JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 67 (1996) 141–146

Nuttall–Pommerenke theorems for homogeneous Padé approximants

A. Cuyt[a], K. Driver[b], D.S. Lubinsky[b,*]

[a] *Department of Mathematics, UIA, University of Antwerp. Universiteitsplein 1, B-2610 Wilrijk, Belgium*
[b] *Department of Mathematics, Witwatersrand University, P.O. Wits 2050, South Africa*

Received 30 August 1994

Abstract

We investigate Nuttall–Pommerenke theorems for several variable homogeneous Padé approximants using ideas of Goncar, Karlsson and Wallin.

Keywords: Multivariate Padé approximants; Homogeneous approximants

1. Nuttall–Pommerenke theorems for homogeneous Padé approximants

We begin by recalling the definition of homogeneous Padé approximants. Let $f(z)$ denote a power series of k variables $z_1, z_2, \ldots, z_k$ convergent in a neighbourhood of $\mathbf{0}$. We can rearrange the Maclaurin series of f into a homogeneous expansion

$$f(z) = \sum_{j=0}^{\infty} C_j(z), \tag{1}$$

where $C_j(z)$ is a *homogeneous polynomial* of degree j, that is, is a sum of terms of the form $cz_1^{j_1} z_2^{j_2} \cdots z_k^{j_k}$, with $j_1 + j_2 + \cdots + j_k = j$. The homogeneous Padé approximant of type (m,n), denoted $[m/n] = P/Q$, is a rational function of $z = (z_1, z_2, \ldots, z_k)$ such that

$$f(z)Q(z) - P(z) = \sum_{j=mn+m+n+1}^{\infty} D_j(z), \tag{2}$$

where

$$P(z) = \sum_{j=0}^{m} A_{j+mn}(z); \qquad Q(z) = \sum_{j=0}^{n} B_{j+mn}(z) \tag{3}$$

* Corresponding author. E-mail: 036dsl@cosmos.wits.ac.za.

SSDI 0377-0427(94)00115-4

and each A_j, B_j, D_j is a homogeneous polynomial of degree j. At first sight, the form of P, Q and the remainder $fQ - P$ may seem strange, but these are the natural forms for homogeneous Padé approximants. See [5–7] for further orientation.

In [10, 11] Nuttall–Pommerenke theorems were established for nonhomogeneous Padé approximants, but the results of Goncar, Karlsson and Wallin do not formally cover our approximants. We can use several of their ideas, once we take note of the following crucial *projection property* of $[m/n]$: On each complex line through the origin in $\mathbb{C}^k$, $[m/n]$ is an ordinary one-variable Padé approximant to the projection of f onto that line. To be more precise, let

$$\boldsymbol{\lambda} := (\lambda_2, \lambda_3, \ldots, \lambda_k) \in \mathbb{C}^{k-1} \tag{4}$$

and

$$f_{\boldsymbol{\lambda}}(z) := f(z, \boldsymbol{\lambda} z), \quad z \in \mathbb{C}. \tag{5}$$

Then [5]

$$[m/n]_{f_{\boldsymbol{\lambda}}}(z) = [m/n]_f(z, \boldsymbol{\lambda} z) \tag{6}$$

(the subscript indicates the function from which the approximant was formed). However, despite this property, the fact that $[m/n](\boldsymbol{z})$ is a rational function whose numerator and denominator can have degree $mn + \max\{m, n\} + 1$ creates difficulties. The main contribution of this paper is to partially circumvent them.

In order to state our results, we need more notation. Throughout m_{2k} denotes Lebesgue measure on $\mathbb{C}^{2k}$. We say that a rational function $r(\boldsymbol{z})$ of k variables $\boldsymbol{z} = (z_1, z_2, \ldots, z_k)$ is of *order* n if its numerator and denominator are polynomials of degree $\leqslant n$ in each variable $z_1, z_2, \ldots, z_k$. The norm $\|\boldsymbol{z}\|$ of $\boldsymbol{z}$ is the usual Euclidean norm. For compact $K \subset \mathbb{C}^k$, and $f: K \to \mathbb{C}$, we let

$$E_n(f; K) := \inf\{\, \|f - r\|_{L_\infty(K)} \colon r \text{ of order } n\} \tag{7}$$

denote the *error of best rational approximation of* f *by rational functions of order* n *on* K.

Let $U \subset \mathbb{C}^k$ be open and connected and $f: U \to \mathbb{C}$ be analytic. We say that f belongs to the *Goncar–Walsh class* on U, and write $f \in R_0(U)$, if, for each compact set $K \subset U$,

$$\lim_{n \to \infty} E_n(f; K)^{1/n} = 0. \tag{8}$$

Cirka [4] has shown that if f is analytic in $\mathbb{C}^k$ outside an analytic set A (that is, a set of the form $A = \{\boldsymbol{z} \colon g(\boldsymbol{z}) = 0\}$ for some entire function g), then $f \in R_0(\mathbb{C}^k \backslash A)$.

It is implicit in the work of Goncar (see the footnote on p. 306 of [10]), that if U is the Weierstrass domain of analytic continuation of f, and (8) holds when K is some closed ball within U, then (8) holds for each compact $K \subset U$. This is well known in the case of one dimension [9], but less obvious in several dimensions, so we avoid this fine point.

In fact, in [10], Goncar's definition of R_0 is a little different. He defined $f \in R_0$ if there exist rational functions r_n of order n, for large enough n, and a ball B such that for each $\varepsilon > 0$,

$$m_{2k}\{\boldsymbol{z} \in B \colon |f - r_n|(\boldsymbol{z}) \geqslant \varepsilon^n\} \to 0, \quad n \to \infty.$$

He showed that if $f_{\boldsymbol{\lambda}}$ belongs to R_0 (as a function of one variable) for a.e. $\boldsymbol{\lambda} \in \mathbb{C}^{k-1}$, then $f \in R_0$. Moreover, if $f \in R_0$, then f is single-valued in its Weierstrass domain of analytic continuation. We emphasise again that our apparently more restrictive definition of R_0 is equivalent to Goncar's, but is adopted in order to simplify proofs.

Following is our Nuttall–Pommerenke theorem:

Theorem 1.1. *Let U be an open connected subset of $\mathbb{C}^k$ containing $\mathbf{0}$. Let $f \in R_0(U)$. Let $\{m_j\}$ and $\{n_j\}$ be sequences of positive integers with*

$$\lim_{j\to\infty} m_j = \infty; \tag{9}$$

and for some $\eta > 1$,

$$\frac{1}{\eta} \leqslant \frac{m_j}{n_j} \leqslant \eta, \quad j \geqslant 1. \tag{10}$$

Then given $r > 0$, $\varepsilon > 0$,

$$m_2\{z \in U\colon \|z\| \leqslant r \text{ and } |f - [m_j/n_j]|(z) > \varepsilon^{m_j+n_j}\} \to 0, \quad j \to \infty. \tag{11}$$

It should be possible to replace $2k$-dimensional Lebesgue measure by product capacity or Favarov's capacity or the general capacities [3, 16] on $\mathbb{C}^k$, but we have avoided this because of the difficulty of moving from estimates on the exceptional sets along lines to estimates on the exceptional sets on $\mathbb{C}^k$ itself.

2. Proof of the result

We record a lemma from [8]:

Lemma 2.1. *Let P be a (one-variable) polynomial of degree $\leqslant n$, normalized by the condition*

$$\|P\|_{L_\infty(|z| \leqslant r)} = 1. \tag{12}$$

Then

$$m_{2k}(\{z\colon |z| \leqslant r \text{ and } |P(z)| \leqslant \varepsilon^n\}) \leqslant 4\pi r^2 \varepsilon^2. \tag{13}$$

Proof. See [8]. There it is also shown that the estimate (13) and its cousin for capacity are sharp. A weaker estimate, which is still sufficient for our purposes, is given in [14, p. 777]. Related estimates appear in [1, 2, 12, 13, 15]. □

We proceed to the proof of Theorem 1.1. We break it into several steps:

Step 1: *Replace U in* (11). We can choose finitely many closed balls contained in U, with union K say, such that $m_{2k}(U\backslash K)$ is arbitrarily small. So we may prove (11) with U replaced by K. We can choose slightly larger concentric open balls, whose union is a set V, say. We may assume that $\bar{V} \subset U$, that $\mathbf{0} \in V$, and that f is analytic in $\bar{V}$. Let δ denote the distance from $\mathbb{C}\backslash V$ to K.

Step 2: Convergence along a slice. For fixed $\boldsymbol{\lambda} \in \mathbb{C}^{k-1}$, let

$$f_\lambda(z) := f(z, \boldsymbol{\lambda} z),$$

so that

$$[m_j/n_j]_f(z, \boldsymbol{\lambda} z) = [m_j/n_j]_{f_\lambda}(z) =: P_j/Q_j(z),$$

say. Let

$$K_\lambda := \{z\colon (z, \boldsymbol{\lambda} z) \in K\}; \qquad V_\lambda := \{z\colon (z, \boldsymbol{\lambda} z) \in V\}.$$

Then K_λ is a compact subset of open V_λ, and the boundary ∂V_λ has distance at least $\delta/\|(1, \boldsymbol{\lambda})\|$ to K_λ. Let

$$\sigma_j := \min\{m_j, n_j\}, \quad j \geqslant 1.$$

Choose rational functions $r_j(z)$ of order σ_j such that

$$\|f - r_j\|_{L_\infty(\bar{V})} = E_{\sigma_j}(f; \bar{V}).$$

By our hypotheses on f and $\{m_j\}$, $\{n_j\}$,

$$E_{\sigma_j}(f; \bar{V})^{1/(m_j+n_j)} \to 0, \quad j \to \infty. \tag{14}$$

Let

$$r_j^*(z) := r_j(z, \boldsymbol{\lambda} z) = p_j^*(z)/q_j^*(z),$$

say. Cauchy's integral formula gives for $z \in K_\lambda$,

$$\begin{aligned} \frac{q_j^*(fQ_j - P_j)(z)}{z^{m_j+n_j+1}} &= \frac{1}{2\pi \mathrm{i}} \int_{\partial V_\lambda} \frac{q_j^*(fQ_j - P_j)(t)}{t^{m_j+n_j+1}} \frac{\mathrm{d}t}{t-z} \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\partial V_\lambda} \frac{Q_j(fq_j^* - p_j^*)(t)}{t^{m_j+n_j+1}} \frac{\mathrm{d}t}{t-z}. \end{aligned}$$

Here we have used that $p_j^* Q_j - P_j q_j^*$ *has* degree at most $m_j + n_j$, so the part of the integral involving it is identically 0. Thus

$$(f_\lambda - [m_j/n_j]_{f_\lambda})(z) = \frac{1}{2\pi i} \int_{\partial V_\lambda} \frac{(q_j^* Q_j)(t)}{(q_j^* Q_j)(z)} \frac{(f - r_j^*)(t)}{t - z} \left(\frac{z}{t}\right)^{m_j+n_j+1} \mathrm{d}t,$$

for $z \in K_\lambda$. Then for such z,

$$|f_\lambda - [m_j/n_j]_{f_\lambda}|(z) \leqslant C_1 E_{\sigma_j}(f; \bar{V}) \frac{\|q_j^* Q_j\|_{L_\infty(|t|=R)}}{|q_j^* Q_j|(z)} C_2^{m_j+n_j+1}.$$

We note that inasmuch as V is a finite union of balls, the length of ∂V_λ is bounded independently of $\boldsymbol{\lambda}$. Thus C_1, C_2 depend on V and K, but not on j, z or $\boldsymbol{\lambda}$. Also R is chosen so that the polydisc

$\{z\colon |z_j| \leqslant R\}$ contains V. Note that for $\boldsymbol{z} \in V$, $R \geqslant \|\boldsymbol{z}\| = |z|\,\|(1,\boldsymbol{\lambda})\|$, so in the contour integral,

$$\frac{1}{|t-z|} \leqslant \frac{\|(1,\boldsymbol{\lambda})\|}{\delta} \leqslant \frac{R}{|z|}.$$

By Lemma 2.1, if $0 < \varepsilon < 1$,

$$\frac{\|q_j^* Q_j\|_{L_\infty(|t|=R)}}{|q_j^* Q_j|(z)} \leqslant \varepsilon^{-2n_j}, \quad |z| \leqslant R, \ z \notin \mathscr{E}_{j,\boldsymbol{\lambda}}$$

where

$$m_2(\mathscr{E}_{j,\boldsymbol{\lambda}}) \leqslant 4\pi R^2 \varepsilon^2. \tag{15}$$

In view of (14) and the fact that $E_{\sigma_j}(f;\bar{V})$ does not depend on $\boldsymbol{\lambda}$, we obtain for $j \geqslant j_0$, $z \in K_{\boldsymbol{\lambda}} \backslash \mathscr{E}_{j,\boldsymbol{\lambda}}$,

$$|f_{\boldsymbol{\lambda}} - [m_j/n_j]_{f_{\boldsymbol{\lambda}}}|(z) \leqslant \varepsilon^{m_j+n_j}. \tag{16}$$

The crucial thing is that j_0 is independent of $\boldsymbol{\lambda}$.

Step 3: *Patch the exceptional sets together.* We can reformulate (16) as

$$|f - [m_j/n_j]|(\boldsymbol{z}) \leqslant \varepsilon^{m_j+n_j}, \quad \boldsymbol{z} \in K \backslash \mathscr{E}_j,$$

where

$$\mathscr{E}_j := \bigcup_{\boldsymbol{\lambda} \in \mathbb{C}^{k-1}} \{(z, \boldsymbol{\lambda} z)\colon z \in \mathscr{E}_{j,\boldsymbol{\lambda}}\}.$$

To transform the estimates on the size of $\mathscr{E}_{j,\boldsymbol{\lambda}}$ to $\mathscr{E}_j$, we need the Jacobian of the transformation $\boldsymbol{z} \to (z, \boldsymbol{\lambda})$. Here $\boldsymbol{z} \in \mathbb{C}^k$, $z \in \mathbb{C}$, $\boldsymbol{\lambda} \in \mathbb{C}^{k-1}$. Let us write

$$\boldsymbol{z} = (z_1, z_2, \ldots, z_k), \qquad z_j = x_j + \mathrm{i}y_j, \quad 1 \leqslant j \leqslant k;$$

$$\boldsymbol{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_k), \qquad \lambda_j = \mu_j + \mathrm{i}\nu_j, \quad 2 \leqslant j \leqslant k.$$

We see that the coordinates (x_j, y_j) transform to (x_j, y_j) if $j = 1$ and to $(\mu_j x_1 - \nu_j y_1, \nu_j x_1 + \mu_j y_1)$ if $2 \leqslant j \leqslant k$. The $2k \times 2k$ matrix of the transformation is hence

$$A = \begin{bmatrix} I & & & \\ & \Lambda & & \\ & & \ddots & \\ & & & \Lambda \end{bmatrix}; \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad \Lambda = \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix}.$$

Hence the Jacobian of the transformation $\boldsymbol{z} \to (z, \boldsymbol{\lambda})$ is

$$\frac{\partial \boldsymbol{z}}{\partial(z,\boldsymbol{\lambda})} = |\det A| = |z_1|^{2(k-1)} = |z|^{2(k-1)}.$$

To avoid problems for large $\boldsymbol{\lambda}$, let us set

$$B_\eta := \{(z, \boldsymbol{\lambda} z)\colon \|\boldsymbol{z}\| \leqslant R \text{ and } \|(1,\boldsymbol{\lambda})\| \geqslant 1/\eta\}.$$

We see that if χ denotes the characteristic function,

$$\begin{aligned} m_{2k}(\mathscr{E}_j \backslash B_\eta) &= \int_{\mathbb{C}^k} \chi_{\mathscr{E}_j \backslash B_\eta}(z)\, \mathrm{d}m_{2k}(z) \\ &= \int_{\mathbb{C}^{k-1}} \int_{\mathbb{C}} \chi_{\mathscr{E}_j \backslash B_\eta}(z, \lambda z)\, \frac{\partial z}{\partial(z, \lambda)}\, \mathrm{d}m_2(z)\, \mathrm{d}m_{2k-2}(\lambda) \\ &\leqslant \int_{\lambda : \|(1,\lambda)\| \leqslant 1/\eta} 4\pi R^2 \varepsilon^2 R^{2(k-1)}\, \mathrm{d}m_{2k-2}(\lambda) \\ &\leqslant C R^{2k} \varepsilon^2 \eta^{-2(k-1)}, \end{aligned}$$

where C is independent of R, η, ε. Here we have used (15). Since B_η consists of points $(z, \lambda z)$ with $|z| \leqslant R/\|(1, \lambda)\| \leqslant R\eta$, we see that $m_{2k}(B_\eta) = \mathrm{O}(\eta^2)$, $\eta \to 0+$. Choosing η small enough, and then ε small enough, gives the result. □

References

[1] G.A. Baker, *Essentials of Padé Approximants* (Academic Press, New York, 1975).

[2] G.A. Baker and P.R. Graves-Morris, *Padé Approximants, Part I: Basic Theory*, Encyclopaedia Math. Appl. **13** (Addison-Wesley, Reading, MA, 1981).

[3] U. Cegrell, *Capacities in Complex Analysis*, Aspects of Math. **14** (Vieweg, Braunschweig, 1988).

[4] E.M. Cirka, Expansions in series and the rate of rational approximations for holomorphic functions with analytic singularities, *Math. USSR. Sb.* **22** (1974) 323–332.

[5] A. Cuyt, *Padé Approximants for Operators: Theory and Applications*, Lecture Notes in Math. **1065** (Springer, Berlin, 1984).

[6] A. Cuyt, Multivariate Padé approximants revisited, *BIT* **26** (1986) 71–79.

[7] A. Cuyt, A multivariate convergence theorem of the "de Montessus de Ballore" type, *J. Comput. Appl. Math.* **32** (1990) 47–57.

[8] A. Cuyt, K. Driver and D.S. Lubinsky, On the size of lemniscates of polynomials in one and several variables, *Proc. Amer. Math. Soc.*, to appear.

[9] A.A. Goncar, A local condition of single-valuedness of analytic functions, *Math. USSR. Sb.* **18** (1972) 151–167.

[10] A.A. Goncar, A local condition for single-valuedness of analytic functions of several variables, *Math. USSR. Sb.* **22** (1974) 305–322.

[11] J. Karlsson and H. Wallin, Rational approximation by an interpolation procedure in several variables, in: P.R. Graves-Morris, E.B. Saff and R.S. Varga, Eds., *Padé and Rational Approximation* (Academic Press, New York, 1977) 83–100.

[12] D.S. Lubinsky, Diagonal Padé approximants and capacity, *J. Math. Anal. Appl.* **78** (1980) 58–67.

[13] J. Nuttall, The convergence of Padé approximants of meromorphic functions, *J. Math. Anal. Appl.* **31** (1970) 147–153.

[14] C. Pommerenke, Padé approximants and convergence in capacity, *J. Math. Anal. Appl.* **41** (1973) 775–780.

[15] H. Stahl, General convergence results for Padé approximants, in: C.K. Chui, L.L. Schumaker and J.D. Ward, Eds., *Approximation Theory VI* (Academic Press, San Diego, 1989) 605–634.

[16] V.P. Zaharjuta, Transfinite diameter, Chebyshev constants, and capacity for compacta in $\mathbb{C}^n$, *Math. USSR. Sb.* **25** (1975) 350–364.