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A direct approach to convergence of multivariate, nonhomogeneous, Padé approximants

A. Cuyt[a,1], K.A. Driver[b], D.S. Lubinsky[b,*]

[a]*Department of Mathematics, University of Antwerp, Universiteitsplein 1, B-2610, Wilrijk, Belgium*
[b]*Department of Mathematics, Witwatersrand University, P.O. Wits 2050, South Africa*

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Abstract

We present a direct approach for proving convergence in measure/product capacity of multivariate, nonhomogeneous, Padé approximants. Previous approaches have involved projection onto Padé-type approximation in one variable, and only yielded convergence in (Lebesgue) measure.

Keywords: Multivariate Padé approximants; Nonhomogeneous approximants; Nuttall–Pommerenke theorems

1. Introduction and results

The convergence theory of multivariate Padé approximation has received much attention in recent years. Usually, researchers have distinguished between homogeneous [14], and nonhomogeneous [2, 10, 15, 17] approximants. The definition of the homogeneous multivariate Padé approximants is in some respects very close to the univariate definition: it can be computed using the univariate epsilon and qd-algorithms [9, 10], and reduces to the univariate Padé approximant on every complex line through the origin [6]. However it introduces a high-order singularity in the neighbourhood of the origin.

This drawback is taken care of in the definition of the nonhomogeneous multivariate Padé approximant at the expense of some elegant univariate properties. In this paper, we present a direct approach for proving convergence in measure/capacity of nonhomogeneous approximants. This is made possible by a recent estimate of the authors [13] on the size of the lemniscate of a suitably normalized multivariate polynomial. See [3, 4, 19, 22, 24] for results and references on convergence in measure/capacity for univariate Padé approximants.

* Corresponding author. E-mail: 036dsl@cosmos.wits.ac.za.

[1] NFWO, Belgium.

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Throughout

$$\underline{z} := (z_1, z_2, z_3, \ldots, z_l) \in \mathbb{C}^l;$$

$\mathbb{N}$ denotes the set of nonnegative integers, and for

$$\underline{j} = (j_1, j_2, \ldots, j_l) \in \mathbb{N}^l,$$

we set

$$\underline{z}^{\underline{j}} := z_1^{j_1} z_2^{j_2} \cdots z_l^{j_l}.$$

The size of $\underline{j}$ is

$$|\underline{j}| := j_1 + j_2 + \cdots + j_l.$$

A multivariate polynomial $S(\underline{z})$ is naturally *associated* with a finite set $\mathscr{S} \subset \mathbb{N}^l$:

$$S(\underline{z}) = \sum_{\underline{j} \in \mathscr{S}} c_{\underline{j}} \underline{z}^{\underline{j}} \quad (c_{\underline{j}} \in \mathbb{C}).$$

The index set $\mathscr{S}$ contains the nonzero coefficients of S, but possibly also some vanishing coefficients. We define ∂S to be the *maximum partial degree* of S, so that

$$\partial S := \max \left\{ \max_{1 \leqslant k \leqslant l} j_k : \underline{j} = (j_1, j_2, \ldots, j_l) \in \mathscr{S} \text{ and } c_{\underline{j}} \neq 0 \right\}.$$

If $T(\underline{z})$ is another polynomial, associated with, say $\mathscr{T}$, then to describe the product polynomial $(ST)(\underline{z})$, we need

$$\mathscr{S} * \mathscr{T} := \{ \underline{j} + \underline{k} : \underline{j} \in \mathscr{S}, \underline{k} \in \mathscr{T} \}. \tag{1}$$

Thus,

$$(ST)(\underline{z}) = \sum_{\underline{j} \in \mathscr{S} * \mathscr{T}} d_{\underline{j}} \underline{z}^{\underline{j}}$$

and ST is associated with $\mathscr{S} * \mathscr{T}$. We say that S/T is a rational function of type $\mathscr{S}/\mathscr{T}$.

