Exploring multivariate Padé approximants for multiple hypergeometric series

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We investigate the approximation of some hypergeometric functions of two variables, namely the Appell functions F_i, $i = 1, \dots, 4$, by multivariate Padé approximants. Section 1 reviews the results that exist for the projection of the F_i onto $x = 0$ or $y = 0$, namely, the Gauss function ${}_2F_1(a, b; c; z)$, since a great deal is known about Padé approximants for this hypergeometric series. Section 2 summarizes the definitions of both homogeneous and general multivariate Padé approximants. In section 3 we prove that the table of homogeneous multivariate Padé approximants is normal under similar conditions to those that hold in the univariate case. In contrast, in section 4, theorems are given which indicate that, already for the special case $F_1(a, b, b'; c; x, y)$ with $a = b = b' = 1$ and $c = 2$, there is a high degree of degeneracy in the table of general multivariate Padé approximants. Section 5 presents some concluding remarks, highlighting the difference between the two types of multivariate Padé approximants in this context and discussing directions for future work.

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1. Introduction

There are four classes of functions, each called Appell series, that arise as the natural generalization to two variables of the Gauss hypergeometric function. For any positive integer i, let

$$(a)_i := \begin{cases} a(a+1)(a+2)\cdots(a+i-1), & i \geqslant 1, \\ 1, & i = 0. \end{cases} \tag{1}$$

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Then the Gauss or ordinary hypergeometric function is defined by [24]

$$ {}_2F_1(a,b;c;z) := \sum_{i=0}^{\infty} \frac{(a)_i(b)_i}{(c)_i i!} z^i, \tag{2} $$

where the parameters a, b, c and z may be real or complex. Some of the vast literature pertaining to the properties and applications of hypergeometric functions of all types, including the Gauss function, may be accessed via the bibliographies in [11,22,24].

Following Appell's approach, if we consider the simple product of two Gauss functions

$$ {}_2F_1(a,b;c;x)\,{}_2F_1\bigl(a',b';c';y\bigr) = \sum_{i,j=0}^{\infty} \frac{(a)_i(a')_j(b)_i(b')_j x^i y^j}{(c)_i(c')_j i! j!} $$

and replace pairs of products $(a)_i(a')_j$ by the composite product $(a)_{i+j}$, four new functions of two variables are generated. These four functions bear Appell's name and are defined by

$$ F_1\bigl(a,b,b';c;x,y\bigr) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_i(b')_j x^i y^j}{(c)_{i+j} i! j!}, \tag{3} $$

$$ F_2\bigl(a,b,b';c,c';x,y\bigr) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_i(b')_j x^i y^j}{(c)_i(c')_j i! j!}, \tag{4} $$

$$ F_3\bigl(a,a',b,b';c;x,y\bigr) = \sum_{i,j=0}^{\infty} \frac{(a)_i(a')_j(b)_i(b')_j x^i y^j}{(c)_{i+j} i! j!}, \tag{5} $$

$$ F_4\bigl(a,b;c,c';x,y\bigr) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_{i+j} x^i y^j}{(c)_i(c')_j i! j!}. \tag{6} $$

All four Appell functions reduce to the Gauss function if one of the variables is equal to zero. Moreover [24], we have that

$$ F_1\bigl(a,b,b';c;x,x\bigr) = {}_2F_1\bigl(a,b+b';c;x\bigr). \tag{7} $$

The study of generalized hypergeometric functions of several variables has been extensive due to their frequent occurrence in the solution of statistical and physical problems [11].

In several papers [10,21,25,26], properties of Padé approximants for the univariate function ${}_2F_1(a,b;c;z)$ have been studied. The Padé approximation problem of order (n,m) for a function f, given by its formal Taylor series expansion

$$ f(z) = c_0 + c_1 z + c_2 z^2 + \cdots, \quad c_0 \neq 0, \tag{8} $$

consists in finding polynomials

$$p(z) = \sum_{i=0}^{n} a_i z^i \quad \text{and} \quad q(z) = \sum_{i=0}^{m} b_i z^i$$

such that in the power series $(fq - p)(z)$ the coefficients of z^i for $i = 0, \dots, n+m$ disappear, in other words,

$$\partial p \leqslant n, \qquad \partial q \leqslant m, \qquad \omega(fq - p) \geqslant n + m + 1, \tag{9}$$

where ∂p denotes the exact degree of a polynomial p and ωp stands for the order of a power series p. Condition (9) is equivalent to the following two systems of linear equations:

$$\begin{cases} c_0 b_0 = a_0, \\ c_1 b_0 + c_0 b_1 = a_1, \\ \quad \vdots \\ c_n b_0 + c_{n-1} b_1 + \cdots + c_{n-m} b_m = a_n, \end{cases} \tag{10a}$$

$$\begin{cases} c_{n+1} b_0 + c_n b_1 + \cdots + c_{n-m+1} b_m = 0, \\ \quad \vdots \\ c_{n+m} b_0 + c_{n+m-1} b_1 + \cdots + c_n b_m = 0 \end{cases} \tag{10b}$$

with $c_i = 0$ for $i < 0$. Since the system (10b) is a homogeneous system of m equations in $m + 1$ unknowns, it has at least one nontrivial solution. Moreover, all nontrivial solutions of (10) supply the same irreducible form after normalization [2]. If $p(z)$ and $q(z)$ satisfy (10) we shall denote by $[n/m]^f = p_{n,m}/q_{n,m}$ the irreducible form of p/q normalized such that $q_{n,m}(0) = 1$. This rational function $[n/m]^f(z)$ is called the Padé approximant of order (n, m) for $f(z)$. When no confusion is possible, we shall omit the superscript f in $[n/m]^f$.

The Padé approximants $[n/m]$ for f can be ordered in a table for different values of n and m:

$$\begin{array}{ccccc} [0/0] & [0/1] & [0/2] & [0/3] & \dots \\ [1/0] & [1/1] & [1/2] & [1/3] & \dots \\ [2/0] & [2/1] & [2/2] & [2/3] & \dots \\ \vdots & \vdots & \vdots & \ddots & \end{array}$$

This table is called the Padé table of f. The first column consists of the partial sums of f. The first row contains the reciprocals of the partial sums of $1/f$. We call a Padé approximant normal if it occurs only once in the Padé table and we call the Padé table normal if each entry in the table is normal.

Several generalizations of the concept of univariate Padé approximant exist for multivariate functions. In section 2 we recall two important approaches, which will be used to construct Padé approximants for the Appell functions in sections 3 and 4. Let us first review the results which exist for the Padé approximants of the univariate function ${}_2F_1(a, b; c; z)$. These results are summarized below.

Theorem 1.1 [10]. The Padé table for the hypergeometric series ${}_2F_1(a,1;c;z)$ with $c > a > 0$ is normal.

Theorem 1.2 [10,26]. The Padé approximants $[n/m]$ for the hypergeometric function ${}_2F_1(a,1;c;z)$ with $m \leqslant n$ and $a, c, c-a \notin \mathbb{Z}^-$ are normal.

Theorem 1.3 [21,25]. Let $c \notin \mathbb{Z}^-$ and let $m \leqslant n+1$. Then the denominator of the (n,m) Padé approximant for the function ${}_2F_1(a,1;c;z)$ is given by

$$q_{n,m}(z) = {}_2F_1(-m, -a-n; -c-n-m+1; z).$$

2. Multivariate Padé approximants

During the last two decades many efforts have been made to generalize the concept of univariate Padé approximant to the multivariate case. In section 2.1 we shall first review the definition of general multivariate Padé approximants. This definition includes all definitions for multivariate Padé approximants based on the use of a linear system of defining equations for the numerator and denominator coefficients, such as those found in [4,7,12,15,16,18,19]. In section 2.2 we then consider the so-called homogeneous multivariate Padé approximants [6], which have the advantage that they preserve the properties and the nature of univariate Padé approximants even better than the general Padé approximants. This will also turn out to be the case when looking at the general and homogeneous multivariate Padé approximants for the Appell functions F_i. Other ways to define multivariate Padé approximants can be found in [3], where an algebraic approach is taken, and in [9,17,20,23], where the generalization is based on the use of branched continued fractions. However, we shall not consider these multivariate Padé approximants here.

In the sequel of this section we restrict ourselves to the case of two variables because the generalization to functions of more variables is more difficult only in notation.

2.1. General multivariate Padé approximants

Let a bivariate function $f(x,y)$ be given by the formal Taylor series expansion

$$f(x,y) = \sum_{(i,j)\in\mathbb{N}^2} c_{ij} x^i y^j$$

with

$$c_{ij} = \frac{1}{i!}\frac{1}{j!}\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\bigg|_{(0,0)}.$$

Let N, D and E be finite subsets of $\mathbb{N}^2$ such that

$$N \subseteq E, \tag{11a}$$

$$\#(E \backslash N) = m = \#D - 1, \tag{11b}$$

$$E \text{ satisfies the inclusion property,} \tag{11c}$$

where (11c) means that when a point belongs to the index set E, the rectangular subset of points emanating from the origin with the given point as its furthermost corner is also contained in E. Our purpose is to determine polynomials $p(x,y)$ and $q(x,y)$ of the form

$$p(x,y) = \sum_{(i,j)\in N} a_{ij} x^i y^j, \tag{12a}$$

$$q(x,y) = \sum_{(i,j)\in D} b_{ij} x^i y^j \tag{12b}$$

such that

$$(fq - p)(x,y) = \sum_{(i,j)\in \mathbb{N}^2 \backslash E} d_{ij} x^i y^j. \tag{12c}$$

Condition (11a) enables us to split the system of equations

$$d_{ij} = 0, \quad (i,j) \in E,$$

into an inhomogeneous linear system defining the numerator coefficients a_{ij},

$$\sum_{\mu=0}^{i} \sum_{\nu=0}^{j} c_{\mu\nu} b_{i-\mu,j-\nu} = a_{ij}, \quad (i,j) \in N, \tag{13}$$

and a homogeneous linear system defining the denominator coefficients b_{ij},

$$\sum_{\mu=0}^{i} \sum_{\nu=0}^{j} c_{\mu\nu} b_{i-\mu,j-\nu} = 0, \quad (i,j) \in E \backslash N. \tag{14}$$

By convention $b_{kl} = 0$ for $(k,l) \notin D$. Condition (11c) takes care of the Padé approximation property, namely

$$\left(f - \frac{p}{q}\right)(x,y) = \sum_{(i,j)\in \mathbb{N}^2 \backslash E} e_{ij} x^i y^j$$

provided $q(0,0) \neq 0$. Therefore we call the rational function p/q, with p, q satisfying (12), a general multivariate Padé approximant of order (N, D, E) to $f(x,y)$ and we denote it by $[N/D]_E^f$. The superscript f is usually omitted when no confusion can arise. It is clear from (11b) that a nontrivial general multivariate Padé approximant always exists and that it will be unique up to a constant factor in the numerator and denominator if the coefficient matrix of the linear system (14) has maximal rank. In this case the general Padé approximant $[N/D]_E$ is called nondegenerate. If the rank of (14) is not maximal then multiple solutions of (12) exist and we refer to [1] for a discussion of the relationships between these solutions and solutions of Padé approximation problems of "higher" degree or "higher" order. For all definitions covered by the general definition given here, one cannot guarantee the existence of a "unique" irreducible form if multiple solutions of (12) exist.

In order to define a table of general multivariate Padé approximants, we impose an enumeration on the index sets N, D and E:

$$N = \{(i_0, j_0), \dots, (i_n, j_n)\} \equiv N_n,$$
$$D = \{(d_0, e_0), \dots, (d_m, e_m)\} \equiv D_m,$$
$$E = N \cup \{(i_{n+1}, j_{n+1}), \dots, (i_{n+m}, j_{n+m})\} = E_{n+m}.$$

The numbering imposed on the sets N, D and E now makes it possible to set up a general multivariate Padé table where the column index indicates the denominator "degree", i.e., $\#D$, and the row index indicates the numerator "degree", i.e., $\#N$:

$$\begin{array}{cccc} [N_0/D_0]_{E_0} & [N_0/D_1]_{E_1} & [N_0/D_2]_{E_2} & \dots \\ [N_1/D_0]_{E_1} & [N_1/D_1]_{E_2} & [N_1/D_2]_{E_3} & \dots \\ [N_2/D_0]_{E_2} & [N_2/D_1]_{E_3} & [N_2/D_2]_{E_4} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

It is clear that a table of general multivariate Padé approximants depends on the numbering chosen for the index sets N, D and E. Interchanging index points influences columns or rows in the table, but not the intrinsic properties of particular approximants [8,13,14]. The most frequently used enumerations are the "triangle" numbering and the "square" numbering. In the triangle numbering, index points are ordered along upward sloping diagonals in $\mathbb{N}^2$, namely

$$\begin{array}{cccccccccc} (0,0) & (1,0) & (0,1) & (2,0) & (1,1) & (0,2) & (3,0) & (2,1) & (1,2) & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots \end{array}$$

while the square numbering is carried out along squares in $\mathbb{N}^2$:

$$\begin{array}{cccccccccc} (0,0) & (1,0) & (0,1) & (1,1) & (2,0) & (2,1) & (0,2) & (1,2) & (2,2) & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots \end{array}$$

2.2. *Homogeneous multivariate Padé approximants*

The approach we have taken in the previous section to define and construct multivariate Padé approximants is essentially based on rewriting the double series expansion

$$\sum_{(i,j)\in\mathbb{N}^2} c_{ij} x^i y^j \tag{15}$$

as the single sum

$$\sum_{r(i,j)=0}^{+\infty} c_{ij} x^i y^j,$$

where a numbering $r(i,j)$ of $\mathbb{N}^2$ places the points in $\mathbb{N}^2$ one after the other. By doing so, the dimension of the problem description is reduced: the input is indexed by integer numbers $r(i,j) \in \mathbb{N}$ and not by multi-indices $(i,j) \in \mathbb{N}^2$.

Another way to work with the bivariate power series (15) is the following. Rewrite

$$\sum_{(i,j)\in\mathbb{N}^2} c_{ij} x^i y^j = \sum_{\ell=0}^{\infty} \left(\sum_{i+j=\ell} c_{ij} x^i y^j \right).$$

This approach is taken in [6, p. 59–62] to construct homogeneous multivariate Padé approximants. For chosen ν and μ, which are the homogeneous counterparts of the univariate degrees n and m in (9), we introduce the notations

$$\begin{aligned}
A_\ell(x,y) &= \sum_{i+j=\nu\mu+\ell} a_{ij} x^i y^j, \quad && \ell = 0,\ldots,\nu, \\
B_\ell(x,y) &= \sum_{i+j=\nu\mu+\ell} b_{ij} x^i y^j, \quad && \ell = 0,\ldots,\mu, \\
C_\ell(x,y) &= \sum_{i+j=\ell} c_{ij} x^i y^j, && \ell = 0,1,2\ldots.
\end{aligned}$$

Let a function $f(x,y)$ again be given by its formal Taylor series expansion

$$f(x,y) = \sum_{\ell=0}^{\infty} C_\ell(x,y).$$

Then the homogeneous multivariate Padé approximation problem consists in finding polynomials

$$p(x,y) = \sum_{\ell=0}^{\nu} A_\ell(x,y), \tag{16a}$$

$$q(x,y) = \sum_{\ell=0}^{\mu} B_\ell(x,y) \tag{16b}$$

such that

$$(fq-p)(x,y) = \sum_{i+j\geqslant\nu\mu+\nu+\mu+1} d_{ij} x^i y^j. \tag{16c}$$

This problem can be reformulated as

$$\begin{cases} C_0(x,y)B_0(x,y) \equiv A_0(x,y), \\ C_1(x,y)B_0(x,y) + C_0(x,y)B_1(x,y) \equiv A_1(x,y), \\ \quad \vdots \\ C_\nu(x,y)B_0(x,y) + \cdots + C_{\nu-\mu}(x,y)B_\mu(x,y) \equiv A_\nu(x,y), \end{cases}$$

$$\begin{cases} C_{\nu+1}(x,y)B_0(x,y) + \cdots + C_{\nu+1-\mu}(x,y)B_\mu(x,y) \equiv 0, \\ \quad \vdots \\ C_{\nu+\mu}(x,y)B_0(x,y) + \cdots + C_\nu(x,y)B_\mu(x,y) \equiv 0, \end{cases}$$

where $C_\ell(x,y) \equiv 0$ if $\ell < 0$. This is exactly the system of defining equations for univariate Padé approximants if the univariate terms $c_\ell x^\ell$, $a_\ell x^\ell$ and $b_\ell x^\ell$ are substituted, respectively, by $C_\ell(x,y)$, $A_\ell(x,y)$ and $B_\ell(x,y)$.

It is proven in [6, p. 60] that a nontrivial solution of (16) always exists and that, in contrast to the general multivariate Padé approximation problem, all solutions of (16) have the same irreducible form. If $p(x,y)$ and $q(x,y)$ satisfy (16), we can therefore denote by $[\nu/\mu]_H^f = p_{\nu,\mu}/q_{\nu,\mu}$ the properly normalized irreducible form of p/q. The rational function $[\nu/\mu]_H^f$, or $[\nu/\mu]_H$ when no confusion is possible, is called the homogeneous multivariate Padé approximant of order (ν,μ) for f. Here the subscript H does not refer to an equation index set but merely to the adjective "homogeneous". As in the univariate case, the homogeneous multivariate Padé approximant can be ordered in a table for different values of ν and μ:

$$\begin{array}{cccc} [0/0]_H & [0/1]_H & [0/2]_H & \dots \\ [1/0]_H & [1/1]_H & [1/2]_H & \dots \\ [2/0]_H & [2/1]_H & [2/2]_H & \dots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Each entry $[\nu/\mu]_H$ in this table is a homogeneous multivariate Padé approximant of which the numerator and denominator polynomials have homogeneous degree at most $\nu\mu+\nu$, respectively $\nu\mu+\mu$. As in the univariate case, the order (ν,μ) of the homogeneous Padé approximant $[\nu/\mu]_H$ completely determines its row and column index in the table of homogeneous Padé approximants. This is in contrast with the general Padé approximation table where the order (N,D,E) of the general Padé approximant $[N/D]_E$ is not sufficient to determine its row and column index, since here also the numbering $r(i,j)$ of the multi-indices plays a role.

3. Homogeneous Padé approximants for the Appell series

After reviewing some existing results for Padé approximants of the Gauss hypergeometric function ${}_2F_1(a,1;c;z)$ in section 1, we want to discuss the generalization of these theorems to homogeneous multivariate Padé approximants for the Appell functions. In this respect the projection property of the homogeneous multivariate Padé approximants plays an essential role. To formulate this property we denote for $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, the projection operator by

$$\mathcal{P}_\lambda : f(x,y) \to f(\lambda_1 z, \lambda_2 z) = \mathcal{P}_\lambda(f)(z).$$

Choosing $\lambda = (1,0)$, respectively $(0,1)$, amounts to projecting on $y = 0$, respectively $x = 0$. The following property holds.

Theorem 3.1 [5,6]. Let $[\nu/\mu]_H^f(x,y)$ be the homogeneous Padé approximant of order (ν,μ) for $f(x,y)$. Then for each $\lambda = (\lambda_1,\lambda_2) \in \mathbb{C}^2$, the irreducible form of $\mathcal{P}_\lambda([\nu/\mu]_H^f)$ equals $[\nu/\mu]^{\mathcal{P}_\lambda(f)}$, which is the univariate Padé approximant of order (ν,μ) for $\mathcal{P}_\lambda(f)$.

This projection property, together with the fact that all four Appell series reduce to the Gauss function if one of the variables x or y equals zero, and the results on univariate Padé approximants for the Gauss function ${}_2F_1(a,1;c;z)$, leads to the following theorems.

Theorem 3.2. The table of homogeneous multivariate Padé approximants for each of the following Appell functions:

$$F_1\big(a,1,b';c;x,y\big), \qquad c > a > 0, \tag{17a}$$
$$F_1(a,b,1;c;x,y), \qquad c > a > 0, \tag{17b}$$
$$F_1\big(a,b,b';c;x,y\big), \qquad c > a > 0,\ b + b' = 1, \tag{17c}$$
$$F_2\big(a,1,b';c,c';x,y\big), \qquad c > a > 0, \tag{17d}$$
$$F_2\big(a,b,1;c,c';x,y\big), \qquad c' > a > 0, \tag{17e}$$
$$F_3\big(a,a',1,b';c;x,y\big), \qquad c > a > 0, \tag{17f}$$
$$F_3\big(a,a',b,1;c;x,y\big), \qquad c > a' > 0, \tag{17g}$$
$$F_4\big(a,1;c,c';x,y\big), \qquad c > a > 0 \text{ or } c' > a > 0, \tag{17h}$$

is normal.

Proof. We shall give a proof only for (17a) since a proof for all other Appell functions (17b)–(17h) is completely similar. Let

$$g \equiv F_1\big(a,1,b';c;x,y\big) \quad \text{with } c > a > 0$$

and assume that the table of homogeneous multivariate Padé approximants for g is not normal. This implies that there exist $(\nu_1,\mu_1),(\nu_2,\mu_2) \in \mathbb{N}^2$ with $(\nu_1,\mu_1) \neq (\nu_2,\mu_2)$ such that

$$[\nu_1/\mu_1]_H^g \equiv [\nu_2/\mu_2]_H^g$$

and hence

$$\mathcal{P}_\lambda\big([\nu_1/\mu_1]_H^g\big) \equiv \mathcal{P}_\lambda\big([\nu_2/\mu_2]_H^g\big) \tag{18}$$

for any $\lambda \in \mathbb{C}^2$. Because of the projection property of homogeneous Padé approximants, (18) implies

$$[\nu_1/\mu_1]^{\mathcal{P}_\lambda(g)} = [\nu_2/\mu_2]^{\mathcal{P}_\lambda(g)}. \tag{19}$$

Since, for $\lambda = (1,0)$, we have that

$$\mathcal{P}_{(1,0)}(g) = {}_2F_1(a,1;c;z) \quad \text{with } c > a > 0, \tag{20}$$

equation (19) is in contradiction with the fact that, according to theorem 1.1, the Padé table for (20) is normal. This completes the proof. ☐

Theorem 3.3. The homogeneous multivariate Padé approximants $[\nu/\mu]_H$ of order (ν,μ) with $\mu \leqslant \nu$, for each of the following Appell functions:

$$\begin{array}{ll}
F_1\bigl(a,1,b';c;x,y\bigr), & a,c,c-a \notin \mathbb{Z}^-,\\
F_1(a,b,1;c;x,y), & a,c,c-a \notin \mathbb{Z}^-,\\
F_1\bigl(a,b,b';c;x,y\bigr), & a,c,c-a \notin \mathbb{Z}^-,\ b+b'=1,\\
F_2\bigl(a,1,b';c,c';x,y\bigr), & a,c,c-a \notin \mathbb{Z}^-,\\
F_2\bigl(a,b,1;c,c';x,y\bigr), & a,c',c'-a \notin \mathbb{Z}^-,\\
F_3\bigl(a,a',1,b';c;x,y\bigr), & a,c,c-a \notin \mathbb{Z}^-,\\
F_3\bigl(a,a',b,1;c;x,y\bigr), & a',c,c-a' \notin \mathbb{Z}^-,\\
F_4\bigl(a,1;c,c';x,y\bigr), & a,c,c-a \notin \mathbb{Z}^- \text{ or } a,c',c'-a \notin \mathbb{Z}^-,
\end{array}$$

are normal.

Proof. Our proof is again by contradiction and completely similar to that of theorem 3.2. ☐

4. General Padé approximants for the Appell series

In this section we attempt to generalize the results that exist for Padé approximants of the Gauss function ${}_2F_1(a,1;c;z)$ to general multivariate Padé approximants for the Appell series. Unfortunately, unlike for homogeneous Padé approximants, a projection property only partially holds for general multivariate Padé approximants. The following theorem makes this more explicit. Let us introduce, for any finite subset S of $\mathbb{N}^2$, the notations

$$S_x = \max\bigl\{i \mid (i,j) \in S\bigr\}, \qquad S_y = \max\bigl\{j \mid (i,j) \in S\bigr\}$$

and, as special cases of the projection operator $\mathcal{P}$,

$$\mathcal{P}_x = \mathcal{P}_{(1,0)}, \qquad \mathcal{P}_y = \mathcal{P}_{(0,1)}.$$

Theorem 4.1 [16]. Let the solution of the general multivariate Padé approximation problem (12) of order (N,D,E) for $f(x,y)$ be nondegenerate. Then:

(1) If and only if $E_x \geqslant N_x + D_x$, the irreducible form of $\mathcal{P}_x([N/D]_E^f)$ equals $[N_x/D_x]^{\mathcal{P}_x(f)}$, which is the univariate Padé approximant of order (N_x, D_x) for $\mathcal{P}_x(f)$.

(2) If and only if $E_y \geqslant N_y + D_y$, the irreducible form of $\mathcal{P}_y([N/D]_E^f)$ equals $[N_y/D_y]^{\mathcal{P}_y(f)}$, which is the univariate Padé approximant of order (N_y, D_y) for $\mathcal{P}_y(f)$.

Even when the conditions stated in theorem 4.1 are satisfied, it should be noted that two general multivariate Padé approximants of different orders, $[N^1/D^1]_{E^1}$ and $[N^2/D^2]_{E^2}$, can be projected on a univariate Padé approximant of the same order. This easily follows from the fact that for different degree sets S^1 and S^2, one can have $S_x^1 = S_x^2$. From these considerations it should be clear that one cannot, as for homogeneous Padé approximants, deduce results on the normality of general Padé approximants for the Appell series, based on a projection property.

In order to deduce the (non-)normality of general multivariate Padé approximants to the Appell series, we therefore need to take a different approach. To this end, we first study general multivariate Padé approximants for the simplest of all Appell series, namely the function $F_1(1,1,1;2;x,y)$, of which the projection $\mathcal{P}_x(F_1(1,1,1;2;x,y))$ satisfies $c > a > 0$ since $c = 2$ and $a = 1$. We know that for $|x| < 1$, $|y| < 1$ and $x \neq y$, we have

$$F_1(1,1,1;2;x,y) = \sum_{i,j=0}^{+\infty} \frac{x^i y^j}{i+j+1} = \frac{\ln(1-x) - \ln(1-y)}{y-x}. \tag{21}$$

Based on the Taylor series expansion (21), it is easy to construct different general multivariate Padé approximants to the Appell series $F_1(1,1,1;2;x,y)$. The results of these computations are given in section 5. However, more generally one can state the following theorems, which say that a significant amount of degenerate and hence non-normal behaviour is present in a table of general multivariate Padé approximants for the function $F_1(1,1,1;2;x,y)$. Surprising though this may seem, it should be clear from the discussion on the projection property for general multivariate Padé approximants that this is not in contradiction with the fact that the Padé table for the univariate function $\mathcal{P}_x(F_1(1,1,1;2;x,y))$ is normal.

In the sequel, the index sets N and E are always enumerated using the triangle numbering, while we shall allow both triangle and square enumeration for the denominator index set D, denoted respectively by $D^{(t)}$ for triangle and $D^{(s)}$ for square. Theorem 4.2 below discusses the table of general multivariate Padé approximants for the function $F_1(1,1,1;2;x,y)$, where the index set D is enumerated according to the triangle numbering. We then formulate a similar theorem for the table of general multivariate Padé approximants, where the denominator index set is enumerated according to the square numbering.

If (i, j) is the mth point in $\mathbb{N}^2$ according to some numbering $r(i, j)$, in other words $r(i, j) = m$, we let

$$\delta_m = i + j, \qquad \omega_m = j.$$

In other words, δ_m indicates the diagonal on which the mth point in the numbering of $\mathbb{N}^2$ is located and ω_m is the ordinate of the mth point in the numbering. Any index set $D_m^{(t)}$ with $m \geqslant 0$ is then uniquely characterized by the two integers δ_m and ω_m as follows:

$$D_m^{(t)} = \big\{(i, j) \mid 0 \leqslant i+j \leqslant \delta_m - 1\big\} \cup \big\{(\delta_m, 0), (\delta_m - 1, 1), \ldots, (\delta_m - \omega_m, \omega_m)\big\}. \quad (22)$$

It is easy to verify that when the triangle numbering is used for $\mathbb{N}^2$, the following relation holds between m, δ_m and ω_m:

$$m + 1 = \frac{\delta_m(\delta_m + 1)}{2} + \omega_m + 1, \quad 0 \leqslant \omega_m \leqslant \delta_m. \quad (23)$$

Theorem 4.2. Let the triangle numbering be used to enumerate the sets E, N and D and let

$$n(m) = \begin{cases} 2\delta_m(\delta_m + 1) - m, & 0 \leqslant \omega_m \leqslant \delta_m - 1, \\ 2\delta_m(\delta_m + 2) - m + 2, & \omega_m = \delta_m. \end{cases}$$

Then, in the general multivariate Padé table for $F_1(1, 1, 1; 2; x, y)$ we have:

- If $2 \leqslant m$, then already the entry $[N_{n(m)}/D_m^{(t)}]$ is degenerate.
- If $8 \leqslant m$, then, moreover, the mth column contains only a finite number of uniquely determined general multivariate Padé approximants since all entries $[N_k/D_m^{(t)}]$, $k > n(m)$, are also degenerate.

Proof. For fixed $m \geqslant 0$, the matrix determining the denominator coefficients of $[N_k/D_m^{(t)}]_{E_{k+m}}$, $k \geqslant 0$, is given by

$$M = \begin{pmatrix} c_{i_{k+1}-d_0, j_{k+1}-e_0} & c_{i_{k+1}-d_1, j_{k+1}-e_1} & \cdots & c_{i_{k+1}-d_m, j_{k+1}-e_m} \\ c_{i_{k+2}-d_0, j_{k+2}-e_0} & c_{i_{k+2}-d_1, j_{k+2}-e_1} & \cdots & c_{i_{k+2}-d_m, j_{k+2}-e_m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{i_{k+m}-d_0, j_{k+m}-e_0} & c_{i_{k+m}-d_1, j_{k+m}-e_1} & \cdots & c_{i_{k+m}-d_m, j_{k+m}-e_m} \end{pmatrix}.$$

We will now determine for which values of k this matrix is rank-deficient, i.e., which multivariate Padé approximants in the mth column are degenerate. To this end we recall that the Taylor coefficients of $F_1(1, 1, 1; 2; x, y)$ are given by

$$c_{ij} = \begin{cases} \dfrac{1}{1 + i + j}, & i \geqslant 0,\ j \geqslant 0, \\ 0, & \text{otherwise}. \end{cases}$$

Given these Taylor coefficients and assuming that $m \geqslant 2$, it is clear that if two rows l_1 and l_2 in the matrix M are such that both

(1) (i_{k+l_1}, j_{k+l_1}) and (i_{k+l_2}, j_{k+l_2}) are on a same diagonal, i.e.,

$$\begin{aligned}&\{(i_{k+l_1}, j_{k+l_1}), (i_{k+l_2}, j_{k+l_2})\} \subseteq \text{diag}(\lambda) = \{(i,j) \mid i+j=\lambda\},\\ &\quad \text{for some } \lambda \geqslant 0,\end{aligned} \tag{24}$$

and

(2) all elements in rows l_1 and l_2 are different from zero, i.e.,

$$\begin{aligned}c_{i_{k+l_1}-d_s, j_{k+l_1}-e_s} &\neq 0, \quad s=0,\ldots,m,\\ c_{i_{k+l_2}-d_s, j_{k+l_2}-e_s} &\neq 0, \quad s=0,\ldots,m,\end{aligned} \tag{25}$$

then rows l_1 and l_2 in M are identical and hence the matrix M is rank-deficient. Condition (25) is satisfied if and only if

$$\begin{aligned}i_{k+l_1} \geqslant d_s, \quad j_{k+l_1} \geqslant e_s, \quad s=0,\ldots,m,\\ i_{k+l_2} \geqslant d_s, \quad j_{k+l_2} \geqslant e_s, \quad s=0,\ldots,m.\end{aligned} \tag{26}$$

We now first consider a fixed mth column in the table of general multivariate Padé approximants, with m such that $0 \leqslant \omega_m \leqslant \delta_m - 1$. Afterwards we shall consider the case where m is such that $\omega_m = \delta_m$. So, assuming $0 \leqslant \omega_m \leqslant \delta_m - 1$, it is easy to see that, because of (22), condition (26) is equivalent to

$$\begin{aligned}i_{k+l_1} \geqslant \delta_m, \qquad j_{k+l_1} \geqslant \delta_m - 1,\\ i_{k+l_2} \geqslant \delta_m, \qquad j_{k+l_2} \geqslant \delta_m - 1.\end{aligned} \tag{27}$$

Joining conditions (24) and (27), one can now verify that the matrix M is rank-deficient if the set

$$E_{k+m} \setminus N_k = \{(i_{k+1}, j_{k+1}), (i_{k+2}, j_{k+2}), \ldots, (i_{k+m}, j_{k+m})\},$$

which indexes the rows of the matrix M, satisfies

$$\#\big(E_{k+m} \setminus N_k \cap \text{diag}(\lambda) \cap \{(i,j) \mid i \geqslant \delta_m,\ j \geqslant \delta_m - 1\}\big) \geqslant 2 \tag{28}$$

for some $\lambda \geqslant 0$. Letting $n_1(m)$ denote the smallest value of k for which (28) is satisfied, we find that

$$\begin{aligned}E_{n_1(m)+m} = {} & \{(i,j) \mid 0 \leqslant i+j \leqslant 2\delta_m - 1\}\\ & \cup \{(2\delta_m, 0), \ldots, (\delta_m + 1, \delta_m - 1), (\delta_m, \delta_m)\},\end{aligned} \tag{29}$$

where we have used the fact that the triangle numbering is used to enumerate the equation set E. From (29) the value of $n_1(m)$ can be deduced

$$\#E_{n_1(m)+m} = n_1(m) + m + 1 = \frac{2\delta_m(2\delta_m + 1)}{2} + \delta_m + 1$$

and hence

$$n_1(m) = 2\delta_m(\delta_m + 1) - m.$$

So far we can conclude that in the mth column of the Padé table, with $m \geqslant 2$ and $0 \leqslant \omega_m \leqslant \delta_m - 1$, the Padé approximant $[N_{n_1(m)}/D_m]$ is degenerate. The question that we shall now answer is whether the elements $[N_k/D_m]$ are also degenerate for all $k > n_1(m)$. Assume there exists $k > n_1(m)$ such that $[N_k/D_m]$ is nondegenerate. Then according to (28) and (29), we must have that

$$\forall\lambda \geqslant 0:\quad \#\bigl(E_{k+m} \setminus N_k \cap \mathrm{diag}(2\delta_m + \lambda) \cap \bigl\{(i,j) \mid i \geqslant \delta_m,\ j \geqslant \delta_m - 1\bigr\}\bigr) < 2$$

or, equivalently,

$$\forall\lambda \geqslant 0:\quad \#\bigl(E_{k+m} \setminus N_k \cap \bigl\{(\delta_m + 1 + \lambda, \delta_m - 1), (\delta_m + \lambda, \delta_m), \ldots, (\delta_m, \delta_m + \lambda)\bigr\}\bigr) < 2.$$

Using the fact that the triangle numbering is used to enumerate the elements in E, this condition amounts to

$$\exists\lambda \geqslant 0:\quad E_{k+m} \setminus N_k \subseteq \bigl\{(\delta_m, \delta_m + \lambda), \ldots, (0, 2\delta_m + \lambda),\\ (2\delta_m + 1 + \lambda, 0), \ldots, (\delta_m + 2 + \lambda, \delta_m - 1)\bigr\}. \tag{30}$$

Since $\#(E_{k+m} \setminus N_k) = m$, condition (30) can never be satisfied when $m > 2\delta_m + 1$. This leads to the conclusion that if m is such that $0 \leqslant \omega_m \leqslant \delta_m - 1$ and

- $m > 2\delta_m + 1$, or, equivalently, $m \geqslant 8$, then all multivariate Padé approximants $[N_k/D_m]_{E_{k+m}}$ with $k \geqslant n_1(m)$ are degenerate,
- $2 \leqslant m \leqslant 2\delta_m + 1$, or, equivalently, $2 \leqslant m < 8$, then a necessary condition for the Padé approximant $[N_k/D_m]_{E_{k+m}}$ with $k > n_1(m)$ to be uniquely determined is that E_{k+m} satisfies (30).

This completes the proof for the case $0 \leqslant \omega_m \leqslant \delta_m - 1$. For the case where m is such that $\omega_m = \delta_m$ the proof is completely similar. Taking into account (22), condition (26) is now equivalent to

$$\begin{aligned} i_{k+l_1} &\geqslant \delta_m, & j_{k+l_1} &\geqslant \delta_m,\\ i_{k+l_2} &\geqslant \delta_m, & j_{k+l_2} &\geqslant \delta_m. \end{aligned} \tag{31}$$

Combining conditions (24) and (31), the matrix M is rank-deficient if

$$\#\bigl(E_{k+m} \setminus N_k \cap \mathrm{diag}(\lambda) \cap \bigl\{(i,j) \mid i \geqslant \delta_m,\ j \geqslant \delta_m\bigr\}\bigr) \geqslant 2 \tag{32}$$

for some $\lambda \geqslant 0$. Letting $n_2(m)$ denote the smallest value of k for which (32) is satisfied, we find that

$$E_{n_2(m)+m} = \bigl\{(i,j) \mid 0 \leqslant i + j \leqslant 2\delta_m\bigr\} \cup \bigl\{(2\delta_m + 1, 0), \ldots, (\delta_m + 1, \delta_m), (\delta_m, \delta_m + 1)\bigr\}$$

and hence

$$n_2(m) = 2\delta_m(\delta_m + 2) - m + 2.$$

Again, we can conclude that in the mth column of the Padé table, with m such that $\omega_m = \delta_m$, the Padé approximant $[N_{n_2(m)}/D_m]$ is degenerate. From (32) we can also conclude that for $k > n_2(m)$, $[N_k/D_m]_{E_{k+m}}$ is nondegenerate only if E_{k+m} satisfies

$$\forall\lambda \geqslant 0 \quad \#\bigl(E_{k+m} \setminus N_k \cap \text{diag}(2\delta_m + 1 + \lambda) \cap \bigl\{(i,j) \mid i \geqslant \delta_m,\ j \geqslant \delta_m\bigr\}\bigr) < 2$$

or, equivalently,

$$\begin{aligned}\exists\lambda \geqslant 0\colon \quad E_{k+m} \setminus N_k \subseteq \bigl\{&(\delta_m, \delta_m + 1 + \lambda), \ldots, (0, 2\delta_m + 1 + \lambda),\\ &(2\delta_m + 2 + \lambda, 0), \ldots, (\delta_m + 2 + \lambda, \delta_m)\bigr\}.\end{aligned} \tag{33}$$

Since $\#(E_{k+m} \setminus N_k) = m$, condition (33) can never be satisfied when $m > 2\delta_m + 2$, from which we conclude that if m is such that $\omega_m = \delta_m$ and

- $m > 2\delta_m + 2$, or, equivalently, $m \geqslant 8$, then all multivariate Padé approximants $[N_k/D_m]_{E_{k+m}}$ with $k \geqslant n_2(m)$ are degenerate,
- $2 \leqslant m \leqslant 2\delta_m + 2$, or, equivalently, $2 \leqslant m < 8$, then (33) is a necessary condition for the Padé approximant $[N_k/D_m]_{E_{k+m}}$ with $k > n_2(m)$ to be uniquely determined.

This completes the proof. □

Theorem 4.2 tells us that in the table of general multivariate Padé approximants for the function $F_1(1,1,1;2;x,y)$, with each of the index sets N, D and E enumerated according to the triangle numbering, among others the entries $[N_6/D_2^{(t)}]$, $[N_9/D_3^{(t)}]$, $[N_8/D_4^{(t)}]$, $[N_{13}/D_5^{(t)}]$ and $[N_{18}/D_6^{(t)}]$ are degenerate general multivariate Padé approximants found in respectively the second, the third, the fourth, the fifth and the sixth column. This finding will coincide with the explicit formulas given in section 5 for some general multivariate Padé approximants to $F_1(1,1,1;2;x,y)$.

Let us now turn to the general multivariate Padé table where D is enumerated according to the square numbering, while still enumerating N and E according to the triangle numbering. Any index set $D_m^{(s)}$ with $m \geqslant 0$ is again uniquely characterized by the two integers δ_m and ω_m as follows. For $2\omega_m < \delta_m$:

$$\begin{aligned}D_m^{(s)} = &\bigl\{(i,j) \mid 0 \leqslant i \leqslant \delta_m - \omega_m - 1,\ 0 \leqslant j \leqslant \delta_m - \omega_m - 1\bigr\}\\ &\cup \bigl\{(\delta_m - \omega_m, 0), \ldots, (\delta_m - \omega_m, \omega_m)\bigr\}\end{aligned}$$

and for $2\omega_m \geqslant \delta_m$:

$$D_m^{(s)} = \bigl\{(i,j) \mid 0 \leqslant i \leqslant \omega_m,\ 0 \leqslant j \leqslant \omega_m - 1\bigr\} \cup \bigl\{(0, \omega_m), \ldots, (\delta_m - \omega_m, \omega_m)\bigr\},$$

depending on whether we are adding index points vertically or horizontally in order to complete the square.

Theorem 4.3. Let the triangle numbering be used to enumerate the sets E and N and let the square numbering be used to enumerate the denominator index set D. Moreover, let

$$n(m) = \begin{cases} m + 2(\delta_m - \omega_m) - 2\omega_m, & \omega_m < \delta_m - \omega_m, \\ m + 2\omega_m - 2(\delta_m - \omega_m) + 2, & \omega_m \geqslant \delta_m - \omega_m. \end{cases}$$

Then, in the general multivariate Padé table for $F_1(1,1,1;2;x,y)$ we have

- if $2 \leqslant m$, then already the entry $[N_{n(m)}/D_m^{(s)}]$ is degenerate,
- if $7 \leqslant m$, then, moreover, the mth column contains only a finite number of uniquely determined general multivariate Padé approximants since all entries $[N_k/D_m^{(s)}]$, $k > n(m)$, are also degenerate.

We omit a proof because of its similarity with that of theorem 4.2.

