Numer. Math. 73: 311–327 (1996)

Numerische Mathematik

Kronecker type theorems, normality and continuity of the multivariate Padé operator

Annie Cuyt[1], **Kathy Driver**[2], **Doron Lubinsky**[2]

[1] NWFO (Belgium), Department Wis-Inf, University of Antwerp (UIA), Universiteitsplein 1, B–2610 Antwerp, Belgium
[2] Department Mathematics, Witwatersrand University, P.O. Wits 2050, South Africa

Received December 19, 1994

Summary. For univariate functions the Kronecker theorem, stating the equivalence between the existence of an infinite block in the table of Padé approximants and the approximated function f being rational, is well-known. In [Lubi88] Lubinsky proved that if f is not rational, then its Padé table is normal almost everywhere: for an at most countable set of points the Taylor series expansion of f is such that it generates a non-normal Padé table. This implies that the Padé operator is an almost always continuous operator because it is continuous when computing a normal Padé approximant [Wuyt81].

In this paper we generalize the above results to the case of multivariate Padé approximation. We distinguish between two different approaches for the definition of multivariate Padé approximants: the general order one introduced in [Levi76, CuVe84] and the so-called homogeneous one discussed in [Cuyt84].

Mathematics Subject Classification (1991): 65D15

Introduction

Let $f(x)$ be analytic in the origin with a series development

$$f(x) = \sum_{i=0}^{\infty} c_i x^i$$

and let the Padé approximant $r_{n,m}$ of degree n in the numerator and m in the denominator be defined by

$$\partial p \le n \qquad \partial q \le m \qquad (fq - p)(x) = \sum_{i=n+m+1}^{\infty} d_i x^i$$

where $r_{n,m}$ is the irreducible form of p/q. Then the results of Kronecker and Lubinsky can essentially be summarized as follows.

Summary:

(I)
$$f(x) = \frac{\sum_{i=0}^{n-1} a_i x^i}{\sum_{i=0}^{m-1} b_i x^i}$$

$$b_0 \ne 0 \qquad b_{m-1} \ne 0 \qquad a_{n-1} \ne 0$$

$$f \text{ irreducible}$$

$$\Longleftrightarrow$$

(II)
$$\forall \nu \ge n, \mu \ge m : D(\nu, \mu) := \begin{vmatrix} c_\nu & \dots & c_{\nu-\mu+1} \\ \vdots & \ddots & \vdots \\ c_{\nu+\mu-1} & \dots & c_\nu \end{vmatrix} = 0$$

$$\Longleftrightarrow$$

(III)
$$D_u(n, m) := \det \left[\frac{1}{(n+k-\ell)!} \frac{d^{n+k-\ell} f}{dx^{n+k-\ell}}(u) \right]_{k,\ell=0}^{m-1} \equiv 0$$

for u in a neighbourhood of the origin

In order to easily understand the positive and negative multivariate results of the next sections, we keep the following special case in mind. Take $m = 1$. Then f is a polynomial of degree $n - 1$ according to (I), which means that the series expansion of f reduces to

$$f(x) = \sum_{i=0}^{n-1} c_i x^i$$

while the equivalent statement (II) says for $\mu = m = 1$ that

$$\forall \nu \ge n : c_\nu = \frac{1}{\nu!} \frac{d^\nu f}{dx^\nu}(0) = 0$$

The equivalence with (III) stating that

$$\frac{1}{n!} \frac{d^n f}{dx^n}(u) \equiv 0$$

for u in a neighbourhood of the origin is evident. At the end of the Sects. 2 and 4 we will come back to this example, but then for a multivariate function.

1. Definitions and notations: general order MPA

We restrict our description to the case of two variables because the generalization to functions of more variables is only notationally more difficult. We also prefer not to work with the multi-index notation because the reader may then lose feeling for the difference between the univariate and the multivariate case once we are dealing with the homogeneous approach in Sects. 3.

Given a Taylor series expansion

$$f(x,y) = \sum_{(i,j)\in\mathbb{N}^2} c_{ij} x^i y^j$$

with

$$c_{ij} = \frac{1}{i!}\frac{1}{j!}\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\Big|_{(0,0)}$$

we repeat the definition of general order multivariate Padé approximant p/q to f where $p(x,y)$ and $q(x,y)$ are determined by an accuracy–through–order principle. The polynomials $p(x,y)$ and $q(x,y)$ are of the form

(1a) $$p(x,y) = \sum_{(i,j)\in N} a_{ij} x^i y^j$$

(1b) $$q(x,y) = \sum_{(i,j)\in D} b_{ij} x^i y^j$$

where N and D are finite subsets of $\mathbb{N}^2$. The index sets N and D in a way indicate the degree of the polynomials $p(x,y)$ and $q(x,y)$. Let us denote

$$\partial p = \{(i,j) \mid a_{ij} \neq 0\} \subseteq N \qquad \#N = n+1$$

$$\partial q = \{(i,j) \mid b_{ij} \neq 0\} \subseteq D \qquad \#D = m+1$$

For the construction of a general order multivariate Padé approximant p/q we impose on $p(x,y)$ and $q(x,y)$ that they should satisfy the following conditions for the power series $(fq-p)(x,y)$, namely

(1c) $$(fq-p)(x,y) = \sum_{(i,j)\in\mathbb{N}^2\setminus E} d_{ij} x^i y^j$$

These conditions can be satisfied if, in analogy with the univariate case, the index set E is such that

(2a) $$N \subseteq E$$

(2b) $$\#(E\setminus N) = m = \#D - 1$$

(2c) E satisfies the inclusion property

where (2c) means that when a point belongs to the index set E, then the rectangular subset of points emanating from the origin with the given point as its furthermost corner, also lies in E. Condition (2a) enables us to split the system of equations

$$d_{ij} = 0 \qquad (i,j) \in E$$

in an inhomogeneous part defining the numerator coefficients

(3a) $$\sum_{k=0}^{i} \sum_{\ell=0}^{j} c_{k\ell} b_{i-k,j-\ell} = a_{ij} \qquad (i,j) \in N$$

and a homogeneous part defining the denominator coefficients

(3b) $$\sum_{k=0}^{i} \sum_{\ell=0}^{j} c_{k\ell} b_{i-k,j-\ell} = 0 \qquad (i,j) \in E \backslash N$$

By convention $b_{k\ell} = 0$ if $(k,\ell) \notin D$. Condition (2b) guarantees the existence of a nontrivial denominator $q(x,y)$ because the homogeneous system has one equation less than the number of unknowns and so one unknown coefficient can be chosen freely. Condition (2c) finally takes care of the Padé approximation property, namely

(3c) $$q(0,0) \neq 0 \Longrightarrow (f - \frac{p}{q})(x,y) = \sum_{(i,j)\in \mathbb{N}^2 \backslash E} e_{ij} x^i y^j$$

If E does not satisfy the inclusion property, then (1c) cannot imply (3c).

Let us denote the set of solutions p/q of the general order multivariate Padé approximation problem (1) to f by $[N/D]_E^f$. If the homogeneous system (3b) defining the denominator coefficients has maximal rank, then the solution p/q is unique. The general order multivariate Padé approximants can also be ordered in a table [Cuyt92] in the following way. The size of the numerator index set N will play the role of row number and the size of the denominator index set D that of column number:

$$N = N_n = \{(i_0, j_0), \ldots, (i_n, j_n)\} \qquad \#N = n + 1$$
$$D = D_m = \{(d_0, e_0), \ldots, (d_m, e_m)\} \qquad \#D = m + 1$$
$$E = E_{n+m} = \{(i_0, j_0), \ldots, (i_{n+m}, j_{n+m})\} \qquad \#E_{n+m} = n + m + 1$$

When updating or downdating the index sets N, D and E, the order of the indices is preserved. For more detailed information we refer to [Cuyt92].

2. Results for general order multivariate Padé approximants

A main difference between the general order (introduced in the previous section) and the homogeneous (introduced in the next section) multivariate Padé approximation problem is the consistency property. The consistency property states that if f is a rational function then f is reconstructed when choosing the appropriate degrees for its Padé approximant. This property is satisfied for the general order definition only if the system of Padé conditions is nonsingular [AbCu93]. This explains the slight difference in formulation between Theorem 1 and Theorem 4.

Theorem 1. *If the coefficient matrix of the homogeneous system of linear equations (3b) for the general order multivariate Padé approximant with numerator degree set N_{n-1} and denominator degree set D_{m-1} has maximal rank, then*

$$f(x,y) = \frac{\sum_{k=0}^{n-1} a_{i_k j_k} x^{i_k} y^{j_k}}{\sum_{k=0}^{m-1} b_{d_k e_k} x^{d_k} y^{e_k}}$$

$$d_0 = 0 = e_0 \qquad b_{00} \neq 0 \qquad a_{i_{n-1} j_{n-1}} \neq 0 \qquad b_{d_{m-1} e_{m-1}} \neq 0$$

$$f \text{ irreducible}$$

$$\Longleftrightarrow$$

$$\forall \nu \geq n, \mu \geq m:$$

$$\begin{aligned} \Delta(\nu,\mu) &:= \det \left[c_{i_{\nu+k}-d_\ell, j_{\nu+k}-e_\ell} \right]_{k,\ell=0}^{\mu-1} \\ &= \begin{vmatrix} c_{i_\nu - d_0, j_\nu - e_0} & \cdots & c_{i_\nu - d_{\mu-1}, j_\nu - e_{\mu-1}} \\ \vdots & \ddots & \vdots \\ c_{i_{\nu+\mu-1}-d_0, j_{\nu+\mu-1}-e_0} & \cdots & c_{i_{\nu+\mu-1}-d_{\mu-1}, j_{\nu+\mu-1}-e_{\mu-1}} \end{vmatrix} = 0 \end{aligned}$$

Proof. "$\Longrightarrow$": Since $f(x,y)$ is a rational function and the rank of (3b) is maximal, f equals its general order multivariate Padé approximant $[N/D]_E^f = p/q$ with

$$\begin{aligned} N &= \{(i_0,j_0),\ldots,(i_{n-1},j_{n-1})\} \\ D &= \{(d_0,e_0),\ldots,(d_{m-1},e_{m-1})\} \\ E &= \{(i_0,j_0),\ldots,(i_{n+m-2},j_{n+m-2})\} \end{aligned}$$

because of the consistency property discussed in [AbCu93]. Moreover

$$fq - p \equiv 0$$

Now fix $\nu \geq n$ and $\mu \geq m$. Then from $fq - p \equiv 0$, together with the inclusion property (2c), we obtain that the linear system of equations

$$\begin{cases} \sum_{k=0}^{\mu-1} c_{i_\nu - d_k, j_\nu - e_k} b_{d_k e_k} = 0 \\ \vdots \\ \sum_{k=0}^{\mu-1} c_{i_{\nu+\mu-1}-d_k, j_{\nu+\mu-1}-e_k} b_{d_k e_k} = 0 \end{cases}$$

has a nontrivial solution for the $b_{d_k e_k}$, with $b_{d_m e_m} = \ldots = b_{d_{\mu-1} e_{\mu-1}} = 0$ if $m < \mu$. Hence

$$\forall \nu \geq n, \mu \geq m :$$
$$\Delta(\nu,\mu) = \begin{vmatrix} c_{i_\nu - d_0, j_\nu - e_0} & \cdots & c_{i_\nu - d_{\mu-1}, j_\nu - e_{\mu-1}} \\ \vdots & \ddots & \vdots \\ c_{i_{\nu+\mu-1} - d_0, j_{\nu+\mu-1} - e_0} & \cdots & c_{i_{\nu+\mu-1} - d_{\mu-1}, j_{\nu+\mu-1} - e_{\mu-1}} \end{vmatrix} = 0$$

"$\Leftarrow$": Since the rank of the homogeneous system (3b) is maximal we can compute the unique general order multivariate Padé approximant $[N/D]^f_E$ satisfying

$$\begin{cases} \sum_{k=0}^{m-1} c_{i_\ell - d_k, j_\ell - e_k} b_{d_k e_k} = a_{i_\ell j_\ell} & \ell = 0, \ldots, n-1 \\ \sum_{k=0}^{m-1} c_{i_\ell - d_k, j_\ell - e_k} b_{d_k e_k} = 0 & \ell = n, \ldots, n+m-2 \end{cases}$$

Because $\Delta(n,m) = 0$ and the rank of (3b) is maximal, the last row of the matrix

$$\begin{pmatrix} c_{i_n - d_0, j_n - e_0} & \cdots & c_{i_n - d_{m-1}, j_n - e_{m-1}} \\ \vdots & \ddots & \vdots \\ c_{i_{n+m-1} - d_0, j_{n+m-1} - e_0} & \cdots & c_{i_{n+m-1} - d_{m-1}, j_{n+m-1} - e_{m-1}} \end{pmatrix}$$

is linearly dependent of the other ones and hence

$$\sum_{k=0}^{m-1} c_{i_{n+m-1} - d_k, j_{n+m-1} - e_k} b_{d_k e_k} = 0$$

We can now prove by induction that

$$\left(c_{i_{s+m-1} - d_0, j_{s+m-1} - e_0}, \ldots, c_{i_{s+m-1} - d_{m-1}, j_{s+m-1} - e_{m-1}}\right) \qquad s \geq n$$

is linearly dependent of the first $m-1$ rows of $\Delta(n,m)$ and consequently

$$\sum_{k=0}^{m-1} c_{i_{s+m-1} - d_k, j_{s+m-1} - e_k} b_{d_k e_k} = 0$$

Assume it is true for all $n + m - 1 \leq s + m - 1 \leq \nu + m - 2$ with $n < \nu$. Then consider $\Delta(\nu, m)$. Since $\Delta(\nu, m) = 0$, there exist $\lambda_0, \ldots, \lambda_{m-1}$ such that

$$\sum_{\ell=0}^{m-1} \lambda_\ell \sum_{k=0}^{m-1} c_{i_{\nu+\ell} - d_k, j_{\nu+\ell} - e_k} b_{d_k e_k} = 0$$

or, because of the induction hypothesis

$$\sum_{\ell=0}^{m-2} \tilde{\lambda}_\ell \sum_{k=0}^{m-1} c_{i_{n+\ell} - d_k, j_{n+\ell} - e_k} b_{d_k e_k} + \lambda_{m-1} \sum_{k=0}^{m-1} c_{i_{\nu+m-1} - d_k, j_{\nu+m-1} - e_k} b_{d_k e_k} = 0$$

Clearly $\lambda_{m-1} \neq 0$ because otherwise the rows of the coefficient matrix in (3b) would be linearly dependent and thus

$$\sum_{k=0}^{m-1} c_{i_{\nu+m-1}-d_k, j_{\nu+m-1}-e_k} b_{d_k e_k} = 0$$

which concludes the induction part. Now that the above is true for all $s \geq n$, we have

$$fq - p \equiv 0$$

or $f = p/q$. □

Theorem 2. *If the coefficient matrix of the homogeneous system of linear equations (3b) for the general order multivariate Padé approximant with numerator degree set N_{n-1} and denominator degree set D_{m-1} has maximal rank, then*

$$f(x,y) = \frac{\sum_{k=0}^{n-1} a_{i_k j_k} x^{i_k} y^{j_k}}{\sum_{k=0}^{m-1} b_{d_k e_k} x^{d_k} y^{e_k}}$$

$$d_0 = 0 = e_0 \qquad b_{00} \neq 0 \qquad a_{i_{n-1} j_{n-1}} \neq 0 \qquad b_{d_{m-1} e_{m-1}} \neq 0$$

$$f \text{ irreducible}$$

$$\Longrightarrow$$

$$\Delta_{(u,v)}(n,m) := \det \left[\frac{1}{(i_{n+k} - d_\ell)!(j_{n+k} - e_\ell)!} \frac{\partial^{i_{n+k} - d_\ell + j_{n+k} - e_\ell} f}{\partial x^{i_{n+k} - d_\ell} \partial y^{j_{n+k} - e_\ell}}(u,v) \right]_{k,\ell=0}^{m-1} \equiv 0$$

for (u,v) in a neighbourhood of the origin

Proof. "$\Longrightarrow$": Because the coefficient matrix of the homogeneous system of linear equations (3b) has maximal rank, it has also maximal rank when the Taylor coefficients are not taken at the origin but at a nearby point (u,v). This is a direct consequence of the continuity of all the $(m-1) \times (m-1)$ determinants formed with the columns of the coefficient matrix of (3b). Then the previous theorem implies that the Padé approximation process at the point (u,v) instead of at the origin, reconstructs the rational function $f(x,y)$ with $\Delta_{(u,v)}(n,m) = 0$, and this for all (u,v) in a neighbourhood of the origin where the defining equations retain their maximal rank. □