Definition 1.1. Let

$$f(\underline{z}) = \sum_{\underline{j} \in \mathbb{N}^l} a_{\underline{j}} \underline{z}^{\underline{j}} \quad (a_{\underline{j}} \in \mathbb{C})$$

be a formal power series. Let $\mathscr{N}$, $\mathscr{D}$ and $\mathscr{I}$ be finite subsets of $\mathbb{N}^l$, and $r := P/Q$ be a rational function of type $\mathscr{N}/\mathscr{D}$. We say that *r interpolates f on the index set $\mathscr{I}$ if*

$$(fQ - P)(\underline{z}) = \sum_{\underline{j} \in \mathbb{N}^l \setminus \mathscr{I}} b_{\underline{j}} \underline{z}^{\underline{j}}. \tag{2}$$

The *order of contact of r* with f is defined to be

$$\nu(r) := \min \{ |\underline{j}| : \underline{j} \notin \mathscr{I} \}. \tag{3}$$

The letters $\mathcal{N}$, $\mathcal{D}$ and $\mathcal{I}$ are chosen to indicate numerator, denominator and interpolation index sets respectively. We also need the notion of the *maximum partial degree* of the index sets $\mathcal{N}$, $\mathcal{D}$ and so on:

$$\partial\mathcal{N} := \max\left\{\max_{1 \leqslant k \leqslant l} j_k : \underline{j} = (j_1, j_2, \ldots, j_l) \in \mathcal{N}\right\}. \tag{4}$$

Thus if P is associated with $\mathcal{N}$, $\partial\mathcal{N}$ denotes an upper bound on the highest possible power of any z_j possibly appearing in P, so $\partial\mathcal{N} \geqslant \partial P$. ($\partial P$ may be less than $\partial\mathcal{N}$ if some coefficients corresponding to elements of $\mathcal{N}$ are zero).

Throughout, meas denotes Lebesgue measure on $\mathbb{C}^l$ (equivalent to Lebesgue measure on $\mathbb{R}^{2l}$). We shall also need the product capacity $\text{cap}^{(l)}$ and Favarov's capacity Γ_l^{F}. Recall first the definition of logarithmic capacity: For compact $\mathbf{K} \subset \mathbb{C}$,

$$\text{cap}\,\mathbf{K} := \lim_{n\to\infty} (\min\{\|P\|_{L_\infty(\mathbf{K})}: P \text{ monic of degree } n\})^{1/n}.$$

See [16, 18, 22] for further orientation.

The product capacity $\text{cap}^{(l)}$ is defined inductively on l: For $l = 1$,

$$\text{cap}^{(1)} := \text{cap}.$$

If $\text{cap}^{(l-1)}$ has already been defined, then for Borel measurable $\mathbf{K} \subset \mathbb{C}^l$,

$$\text{cap}^{(l)}(\mathbf{K}) := \int_0^\infty \text{cap}\{z_1 : \text{cap}^{(l-1)}\{\underline{z}' : \underline{z} \in \mathbf{K}\} > s\}\,\text{d}s.$$

Here

$$\underline{z} = (z_1, z_2, \ldots, z_l) \;\Rightarrow\; \underline{z}' = (z_2, \ldots, z_l).$$

This (apparently strange) definition really does yield a product capacity: If we have a Cartesian product

$$\mathbf{K} := \mathbf{K}_1 \times \mathbf{K}_2 \times \cdots \times \mathbf{K}_l$$

where each $\mathbf{K}_j \subset \mathbb{C}$, then

$$\text{cap}^{(l)}(\mathbf{K}) = \prod_{j=1}^{l} \text{cap}\,\mathbf{K}_j.$$

Favarov's capacity involves the product capacity of unitary transformations (in particular, rotations) of the set $\mathbf{K}$. Recall that a unitary transformation A on $\mathbb{C}^l$ is an $l \times l$ matrix with complex entries such that $\bar{A}^{\text{T}}A = I$. Favarov's capacity of $\mathbf{K}$ is

$$\Gamma_l^{\text{F}}(\mathbf{K}) := \sup\{\text{cap}^{(l)}(A(\mathbf{K})): A \text{ unitary}\}.$$

See [5] for further orientation.

Following is our theorem for "nondiagonal" sequences of approximants:

Theorem 1.2. *Let f be analytic at $\underline{0}$ and meromorphic in the polydisc*

$$\mathbf{P} := \{\underline{z}: |z_j| < \rho_j,\ 1 \leqslant j \leqslant l\} \quad (0 < \rho_j \leqslant \infty)$$

in the following sense: There exists a polynomial S associated with a finite set $\mathcal{S}$ *such that* fS *is analytic in* $\mathbf{P}$. *Let* r_k *be a rational function of type* $\mathcal{N}_k/\mathcal{D}_k$ *interpolating* f *on* $\mathcal{I}_k$, $k \geqslant 1$. *Assume, moreover, that*

$$\mathcal{N}_k * \mathcal{S} \subseteq \mathcal{I}_k, \quad \text{for large enough } k, \tag{5}$$

and

$$\lim_{k \to \infty} v(r_k)/\partial \mathcal{D}_k = \infty. \tag{6}$$

Then $\{r_k\}_{k=1}^{\infty}$ *converges in* meas/cap$^{(l)}$/Γ_l^{F} *to* f *in compact subsets of* $\mathbf{P}$. *More precisely, given a compact subset* $\mathbf{K}$ *of* $\mathbf{P}$, $\exists \theta \in (0,1)$ *such that*

$$\operatorname{meas}\{\underline{z} \in \mathbf{K}\colon |f - r_k|(\underline{z}) > \theta^{v(r_k)}\} \to 0, \quad k \to \infty. \tag{7}$$

The same result holds if we replace meas *by* cap$^{(l)}$ *or* Γ_l^{F}.

The easiest way to assimilate (5) and (6) is to reduce them to the univariate ($l = 1$) case: If s denotes the total multiplicity of poles of f in $\mathbf{P} = \{z : |z| < \rho\}$, and $r_k = p_k/q_k$ is a rational function of type n_k/d_k satisfying

$$(fq_k - p_k)(z) = \mathrm{O}(z^{v(r_k)}) \tag{8}$$

then (5) is the requirement that

$$n_k + s \leqslant v(r_k) - 1.$$

Moreover, (6) becomes

$$\lim_{k \to \infty} v(r_k)/d_k = \infty.$$

In the case of univariate Padé approximants $[n_k/d_k]$, for which $v(r_k) = n_k + d_k + 1$, we obtain the usual requirements in convergence theorems for nondiagonal sequences:

$$d_k \geqslant s; \quad \lim_{k \to \infty} n_k/d_k = \infty.$$

An interesting feature of the above result is that only the total order of contact $v(r_k)$ needs to satisfy (6), not the order of contact in individual variables. We note that our hypotheses above guarantee convergence, but to ensure additional properties of the approximants, such as consistency with the Padé property, one needs additional restrictions on $\mathcal{N}_k$, $\mathcal{D}_k$, $\mathcal{I}_k$. The reader may refer to [1, 11]. In any event, large classes of Padé approximants satisfy (5) and (6).

In formulating our theorem for "diagonal" sequences, we need the notion of the *inclusion rule*: We say that $\mathcal{I} \subseteq \mathbb{N}^l$ satisfies the inclusion rule if

$$\underline{j} = (j_1, j_2, \ldots, j_l) \in \mathcal{I}$$

and

$$0 \leqslant n_i \leqslant j_i, \quad 1 \leqslant i \leqslant l$$

implies

$$\underline{n} := (n_1, n_2, \ldots, n_l) \in \mathscr{I}.$$

Thus if an l-tuple $\underline{j}$ belongs to $\mathscr{I}$, then so do all l-tuples lying in the smallest hypercube in $\mathbb{N}^l$ containing $\underline{0}$ and $\underline{j}$. We shall also need the "box" or hypercube index set

$$\mathscr{R}_k := \{\underline{j} = (j_1, j_2, \ldots, j_l)\colon 0 \leqslant j_i \leqslant k,\ 1 \leqslant i \leqslant l\}.$$

Throughout, $\langle x \rangle$ denotes the greatest integer $\leqslant x$.

Theorem 1.3. *Let f be analytic at $\underline{0}$ and meromorphic in $\mathbb{C}^l$ in the following sense: For each $\rho > 0$, there exists a polynomial S such that fS is analytic in the polydisc*

$$\mathbf{P} := \{\underline{z}\colon |z_j| < \rho,\ 1 \leqslant j \leqslant l\}. \tag{9}$$

Let r_k be a rational function of type $\mathscr{N}_k/\mathscr{D}_k$ interpolating f on $\mathscr{I}_k$, satisfying the inclusion rule, $k \geqslant 1$. Let

$$L_k := \max\{\partial\mathscr{N}_k, \partial\mathscr{D}_k\} \to \infty, \quad k \to \infty \tag{10}$$

and assume $\exists \eta > 0$ such that for large enough k,

$$\mathscr{N}_k * \mathscr{R}_{\langle \eta L_k \rangle} \subseteq \mathscr{I}_k; \qquad \mathscr{D}_k * \mathscr{R}_{\langle \eta L_k \rangle} \subseteq \mathscr{I}_k. \tag{11}$$

Then $\{r_k\}_{k=1}^{\infty}$ converges in meas/cap$^{(l)}$/Γ_l^{F} *in compact subsets of $\mathbb{C}^l$. More precisely, given $\varepsilon > 0$, and a compact subset $\mathbf{K}$ of $\mathbb{C}^l$,*

$$\operatorname{meas}\{\underline{z} \in \mathbf{K}\colon |f - r_k|(\underline{z}) > \varepsilon^{L_k}\} \to 0, \quad k \to \infty. \tag{12}$$

The same result holds if we replace meas *by* cap$^{(l)}$ *or* Γ_l^{F}.

For the univariate case ($l = 1$) and the Padé case $r_k = [n_k/d_k]$, the condition (11) may be reformulated as nothing more than the familiar condition in Nuttall–Pommerenke theorems:

$$\frac{1}{\lambda} \leqslant \frac{n_k}{d_k} \leqslant \lambda, \quad \text{some } \lambda > 1.$$

As an illustration of the result in $l > 1$ dimensions, let us suppose that

$$\mathscr{N}_k = \mathscr{D}_k = \{\underline{j}\colon |\underline{j}| \leqslant k\}.$$

This and (2) allow us to choose for large enough k,

$$\mathscr{I}_k \supseteq \{\underline{j}\colon |\underline{j}| \leqslant (1 + \varepsilon)k\},$$

if $0 < \varepsilon < 2^{1/l} - 1$. It is then easy to see that we can choose η satisfying (11) for large enough k.

In comparing the above result to those of Goncar [15] for the diagonal nonhomogeneous approximants, and that of the authors for the diagonal homogeneous case [14], we note that the conditions on f in [14, 15] allowed for far more general types of singularity. However, our method allows for convergence in cap$^{(l)}$ and it seems unlikely that the methods of [14, 15] which involve projection onto Padé-type approximation in one variable, can give anything more than convergence in meas.

We prove the results in Section 2.

2. Proofs

Our basic estimate for the proof of Theorem 1.2 is contained in the following lemma:

Lemma 2.1. *Let f be analytic at* $\underline{0}$ *and meromorphic in the polydisc*

$$\mathbf{P} := \{\underline{z}: |z_j| < \rho_j,\ 1 \leqslant j \leqslant l\} \quad (0 < \rho_j \leqslant \infty)$$

in the following sense: There exists a polynomial S associated with a finite set $\mathscr{S}$ *such that fS is analytic in* $\mathscr{P}$. *Let* $r_k = P_k/Q_k$ *be a rational function of type* $\mathscr{N}_k/\mathscr{D}_k$ *interpolating f on* $\mathscr{I}_k$, $k \geqslant 1$. *Assume, moreover, that* (5) *holds for large enough k. Let*

$$0 < \theta_1 < \theta_2 < 1; \qquad \frac{\theta_1}{\theta_2} < \theta_3 < 1.$$

Let

$$\mathbf{P}_k := \{\underline{z}: |z_j| \leqslant \theta_k \rho_j,\ 1 \leqslant j \leqslant l\}, \quad k = 1, 2. \tag{13}$$

Then for $\underline{z} \in \mathscr{P}_1$, *and some C independent of* $\underline{z}$ *and k*,

$$|f - r_k|(\underline{z}) \leqslant C \frac{\| