}(1) \end{pmatrix} = \begin{pmatrix} f_{12} \\ f_{17} \end{pmatrix}$$

we find $\alpha_1^{(1)}(1)$ and $\alpha_7^{(1)}(1)$. Apparently n_0 equals at least 3, because in the first bunch computed from the samples f_{5j+12k} with $k = 0$ we find two eigenvalues and in the second bunch with $k = 1$ we find one more. Bringing these results together results in the intermediate estimates

$$\begin{aligned} h_1 = 1 &: \alpha_{h_1}^{(1)}(0) = 0, \alpha_{h_1}^{(1)}(1) = e^{\mathtt{i}2\pi 12/100} - e^{\mathtt{i}2\pi 52/100} + e^{\mathtt{i}2\pi 92/100} - e^{\mathtt{i}2\pi 32/100}, \\ h_2 = 5 &: \alpha_{h_2}^{(1)}(0) = e^{\mathtt{i}2\pi 72/100} - e^{\mathtt{i}2\pi 32/100}, \alpha_{h_2}^{(1)}(1) = 0, \\ h_3 = 7 &: \alpha_{h_3}^{(1)}(0) = 1, \alpha_{h_3}^{(1)}(1) = e^{\mathtt{i}2\pi 8/100}, \end{aligned}$$

where the values h_i are merely mentioned as a guideline and are not explicitly computed. Remember that in real-life experiments the indices h_i are not known and need not be known. They are revealed as the algorithm progresses.

Let us turn our attention to larger values of k to have the current estimate $n_0 = 3$ confirmed and to extract all n distinct terms. As described in Section 4 on the disentangling of collisions, we continue sampling at multiples of the shift, namely we collect the $f_{jr+k\rho}$ for $k > 1$. With $k = 2$ we obtain

$$\begin{aligned} f_{5j+24} = &\left(e^{\mathtt{i}2\pi 24/100} - e^{\mathtt{i}2\pi 4/100} + e^{\mathtt{i}2\pi 84/100} - e^{\mathtt{i}2\pi 64/100}\right) \exp(2\pi \mathtt{i} 5j/100) + \\ &\left(e^{\mathtt{i}2\pi 36/100} - e^{\mathtt{i}2\pi 76/100}\right) \exp(2\pi \mathtt{i} 55j/100) + e^{\mathtt{i}2\pi 16/100} \exp(2\pi \mathtt{i} 45j/100) \end{aligned} \tag{24}$$

and

$$\text{rank } H_4^{(24)} = \text{rank} \begin{pmatrix} f_{24} & f_{29} & f_{34} & f_{39} \\ \vdots & \iddots & & \vdots \\ f_{39} & \cdots & & f_{54} \end{pmatrix} = 3$$

with rank $H_N^{(24)} = 3, N \geq 4$. Merely for completeness we compute the generalized eigenvalues satisfying

$$\begin{pmatrix} f_{29} & \cdots & f_{39} \\ \vdots & \iddots & \vdots \\ f_{39} & \cdots & f_{49} \end{pmatrix} v = \lambda \begin{pmatrix} f_{24} & \cdots & f_{34} \\ \vdots & \iddots & \vdots \\ f_{34} & \cdots & f_{44} \end{pmatrix} v.$$

We find $\lambda_1^{(0)} = \exp(2\pi \mathtt{i}5/100), \lambda_5^{(0)} = \exp(2\pi \mathtt{i}55/100), \lambda_7^{(0)} = \exp(2\pi \mathtt{i}45/100)$, which confirms our earlier obtained combined result. Hence $n_0 = 3$. We also compute the values for $\alpha_i^{(1)}(2), i = 1, 5, 7$ from the Vandermonde system

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1^{(0)} & \lambda_5^{(0)} & \lambda_7^{(0)} \\ (\lambda_1^{(0)})^2 & (\lambda_5^{(0)})^2 & (\lambda_7^{(0)})^2 \end{pmatrix} \begin{pmatrix} \alpha_1^{(1)}(2) \\ \alpha_5^{(1)}(2) \\ \alpha_7^{(1)}(2) \end{pmatrix} = \begin{pmatrix} f_{24} \\ f_{29} \\ f_{34} \end{pmatrix}.$$

The purpose now is to find out how many terms are in the expressions $\alpha_i^{(1)}(k)$ for each i retrieved so far. We compute $\alpha_1^{(1)}(k), \alpha_5^{(1)}(k), \alpha_7^{(1)}(k)$ for $k \geq 3$ from

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1^{(0)} & \lambda_5^{(0)} & \lambda_7^{(0)} \\ (\lambda_1^{(0)})^2 & (\lambda_5^{(0)})^2 & (\lambda_7^{(0)})^2 \end{pmatrix} \begin{pmatrix} \alpha_1^{(1)}(k) \\ \alpha_5^{(1)}(k) \\ \alpha_7^{(1)}(k) \end{pmatrix} = \begin{pmatrix} f_{12k} \\ f_{5+12k} \\ f_{10+12k} \end{pmatrix}, \qquad k = 3, 4, \ldots,$$

which reuses the $n_0 \times n_0$ Vandermonde coefficient matrix from above.