Why the implication in the other direction isn't true, is easy to understand from the special case that we discussed in the introduction. Take $m = 1$ and $d_0 = 0 = e_0$. Then $f(x,y)$ is a bivariate polynomial because the denominator contains only a constant term. The assumption about the rank of the homogeneous system (3b) is fulfilled because for $m = 1$ the system is empty and hence all the $(m-1) \times (m-1)$ determinants equal 1. Theorem 1 tells us that

$$\forall \nu \geq n : c_{i_\nu j_\nu} = \frac{1}{i_\nu!} \frac{1}{j_\nu!} \frac{\partial^{i_\nu + j_\nu} f}{\partial x^{i_\nu} \partial y^{j_\nu}}(0,0) = 0$$

which translates to

$$f(x,y) = \sum_{k=0}^{n-1} c_{i_k j_k} x^{i_k} y^{j_k}$$

Theorem 2 states that this implies

$$\frac{1}{i_n!}\frac{1}{j_n!}\frac{\partial^{i_n+j_n} f}{\partial x^{i_n}\partial y^{j_n}}(u,v) \equiv 0$$

for (u,v) in a neighbourhood of the origin. However the implication in the other direction doesn't hold in general. If it were true for instance for $m = 1$, then having a partial derivative vanish in a neighbourhood of the origin would imply that we are dealing with a polynomial function f. That this is incorrect can easily be seen. Just take $f(x,y) = h(x)$ with $h(x)$ not a polynomial. Take $(i_0, j_0) = (0,0), (i_1, j_1) = (1,0), (i_2, j_2) = (0,1), \ldots$ and $n = 2$. Then clearly

$$\frac{1}{i_2!}\frac{1}{j_2!}\frac{\partial^{i_2+j_2} f}{\partial x^{i_2}\partial y^{j_2}}(u,v) = \frac{\partial h}{\partial y}(u,v) \equiv 0$$

although $f(x,y) = h(x)$ is not a polynomial indexed by $N = N_{n-1} = \{(0,0),(1,0)\}$. However, with the single condition $\Delta_{(u,v)}(n,m) \equiv 0$ being replaced by a finite number of conditions $\Delta_{(u,v)}(n_k, m_k) \equiv 0$ with (n_k, m_k) on the boundary of the set $\mathbb{N}^2 \setminus N$ as in Figure 1, the theorem probably remains true. At least for the case $m = 1$ it is easy to see that

$$\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0,0) = 0 \qquad (i,j) \in \mathbb{N}^2 \setminus N$$

is equivalent to

$$\frac{\partial^{n_k+m_k} f}{\partial x^{n_k}\partial y^{m_k}}(u,v) \equiv 0 \qquad (u,v) \text{ in a neighbourhood of } (0,0)$$

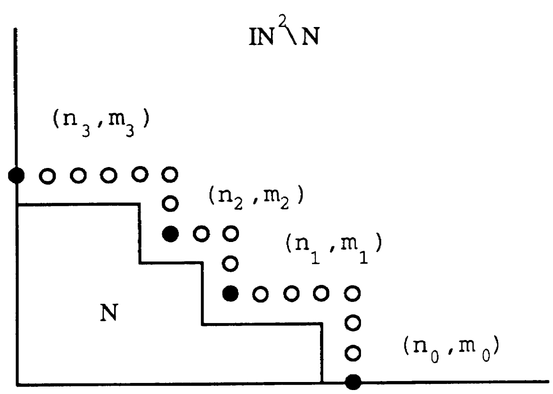

Fig. 1.

Consequently if $f(x,y)$ is not a rational function, one can no longer conclude that its Padé table is normal almost everywhere. In the univariate case normality was easily obtained almost everywhere by slightly shifting the origin if one encountered a nonnormal Padé table at the origin [Lubi88].

With respect to the normality and continuity of the multivariate Padé operator the following still holds. We emphasize that the enumeration of the index sets N, D and E as introduced at the end of Sect. 1, remains fixed when considering Padé approximants of different "degree". When the homogeneous system (3b) has maximal rank, as we assume in the above theorems, then the unique solution of the general order multivariate Padé approximation problem also has a unique irreducible form

$$[N/D]_E^f = r_{N,D}(x,y) = \frac{p_{N,D}}{q_{N,D}}(x,y) = \frac{\sum_{(i,j)\in N'} a_{ij}x^iy^j}{\sum_{(i,j)\in D'} b_{ij}x^iy^j}$$

with

$$\partial p_{N,D} = N' \subseteq N \qquad \partial q_{N,D} = D' \subseteq D$$
$$n' = \max\{k|(i_k,j_k)\in N'\} \quad m' = \max\{\ell|(d_\ell,e_\ell)\in D'\} \quad a_{i_{n'}j_{n'}} \neq 0 \quad b_{i_{m'}j_{m'}} \neq 0$$

We can then define what we call the defect in degree, by

$$\delta_{N,D} = \min(n-n', m-m')$$

Let us now study the influence on the computation of the general order multivariate Padé approximant of small variations in the Taylor coefficients of the function to be approximated. For univariate Padé approximants this problem has been studied in [Wuyt81] and [WeWu83]. This problem is linked to the one discussed above because a shift of the origin implies a change in the Taylor coefficients and hence a change in the coefficients of the Padé approximant. Our question is now, is this change in the coefficients of the Padé approximant dramatic or comparable in size to the change in the Taylor coefficients? The following theorem deals with this continuity problem.

We introduce the seminorm

$$||f(x,y)||_{n+m} = \max_{k=0,\ldots,n+m} |c_{i_kj_k}|$$

for the power series

$$f(x,y) = \sum_{k=0}^{\infty} c_{i_kj_k}x^{i_k}y^{j_k}$$

and use the Tchebyshev-norm

$$||f||_I = \sup_{(x,y)\in I} |f(x,y)|$$

for multivariate functions continuous on compact sets I.

Theorem 3. *If the system of homogeneous equations (3b) for the general order multivariate Padé approximant with numerator index set N and denominator index set D has maximal rank and if $\delta_{N,D} = 0$ and $q_{N,D}(x,y) \neq 0$ in some polyinterval I, then*

$$\forall \epsilon, \exists \delta : ||(f - \tilde{f})(x,y)||_{n+m} < \delta \Longrightarrow ||(r_{N,D} - \tilde{r}_{N,D})(x,y)||_I < \epsilon$$

where $r_{N,D}$ and $\tilde{r}_{N,D}$ are respectively approximating f and $\tilde{f}$ in the sense of (1–3).

Proof. The proof is completely analogous to the one in the univariate literature [WeWu83]. The condition $\delta_{N,D} = 0$ is weaker than a normality condition. □

3. Definitions and notations: homogeneous MPA

The approach we have taken in the previous sections to define and construct multivariate Padé approximants is essentially based on rewriting the double series expansion

(4) $$\sum_{(i,j)\in\mathbb{N}^2} c_{ij} x^i y^j$$

as the single sum

$$\sum_{r_E(i,j)=0}^{\infty} c_{ij} x^i y^j$$

In general, a numbering r_E of $\mathbb{N}^2$ places the points in $\mathbb{N}^2$ one after the other. Another way to work with the bivariate power series (4) is the following:

$$\sum_{(i,j)\in\mathbb{N}^2} c_{ij} x^i y^j = \sum_{\ell=0}^{\infty} \left(\sum_{i+j=\ell} c_{ij} x^i y^j \right)$$

This approach is taken in [Cuyt84, pp. 59–62] to construct homogeneous multivariate Padé approximants. These homogeneous multivariate Padé approximants are a special case of the general definition (3) where for chosen ν and μ in $\mathbb{N}$, which are comparable to the degrees n and m of the univariate Padé approximant, the numerator and denominator index sets N and D are given by

(5a) $$N = \{(i,j) \in \mathbb{N}^2 \mid \nu\mu \leq i + j \leq \nu\mu + \nu\}$$

(5b) $$D = \{(d,e) \in \mathbb{N}^2 \mid \nu\mu \leq d + e \leq \nu\mu + \mu\}$$

while

(5c) $$E = E_{(\nu,\mu)} \cup E_{\Phi}$$

$$E_{(\nu,\mu)} = \{(i,j) \in \mathbb{N}^2 \mid \nu\mu \leq i + j \leq \nu\mu + \nu + \mu\}$$

$$E_{\Phi} = \{(i,j) \in \mathbb{N}^2 \mid 0 \le i+j < \nu\mu\}$$

The conditions (1c) indexed by E_{Φ} are automatically satisfied by the choice of N and D and hence void.

An advantage of homogeneous Padé approximants is that they preserve the properties and the nature of univariate Padé approximants even better than the general order definition. This is for instance reflected in a tremendous simplification of the algorithms for their computation [Cuyt82, Cuyt83]. Let us introduce the notations

(6a) $$A_{\nu\mu+\ell}(x,y)^{\nu\mu+\ell} = \sum_{i+j=\nu\mu+\ell} a_{ij}x^i y^j \qquad \ell = 0,\ldots,\nu$$

(6b) $$B_{\nu\mu+\ell}(x,y)^{\nu\mu+\ell} = \sum_{i+j=\nu\mu+\ell} b_{ij}x^i y^j \qquad \ell = 0,\ldots,\mu$$

$$C_{\ell}(x,y)^{\ell} = \sum_{i+j=\ell} c_{ij}x^i y^j \qquad \ell = 0,1,2\ldots$$

where C_ℓ is the ℓ^{th} Fréchet-derivative of f at the origin and hence an ℓ-linear operator [Rall79]. In fact the notation comes from

$$\sum_{i+j=\ell} c_{ij}x^i y^j = C_\ell \underbrace{\begin{pmatrix} x \\ y \end{pmatrix} \cdots \begin{pmatrix} x \\ y \end{pmatrix}}_{\ell \text{ times}}$$

In the sequel of the text we shall sometimes also write $C_\ell = f^{(\ell)}_{(0,0)}$ for the ℓ^{th} Fréchet derivative of f at the origin. We rewrite

$$p(x,y) = \sum_{(i,j)\in N} a_{ij}x^i y^j = \sum_{\ell=0}^{\nu} A_{\nu\mu+\ell}(x,y)^{\nu\mu+\ell}$$

$$q(x,y) = \sum_{(i,j)\in D} b_{ij}x^i y^j = \sum_{\ell=0}^{\mu} B_{\nu\mu+\ell}(x,y)^{\nu\mu+\ell}$$

such that the conditions

$$(fq-p)(x,y) = \sum_{(i,j)\in\mathbb{N}^2\setminus E} d_{ij}x^i y^j = \sum_{i+j\ge\nu\mu+\nu+\mu+1} d_{ij}x^i y^j$$

can be reformulated as

(7a) $$\begin{cases} C_0(x,y)^0 B_{\nu\mu}(x,y)^{\nu\mu} = A_{\nu\mu}(x,y)^{\nu\mu} \\ C_1(x,y)^1 B_{\nu\mu}(x,y)^{\nu\mu} + C_0(x,y)^0 B_{\nu\mu+1}(x,y)^{\nu\mu+1} = A_{\nu\mu+1}(x,y)^{\nu\mu+1} \\ \quad \vdots \\ C_\nu(x,y)^\nu B_{\nu\mu}(x,y)^{\nu\mu} + \ldots + C_{\nu-\mu}(x,y)^{\nu-\mu} B_{\nu\mu+\mu}(x,y)^{\nu\mu+\mu} \\ \qquad = A_{\nu\mu+\nu}(x,y)^{\nu\mu+\nu} \end{cases}$$

(7b)
$$\begin{cases} C_{\nu+1}(x,y)^{\nu+1}B_{\nu\mu}(x,y)^{\nu\mu} + \ldots + C_{\nu+1-\mu}(x,y)^{\nu+1-\mu}B_{\nu\mu+\mu}(x,y)^{\nu\mu+\mu} = 0 \\ \quad\vdots \\ C_{\nu+\mu}(x,y)^{\nu+\mu}B_{\nu\mu}(x,y)^{\nu\mu} + \ldots + C_{\nu}(x,y)^{\nu}B_{\nu\mu+\mu}(x,y)^{\nu\mu+\mu} = 0 \end{cases}$$

where $C_\ell(x,y)^\ell \equiv 0$ if $\ell < 0$. This is exactly the system of defining equations for univariate Padé approximants if the univariate term $c_\ell x^\ell$ is substituted by

$$C_\ell(x,y)^\ell = \sum_{i+j=\ell} c_{ij}x^i y^j \qquad \ell = 0,1,2\ldots$$

The homogeneous multivariate Padé approximant of order (ν,μ) for $f(x,y)$ can now be defined as the unique irreducible form [Cuyt84]

$$[\nu/\mu]_f = r_{\nu,\mu}(x,y) = \frac{p_{\nu,\mu}(x,y)}{q_{\nu,\mu}(x,y)}$$

of any solution $p(x,y)/q(x,y)$ of (7).

4. Results for homogeneous multivariate Padé approximants

Before proving two other Kronecker-type theorems, let us first point out that the homogeneous multivariate Padé table exhibits a square block structure just like in the univariate case [Cuyt84, pp. 45]. Therefore zero-entries in the table of the $D(n,m)$ as defined below, also occur in square blocks.

Theorem 4. *Let f be analytic in the origin. Then the following are equivalent:*

$$f(x,y) = \frac{\sum_{\ell=0}^{n-1} A_\ell(x,y)^\ell}{\sum_{\ell=0}^{m-1} B_\ell(x,y)^\ell} = \frac{\sum_{\ell=0}^{n-1}\sum_{i+j=\ell} a_{ij}x^i y^j}{\sum_{\ell=0}^{m-1}\sum_{i+j=\ell} b_{ij}x^i y^j}$$

$$B_0 \neq 0 \qquad B_{m-1}(x,y)^{m-1} \not\equiv 0 \qquad A_{n-1}(x,y)^{n-1} \not\equiv 0$$

$$f \textit{ irreducible}$$

$$\Longleftrightarrow$$

$$\forall \nu \geq n, \mu \geq m : D(\nu,\mu) = \begin{vmatrix} C_\nu(x,y)^\nu & \ldots & C_{\nu-\mu+1}(x,y)^{\nu-\mu+1} \\ \vdots & \ddots & \vdots \\ C_{\nu+\mu-1}(x,y)^{\nu+\mu-1} & \ldots & C_\nu(x,y)^\nu \end{vmatrix} \equiv 0$$

Proof. Note that f is a rational function in the usual sense, without the degree of its numerator and denominator being shifted by $\nu\mu$ as in (6).
"$\Longrightarrow$": Since $f(x,y)$ is a rational function of homogeneous degree $n-1$ in the

numerator and homogeneous degree $m - 1$ in the denominator, it equals its homogeneous Padé approximant

$$[n-1/m-1]_f = p/q \qquad p(x,y) = \sum_{i=0}^{n-1} A_i(x,y)^i \qquad q(x,y) = \sum_{i=0}^{m-1} B_i(x,y)^i$$

because of the consistency property for homogeneous Padé approximants proved in [Cuyt84, pp. 65]. Note that for a rational function f the consistency property implies that the shift over $(n-1)(m-1)$ in the degrees of the numerator and denominator of its homogeneous Padé approximant is cancelled when taking the unique irreducible form. Moreover

$$fq - p \equiv 0$$

Now fix $\nu \geq n$ and $\mu \geq m$. Then from $fq - p \equiv 0$ with $B_m(x,y)^m \equiv \ldots \equiv B_{\mu-1}(x,y)^{\mu-1} \equiv 0$ if $m < \mu$, we obtain that the linear system of equations

$$\begin{cases} C_\nu B_0 + \ldots + C_{\nu-\mu+1} B_{\mu-1} \equiv 0 \\ \quad\vdots \\ C_{\nu+\mu-1} B_0 + \ldots + C_\nu B_{\mu-1} \equiv 0 \end{cases}$$

has a nontrivial solution for the $B_i(x,y)^i$. Hence

$$\forall \nu \geq n, \mu \geq m : D(\nu,\mu) = \begin{vmatrix} C_\nu(x,y)^\nu & \ldots & C_{\nu-\mu+1}(x,y)^{\nu-\mu+1} \\ \vdots & \ddots & \vdots \\ C_{\nu+\mu-1}(x,y)^{\nu+\mu-1} & \ldots & C_\nu(x,y)^\nu \end{vmatrix} \equiv 0$$

"$\Longleftarrow$": Let n and m be minimal, in the sense that $D(n, m-1) \not\equiv 0, D(n-1, m) \not\equiv 0$ and $D(n-1, m-1) \not\equiv 0$. Then we can construct the homogeneous Padé approximant $[n-1/m-1]_f$ satisfying

$$\begin{cases} \displaystyle\sum_{i=0}^{m-1} C_{k-i} B_i = A_k \qquad k = 0, \ldots, n-1 \\ \displaystyle\sum_{i=0}^{m-1} C_{k-i} B_i = 0 \qquad k = n, \ldots, n+m-2 \end{cases}$$

Because $D(n,m) \equiv 0$ and $D(n-1, m-1) \not\equiv 0$, the last row of the matrix

$$\begin{pmatrix} C_n(x,y)^n & \ldots & C_{n-m+1}(x,y)^{n-m+1} \\ \vdots & \ddots & \vdots \\ C_{n+m-1}(x,y)^{n+m-1} & \ldots & C_n(x,y)^n \end{pmatrix}$$

is linearly dependent of the other ones and hence

$$C_{n+m-1} B_0 + \ldots + C_n B_{m-1} \equiv 0$$

We can now prove by induction that $\left(C_{s+m-1}(x,y)^{s+m-1}, \ldots, C_s(x,y)^s\right)$ is linearly dependent of the first $m-1$ rows of $D(n,m)$ and consequently

$$C_{s+m-1}B_0 + \ldots + C_s B_{m-1} \equiv 0$$

Assume it is true for all $n+m-1 \le s+m-1 \le \nu+m-2$ with $n < \nu$. Then consider $D(\nu,m)$. Since $D(\nu,m) \equiv 0$, there exist $\lambda_0, \ldots, \lambda_{m-1}$ such that

$$\sum_{i=0}^{m-1} \lambda_i (C_{\nu+i}B_0 + \ldots + C_{\nu+i-m+1}B_{m-1}) \equiv 0$$

or because of the induction hypothesis

$$\sum_{i=0}^{m-2} \tilde{\lambda}_i \sum_{k=0}^{m-1} C_{n+i-k}B_k + \lambda_{m-1} \sum_{k=0}^{m-1} C_{\nu+m-1-k}B_k \equiv 0$$

Clearly $\lambda_{m-1} \not\equiv 0$ because otherwise the rows in $D(n-1, m-1)$ would be linearly dependent and thus

$$C_{\nu+m-1}B_0 + \ldots + C_\nu B_{m-1} \equiv 0$$

which concludes the induction part. Now that the above is true for all $s \ge n$, we have

$$fq - p \equiv 0$$

or $f = p/q$. □

Theorem 5. *Let $f(x,y)$ be analytic in the origin. Then the following are equivalent:*

$$f(x,y) = \frac{\sum_{\ell=0}^{n-1} A_\ell (x,y)^\ell}{\sum_{\ell=0}^{m-1} B_\ell (x,y)^\ell} = \frac{\sum_{\ell=0}^{n-1} \sum_{i+j=\ell} a_{ij} x^i y^j}{\sum_{\ell=0}^{m-1} \sum_{i+j=\ell} b_{ij} x^i y^j}$$

$$B_0 \ne 0 \qquad B_{m-1}(x,y)^{m-1} \not\equiv 0 \qquad A_{n-1}(x,y)^{n-1} \not\equiv 0$$

f irreducible

$$\Longleftrightarrow$$

$$D_{(u,v)}(n,m) := \det \left[f_{(u,v)}^{(n+k-\ell)} (x,y)^{n+k-\ell} \right]_{k,\ell=0}^{m-1} \equiv 0$$

for (u,v) in a neighbourhood of the origin

Proof. "$\Longrightarrow$": Because Theorem 4 is valid at any point (u,v) where f is analytic, it immediately yields the implication in this direction. Irrespective of the point

around which the series development is constructed, the homogeneous Padé approximant of degree $\nu = n-1$ in the numerator and $\mu = m-1$ in the denominator always reconstructs the function f.
"$\Leftarrow$": Let us denote $f^{(k)}_{(u,v)}$ by $\tilde{C}_k$. Take $y = \lambda x$ and $v = \lambda u$ and define $f_\lambda(x) = f(x, \lambda x)$. Then the series development of f_λ around $x = u$ is given by

$$\begin{aligned} f_\lambda(x) &= \sum_{k=0}^{\infty} \tilde{C}_k(x-u, \lambda(x-u))^k \\ &= \sum_{k=0}^{\infty} \left(\sum_{i+j=k} \tilde{c}_{ij}\, \lambda^j \right) (x-u)^k \end{aligned}$$

where

$$\tilde{c}_{ij} = \frac{1}{i!j!} \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(u, \lambda u)$$

Then $D_{(u,v)}(n,m) \equiv 0$ for (u,v) in a neighbourhood of the origin implies that (III) is valid for each $f_\lambda(x)$ and each u in a neighbourhood of the origin. This implies through (I) that for each λ, in other words for each y, f is a rational function of x of degree $n-1$ in the numerator and $m-1$ in the denominator because f_λ is. In the same way we can conclude with the role of x and y interchanged that f is a rational function of y, of the same degrees. Hence f is a rational function of (x,y) [BoMa48, pp. 201]. Taking $\lambda = 1$ implies moreover that the total degree of f cannot exceed $n-1$ in the numerator and $m-1$ in the denominator. □

Let us compare this result to the negative one of Sect. 2 by looking at the special case $m = 1$ again. When f is a polynomial in x and y of homogeneous degree $n-1$, then its series development in the origin reduces to

$$f(x,y) = \sum_{\ell=0}^{n-1} C_\ell(x,y)^\ell$$

which according to Theorem 4 indeed means

$$\forall \nu \geq n : C_n = f^{(n)}_{(0,0)} \equiv 0$$

According to Theorem 5 this is in its turn equivalent to the single condition

$$f^{(n)}_{(u,v)} \equiv 0 \qquad (u,v) \text{ in a neighbourhood of } (0,0)$$

The reason of this success lies in the fact that we are now dealing with Fréchet derivatives and not with partial derivatives.

A straightforward consequence of Theorem 5 is that, except possibly for a set of Lebesgue-measure zero, the multivariate homogeneous Padé table of f at (u,v) is normal.

Corollary. *Let f be analytic in a neighbourhood W of the origin and let f not be a rational function. Then for all $(u,v) \in W \setminus Z$ where Z is a set of Lebesgue-measure zero, and for all $n, m \geq 0$:*

$$D_{(u,v)}(n,m) \not\equiv 0 \ ,$$

meaning that the multivariate homogeneous Padé table of f at (u,v) is normal.

Proof. For $m = 0$, $D_{(u,v)}(n,m) = 1$ and in that case the statement is true. For $m > 0$, $D_{(u,v)}(n,m)$ is an analytic function that does not vanish identically in a neighbourhood of the origin. Its zeros $Z_{n,m}$ constitute a set of Lebesgue-measure zero [Rang86, pp. 31–39]. Hence the set

$$Z := \cup_{(n,m)\in\mathbb{N}^2} Z_{n,m}$$

is a set of Lebesgue-measure zero. Outside Z we have for all n and m that $D_{(u,v)}(n,m) \not\equiv 0$. Combined with the conditions for normality of the multivariate homogeneous Padé approximant as proved in [Cuyt84, pp. 51], the statement is true for all n and m. □

For the sake of completeness we recall the continuity result of the multivariate homogeneous Padé operator proved in [CuWW84], which now becomes an "almost everywhere" continuity result. To this end we introduce the concept of degree defect for the homogeneous case. Let $\omega q_{n,m}$ denote the order of the Padé denominator, in other words: the homogeneous nonzero term of smallest degree in the denominator of the irreducible form $r_{n,m}$ has degree $\omega q_{n,m}$. Because of the shift of degrees introduced for the computation of a solution to the homogeneous Padé approximation problem, it is not always true that for the irreducible form $\omega q_{n,m} = 0$. We define

$$\begin{aligned} n' &= \partial p_{n,m} - \omega q_{n,m} \\ m' &= \partial q_{n,m} - \omega q_{n,m} \\ \delta_{n,m} &= \min(n - n', m - m') \end{aligned}$$

and redefine

$$||f(x,y)||_{n+m} = \max_{0\le k\le n+m} \left|\left| f^{(k)}_{(0,0)} \right|\right|$$

Theorem 6. *If $\delta_{n,m} = 0$ and $q_{n,m}(x,y) \neq 0$ in some poly-interval I, then*

$$\forall\epsilon, \exists\delta : ||(f - \tilde{f})(x,y)||_{n+m} < \delta \Longrightarrow ||(r_{n,m} - \tilde{r}_{n,m})(x,y)||_I < \epsilon$$

where $r_{n,m}$ and $\tilde{r}_{n,m}$ are respectively approximating f and $\tilde{f}$ in the sense of (6–7).

Proof. The proof can be found in [CuWW84]. □

References

[AbCu93] Abouir, J., Cuyt, A. (1993): Multivariate partial Newton-Padé and Newton-Padé type approximants. J. Approx. Th. **72**. 301–316

[BoMa48] Bochner, S., Martin, W.T. (1948): Several complex variables. Princeton University Press, Princeton

[Cuyt82] Cuyt, A. (1982): The epsilon-algorithm and multivariate Padé approximants. Numer. Math. **40**, 39–46

[Cuyt83] Cuyt, A. (1983): The QD-algorithm and multivariate Padé approximants. Numer. Math. **42**, 259–269

[Cuyt84] Cuyt, A. (1984): Padé approximants for operators: theory and applications. LNM **1065**, Springer, Berlin

[Cuyt92] Cuyt, A. (1992): Padé approximation in one and more variables. In: Singh, S.P. (ed.) Approximation Theory, Spline Functions and Applications, Kluwer, Dordrecht. NATO ASI **356**, 41–68

[CuVe84] Cuyt, A., Verdonk, B. (1984): General order Newton-Padé approximants for multivariate functions. Numer. Math. **43**, 293–307

[CuWW84] Cuyt, A., Werner, H., Wuytack, L. (1984): On the continuity of the multivariate Padé operator. J. Comput. Appl. Math. **11**, 95–102

[Levi76] Levin, D. (1976): General order Padé type rational approximants defined from double power series. J. Inst. Math. Appl. **18**, 1–8

[Lubi88] Lubinsky, D. (1988): Power series equivalent to rational functions: a shifting origin Kronecker type theorem and normality of Padé tables. Numer. Math. **54**, 33–39

[Rall79] Rall, L.B. (1979): Computational solution of nonlinear operator equations. Krieger, Huntington

[Rang86] Rang, R.M. (1986): Holomorphic functions and integral representations in several complex variables. Springer, Berlin

[WeWu83] Werner, H., Wuytack, L. (1983): On the continuity of the Padé operator. SIAM J. Numer. Anal. **20**, 1273–1280

[Wuyt81] Wuytack, L. (1981): On the conditioning of the Padé approximation problem. In: Bruin, M., van Rossum, H. (eds.) LNM **888**, 78–89. Springer, Berlin

This article was processed by the author
using the Springer-Verlag TEX QPMZGHB macro package 1991.