We should mention that, although we have only given results on the non-normality of two particular tables of general multivariate Padé approximants for the function $F_1(1,1,1;2;x,y)$, we believe that this non-normality holds for any table of general Padé approximants for $F_1(1,1,1;2;x,y)$. When looking at the given proofs of theorems 4.2 and 4.3, it is already clear that the proofs do not essentially depend on the choice of a particular numbering for the denominator index set D. Choosing a numbering different from the triangle numbering for the index set E is the more difficult generalization.

In the next and final section, we give explicit formulas for some nondegenerate general multivariate Padé approximants and, based on these formulas, we indicate directions for future work.

5. Conclusions and future work

Theorems 4.2 and 4.3 state that there are infinitely many degenerate entries in at least two tables of general multivariate Padé approximants, where respectively the $D^{(t)}$ and the $D^{(s)}$ are the denominator index sets, while in both tables N and E are enumerated according to the triangle numbering. Moreover, in the first 8, respectively 7, columns of these two tables, a behaviour different from the one in the subsequent columns is apparent. Explicit expressions for the nondegenerate entries in these first few columns can easily be computed and are given below for the first five columns of both tables. From the triangle and square enumeration, it is easy to see that the denominator polynomials in these first five columns are indexed by

$$\begin{aligned} &D_0^{(t)} = D_0^{(s)} = \{(0,0)\}, \\ &D_1^{(t)} = D_1^{(s)} = \{(0,0),(1,0)\}, \\ &D_2^{(t)} = D_2^{(s)} = \{(0,0),(1,0),(0,1)\}, \\ &D_3^{(t)} = \{(0,0),(1,0),(0,1),(2,0)\}, \quad D_3^{(s)} = \{(0,0),(1,0),(0,1),(1,1)\}, \\ &D_4^{(t)} = \{(0,0),(1,0),(0,1),(2,0),(1,1)\}, \quad D_4^{(s)} = \{(0,0),(1,0),(0,1),(1,1),(2,0)\}. \end{aligned} \tag{34}$$

As for the numerator polynomials, the triangle numbering of N implies that, in each column of the table, the entry in row (starting with row 0)

$$\bar{t} = \frac{(t+1)(t+2)}{2} - 1, \quad t \geqslant 0,$$

is a general multivariate Padé approximant of homogeneous degree t in the numerator. For convenience we denote this numerator index set by

$$N_{\bar{t}} = \{(i,j) \mid 0 \leqslant i+j \leqslant t\}$$

and subsequent and preceding numerator index sets by

$$N_{\bar{t}+p} = \{(i,j) \mid 0 \leqslant i+j \leqslant t\} \cup \{(t+1,0),(t,1),\ldots,(t-p+2,p-1)\},\\ 1 \leqslant p \leqslant t+2, \tag{35a}$$

respectively

$$N_{\bar{t}-q} = \{(i,j) \mid 0 \leqslant i+j \leqslant t\} \backslash \{(0,t),(1,t-1),\ldots,(q-1,t-q+1)\},\\ 1 \leqslant q \leqslant t+1. \tag{35b}$$

Moreover, we let

$$T_{\bar{t}}(x,y) = \sum_{(i,j)\in N_{\bar{t}}} \frac{x^i y^j}{i+j+1} = \sum_{0\leqslant i+j\leqslant t} \frac{x^i y^j}{i+j+1} \tag{36}$$

denote the homogeneous partial sum of the Taylor series expansion of $F_1(1,1,1;2;x,y)$. The general multivariate Padé approximants for $F_1(1,1,1;2;x,y)$ with numerator index sets given by (35) and denominator index sets $D_i^{(t)}$, $i = 0,\ldots,4$, are listed in table 1 in terms of the partial sums $T_{\bar{t}}(x,y)$.

For the general multivariate Padé approximants $[N_k/D_i^{(s)}]_{E_{k+i}}$, $i = 0,\ldots,4$, a table analogous to table 1 can be constructed. Before doing so, however, we observe that because of (34),

$$[N_k/D_\ell^{(s)}]_{E_{k+\ell}} = [N_k/D_\ell^{(t)}]_{E_{k+\ell}}, \quad \ell = 0,1,2.$$

Moreover, the index sets $D_4^{(t)}$ and $D_4^{(s)}$ are equal, disregarding the order of their elements. The fact that both sets are enumerated differently does not influence the form of the general multivariate Padé approximant and we have

$$[N_k/D_4^{(s)}]_{E_{k+4}} = [N_k/D_4^{(t)}]_{E_{k+4}}.$$

Hence, table 1 is also a listing of the general Padé approximants for $F_1(1,1,1;2;x,y)$ with numerator index set given by (35) and denominator index set $D_i^{(s)}$, $i = 0,1,2,4$. The approximants $[N_k/D_3^{(s)}]_{E_{k+3}}$ for $F_1(1,1,1;2;x,y)$ are given in table 2.

For the singular entries in tables 1 and 2, either “minimal” or “optimal” solutions can be given, depending on the rank deficiency of the linear system (14) of defining equations. The minimal solutions of the Padé approximation problem carry a minimal

Table 1
General Padé approximants for $F_1(1,1,1;2;x,y)$.