Let us write $k = 2\kappa - 2$, so that when we increase κ by 1 then k is increased by 2. We check the rank of the $\kappa \times \kappa$ matrices

$$\begin{pmatrix} \alpha_i^{(1)}(0) & \ldots & \alpha_i^{(1)}(\kappa-1) \\ \vdots & \iddots & \vdots \\ \alpha_i^{(1)}(\kappa-1) & \ldots & \alpha_i^{(1)}(2\kappa-2) \end{pmatrix}, \qquad i = 1, 5, 7.$$

When pursuing the shifts up to $k = 9$, meaning $\kappa = 5$, we find that for $i = 1$ the rank is 4, for $i = 5$ the rank is 2 and for $i = 7$ the rank is 1, leading to a grand total of $n = 7$ distinct terms. We now separate the terms that are hiding in each of the collisions by computing the generalized eigenvalues satisfying

$$\begin{pmatrix} \alpha_i^{(1)}(1) & \ldots & \alpha_i^{(1)}(\kappa) \\ \vdots & \iddots & \vdots \\ \alpha_i^{(1)}(\kappa) & \ldots & \alpha_i^{(1)}(2\kappa-1) \end{pmatrix} v = \lambda \begin{pmatrix} \alpha_i^{(1)}(0) & \ldots & \alpha_i^{(1)}(\kappa-1) \\ \vdots & \iddots & \vdots \\ \alpha_i^{(1)}(\kappa-1) & \ldots & \alpha_i^{(1)}(2\kappa-2) \end{pmatrix} v, \qquad i = 1, 5, 7. \tag{25}$$

We find

$$\begin{aligned} i = 1, \kappa = 4 : \lambda_1^{(1)} &= \exp(2\pi \mathtt{i}12/100), \lambda_2^{(1)} = \exp(2\pi \mathtt{i}52/100), \\ \lambda_3^{(1)} &= \exp(2\pi \mathtt{i}92/100), \lambda_4^{(1)} = \exp(2\pi \mathtt{i}32/100), \\ i = 5, \kappa = 2 : \lambda_5^{(1)} &= \exp(2\pi \mathtt{i}32/100), \lambda_6^{(1)} = \exp(2\pi \mathtt{i}72/100), \\ i = 7, \kappa = 1 : \lambda_7^{(1)} &= \exp(2\pi \mathtt{i}8/100). \end{aligned}$$

At this stage we have all the information to reconstruct the non-aliased generalized eigenvalues:

$$\begin{aligned} \lambda_1^{(0)}, \lambda_1^{(1)} &\to \lambda_1 = \exp(2\pi \mathtt{i}1/100), \\ \lambda_1^{(0)}, \lambda_2^{(1)} &\to \lambda_2 = \exp(2\pi \mathtt{i}21/100), \\ \lambda_1^{(0)}, \lambda_3^{(1)} &\to \lambda_3 = \exp(2\pi \mathtt{i}41/100), \\ \lambda_1^{(0)}, \lambda_4^{(1)} &\to \lambda_4 = \exp(2\pi \mathtt{i}61/100), \end{aligned}$$

$$\lambda_5^{(0)}, \lambda_5^{(1)} \to \lambda_5 = \exp(2\pi \mathtt{i} 11/100),$$
$$\lambda_5^{(0)}, \lambda_6^{(1)} \to \lambda_6 = \exp(2\pi \mathtt{i} 31/100),$$
$$\lambda_7^{(0)}, \lambda_7^{(1)} \to \lambda_7 = \exp(2\pi \mathtt{i} 9/100).$$

Remains to compute the individual linear coefficients of each of the 7 terms. We compute $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ from

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_1^{(1)} & \lambda_2^{(1)} & \lambda_3^{(1)} & \lambda_4^{(1)} \\ (\lambda_1^{(1)})^2 & (\lambda_2^{(1)})^2 & (\lambda_3^{(1)})^2 & (\lambda_4^{(1)})^2 \\ (\lambda_1^{(1)})^3 & (\lambda_2^{(1)})^3 & (\lambda_3^{(1)})^3 & (\lambda_4^{(1)})^3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \begin{pmatrix} \alpha_1^{(1)}(0) \\ \alpha_1^{(1)}(1) \\ \alpha_1^{(1)}(2) \\ \alpha_1^{(1)}(3) \end{pmatrix},$$

the coefficients α_5 and α_6 from

$$\begin{pmatrix} 1 & 1 \\ \lambda_5^{(1)} & \lambda_6^{(1)} \end{pmatrix} \begin{pmatrix} \alpha_5 \\ \alpha_6 \end{pmatrix} \begin{pmatrix} \alpha_5^{(1)}(0) \\ \alpha_5^{(1)}(1) \end{pmatrix}.$$

The coefficient α_7 is given by $\alpha_7 = \alpha_7^{(0)} = \alpha_7^{(1)}(0)$ because there were no collisions in that term.

5.4. Full algorithm in pseudocode

An algorithm covering the eventuality of the above scenarios reads as follows. We assume that $r > 1$ otherwise a classical Prony analysis applies.

So far we used the notation n for the number of exponential terms in the signal, which we often don't know up front. Moreover, the data are usually noisy, so that it is best to add another number of terms in order to model the noise. We denoted the latter in the previous sections by $N - n$ so that the total number of terms we want to identify accumulates to N. To this end at least $2N$ samples are required, even without breaking the Shannon–Nyquist rate. We denote the number of samples collected at the uniformly distributed points t_{jr} by the number $M \geq 2N$. These allow us to build the square Hankel matrices $H_N^{(0)}$ and $H_N^{(1)}$ or somewhat larger rectangular $(M - N) \times N$ versions of these matrices. When sampling at the shifted locations $t_{jr+k\rho}$ we collect for each k not M but m samples where $\lfloor m/2 \rfloor > n$. Using the latter we can build the Hankel matrices $H_{\lfloor m/2 \rfloor}^{(\rho)}$. Often the total number of samples and the amount of undersampling are dictated by the circumstances and the constraints under which the analysis is performed.

We emphasize that n_0 indicates the number of terms in the exponential sum after possible collisions, including the vanished ones due to cancellation in the coefficients. Also, the time step $\Delta \in \mathbb{R}$ satisfies $\Delta \leq 1/\Omega$. With this in mind the algorithm continues as follows.

Algorithm.

Input bounds on n, subsampling factor r and shift term ρ:

- $M, N, m \in \mathbb{N}$ with $M \geq 2N, N \geq n, \lfloor m/2 \rfloor > n$.
- $r \in \mathbb{N}, \rho \in \mathbb{Z}$ with $r > 1$ and $\gcd(r, \rho) = 1$.

A0. Obtain $n_0, \lambda_i^{(0)}, \alpha_i^{(0)}$:

- Collect the samples $f_{jr} = \phi(t_{jr}), j = 0, \ldots, M - 1$ and estimate $n_0 \leq n$ by the numerical rank of the matrix $H_N^{(0)}$.

- For one or more $1 \le k \le 2n-1$ collect the samples $f_{jr+k\rho} = \phi(t_{jr+k\rho}), j = 0, \ldots, m-1$ and compute the numerical rank n_k of $H^{(k\rho)}_{\lfloor m/2 \rfloor}$.
- From these different views on the number of collided terms in the exponential sum, we find that the correct value for n_0 is $n_0 = \max_k n_k$.
- Compute for $i = 1, \ldots, n_0$ the generalized eigenvalues $\lambda_i^{(0)}$ and the coefficients $\alpha_i^{(0)}$ as in the example of Section 5.3.
- Either $N \times N$ Hankel and $2N \times N$ Vandermonde systems are used or their least squares $(M-N) \times N$ and $M \times N$ versions.

A1. Obtain $\alpha_i^{(1)}(k)$ and $h_{i+1} - h_i$ for $i = 1, \ldots, n_0$ and $1 \le k \le K < 2n$:
Put $\alpha_i^{(1)}(0) := \alpha_i^{(0)}, k = 1$ and execute the for loop:

1. compute the $\alpha_i^{(1)}(k)$ from (17) or its $m \times n_0$ least squares version,
2. collect or reuse the samples $f_{jr+(k+1)\rho} = \phi(t_{jr+(k+1)\rho}), j = 0, \ldots, m-1$,
3. compute the $\alpha_i^{(1)}(k+1)$ from (17) or its $m \times n_0$ least squares version,
4. compute the numerical rank $\nu_i(\kappa)$ of the $(\kappa+1) \times (\kappa+1)$ matrix

$$\begin{bmatrix} \alpha_i^{(1)}(0) & \cdots & \alpha_i^{(1)}(\kappa) \\ \vdots & & \vdots \\ \alpha_i^{(1)}(\kappa) & \cdots & \alpha_i^{(1)}(k+1) \end{bmatrix}, \qquad 2\kappa = k+1,$$

5. if $\nu_i(\kappa) = \nu_i(\kappa - 1)$:
   - then $h_{i+1} - h_i = \kappa$,
   - else $k := k+2$, collect or reuse the samples $f_{jr+k\rho}, j = 0, \ldots, m-1$ and goto 1.
6. compute the generalized eigenvalues $\lambda_\ell^{(1)}, \ell = h_i, \ldots, h_{i+1} - 1$ in (18),
7. compute the $\alpha_\ell, \ell = h_i, \ldots, h_{i+1} - 1$ from (16).

End the for loop.

Output number of terms n and the parameters ϕ_i, α_i:
From

- $\lambda_\ell^{(1)}, \ell = h_1, \ldots, h_{n_0+1} - 1$ with $h_1 = 1, h_{n_0+1} - 1 = n$
- and $\lambda_\ell^{(0)}$ with $\lambda_{h_i}^{(0)} = \cdots = \lambda_{h_{i+1}-1}^{(0)}, i = 1, \ldots, n_0$

the $\phi_i, i = 1, \ldots, n$ can be recovered.

The $\alpha_i, i = 1, \ldots, n$ are computed from (16) as in (19).

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