$\left[N_p/D_0^{(t)}\right]_{N_p} = \sum_{(i,j)\in N_p} \frac{x^i y^j}{i+j+1}$	$p \geqslant 0$
$\left[N_{\bar t-1}/D_1^{(t)}\right]_{N_{\bar t}} = T_{\overline{t-1}}(x,y)$	$t \geqslant 1$
$\left[N_{\bar t+p-1}/D_1^{(t)}\right]_{N_{\bar t+p}} = \frac{T_{\bar t}(x,y) - \frac{t+1}{t+2}xT_{\overline{t-1}}(x,y)}{1-\frac{t+1}{t+2}x}$	$1 \leqslant p \leqslant t+1$
$\left[N_{\bar t-2}/D_2^{(t)}\right]_{N_{\bar t}} = \frac{T_{\bar t}(x,y) - \frac{t}{t+1}yT_{\overline{t-1}}(x,y)}{1-\frac{t}{t+1}y}$	$t \geqslant 2$
$\left[N_{\bar t-1}/D_2^{(t)}\right]_{N_{\bar t+1}} = \frac{T_{\bar t}(x,y) - \left(\frac{t+1}{t+2}x + \frac{t}{t+1}y\right)T_{\overline{t-1}}(x,y)}{1-\frac{t+1}{t+2}x-\frac{t}{t+1}y}$	$t \geqslant 1$
$\left[N_{\bar t}/D_2^{(t)}\right]_{N_{\bar t+2}} = \frac{T_{\bar t}(x,y) - \frac{t+1}{t+2}xT_{\overline{t-1}}(x,y)}{1-\frac{t+1}{t+2}x}$	$t \geqslant 1$
$\left[N_{\bar t+p}/D_2^{(t)}\right]_{N_{\bar t+p+2}}$: (14) singular	$t \geqslant 2,\ 1 \leqslant p \leqslant t-1$
$\left[N_{\bar t-3}/D_3^{(t)}\right]_{N_{\bar t}} = \frac{T_{\bar t}(x,y) - \frac{t}{t+1}yT_{\overline{t-1}}(x,y)}{1-\frac{t}{t+1}y}$	$t \geqslant 3$
$\left[N_{\bar t-2}/D_3^{(t)}\right]_{N_{\bar t+1}} = \frac{T_{\bar t}(x,y) - \frac{t}{t+1}yT_{\overline{t-1}}(x,y) - \frac{t}{t+2}x^2T_{\overline{t-2}}(x,y)}{1-\frac{t}{t+1}y-\frac{t}{t+2}x^2}$	$t \geqslant 2$
$\left[N_{\bar t-1}/D_3^{(t)}\right]_{N_{\bar t+2}} = \frac{T_{\overline{t-1}}(x,y) - \frac{t}{t+1}xT_{\overline{t-2}}(x,y)}{1-\frac{t}{t+1}x}$	$t \geqslant 2$
$\left[N_{\bar t+p-1}/D_3^{(t)}\right]_{N_{\bar t+p+2}}$: (14) singular	$t \geqslant 3,\ 1 \leqslant p \leqslant t-1$
$\left[N_{\bar t-3}/D_4^{(t)}\right]_{N_{\bar t+1}} = \frac{T_{\bar t}(x,y) - \left(\frac{t+1}{t+2}x + \frac{t}{t+1}y\right)T_{\overline{t-1}}(x,y) + \frac{t^2-1}{t(t+2)}xyT_{\overline{t-2}}(x,y)}{1-\frac{t+1}{t+2}x-\frac{t}{t+1}y+\frac{t^2-1}{t(t+2)}xy}$	$t \geqslant 3$
$\left[N_{\bar t-2}/D_4^{(t)}\right]_{N_{\bar t+2}} = \frac{T_{\bar t}(x,y) - \frac{t^3x+t(t^2-1)y}{(t-1)(t+1)^2}T_{\overline{t-1}}(x,y)}{1-\frac{t^3x+t(t^2-1)y}{(t-1)(t+1)^2}+\frac{t(2t+1)x^2+t^2(t^2-1)(t+2)xy}{(t-1)(t+2)(t+1)^3}} + \frac{\frac{t(2t+1)x^2+t^2(t^2-1)(t+2)xy}{(t-1)(t+2)(t+1)^3}T_{\overline{t-2}}(x,y)}{1-\frac{t^3x+t(t^2-1)y}{(t-1)(t+1)^2}+\frac{t(2t+1)x^2+t^2(t^2-1)(t+2)xy}{(t-1)(t+2)(t+1)^3}}$	$t \geqslant 2$
$\left[N_{\bar t+p-2}/D_4^{(t)}\right]_{N_{\bar t+p+2}}$: (14) singular	$t \geqslant 3,\ 1 \leqslant p \leqslant t$

number of coefficients in numerator and denominator, in other words, the deficiency has been used for equating to zero as many unknown coefficients in the rational approximant as possible. The optimal solutions, on the other hand, satisfy as many Padé approximation conditions as possible, by adding to (14) the condition(s) that are

Table 2
General Padé approximants for $F_1(1,1,1;2;x,y)$.

$\left[N_{\bar{t}-2}\big/D_3^{(s)}\right]_{N_{\bar{t}+1}} = \dfrac{T_{\bar{t}}(x,y) - \left(\frac{t+1}{t+2}x + \frac{t}{t+1}y\right)T_{\overline{t-1}}(x,y) + \frac{t^2-1}{t(t+2)}xyT_{\overline{t-2}}(x,y)}{1 - \frac{t+1}{t+2}x - \frac{t}{t+1}y + \frac{t^2-1}{t(t+2)}xy}$	$t \geqslant 2$
$\left[N_{\bar{t}-1}\big/D_3^{(s)}\right]_{N_{\bar{t}+2}} = \dfrac{T_{\bar{t}}(x,y) - \left(\frac{t+1}{t+2}x + \frac{t}{t+1}y\right)T_{\overline{t-1}}(x,y) + \frac{t^2}{(t+1)^2}xyT_{\overline{t-2}}(x,y)}{1 - \frac{t+1}{t+2}x - \frac{t}{t+1}y + \frac{t^2}{(t+1)^2}xy}$	$t \geqslant 1$
$\left[N_{\bar{t}+p-1}\big/D_3^{(s)}\right]_{N_{\bar{t}+p+2}}$: (14) singular	$t \geqslant 2,\ 1 \leqslant p \leqslant t$

next in line according to the numbering imposed on the equation index set E. For details we refer to [1, p. 86].

The explicit expressions for the nondegenerate general multivariate Padé approximants are helpful for future investigations. Indeed, summarizing the results we have obtained concerning Padé approximants for the Appell functions, we see that the multivariate counterpart of the univariate theorems 1.1 and 1.2 has been investigated, both for homogeneous Padé approximants and for general multivariate Padé approximants. In the former case the univariate results can be generalized, while in the latter case they cannot. The univariate theorem 1.3, however, which gives explicit expressions for the denominator of the Padé approximant in terms of the Gauss function, still has to be investigated in the multivariate case. It is in this respect that the explicit expressions for the $[N_k/D_i^{(t)}]_{E_{k+i}}$ and $[N_k/D_i^{(s)}]_{E_{k+i}}$ in tables 1 and 2 are helpful. Let us illustrate this in some more detail. If we denote the denominator of

$$\left[N_k\big/D_\ell^{(t)}\right]_{E_{k+\ell}}$$

by $Q_{k,\ell}(x,y)$ then we know from table 1 that, for instance, for $\ell = 2$,

$$\begin{aligned}
Q_{\bar{t}-2,2}(x,y) &= 1 - \frac{t}{t+1}y, && t \geqslant 2\\
Q_{\bar{t}-1,2}(x,y) &= 1 - \frac{t+1}{t+2}x - \frac{t}{t+1}y, && t \geqslant 1,\\
Q_{\bar{t},2}(x,y) &= 1 - \frac{t+1}{t+2}x, && t \geqslant 1.
\end{aligned} \tag{37}$$

The question now arises whether these denominators can be rewritten in terms of the function F_1. Consider the analogy with the univariate case. According to theorem 1.3, the denominator $q_{t,1}(z)$ of the univariate Padé approximant $[t/1]$ for ${}_2F_1(1,1;2;z)$, is given by

$$q_{t,1}(z) = {}_2F_1(-1,-1-t;-2-t;z), \quad t \geqslant 0, \tag{38}$$

or

$$q_{t,1}(z) = 1 - \frac{t+1}{t+2}z, \quad t \geqslant 0. \tag{39}$$

The analogy between (39) and (37) is obvious. For we know from theorem 4.1 that, with $g(x, y) = F_1(1, 1, 1; 2; x, y)$, the irreducible form of both

$$\mathcal{P}_x\big(\big[N_{\bar{t}-1}/D_2^{(t)}\big]\big) \quad \text{and} \quad \mathcal{P}_x\big(\big[N_{\bar{t}}/D_2^{(t)}\big]\big)$$

equals $[t/1]^{\mathcal{P}_x(g)}$ while the irreducible form of

$$\mathcal{P}_y\big(\big[N_{\bar{t}-2}/D_2^{(t)}\big]\big)$$

equals $[t-1/1]^{\mathcal{P}_y(g)}$.

The question of interest is whether, besides the analogy between (39) and (37), there is an analogue of (38) for (37). This and, more generally, the generalization of theorem 1.3 for general Padé approximants is the subject of further investigation. Similarly, the multivariate counterpart of theorem 1.3 for homogeneous multivariate Padé approximants is the subject of future research.

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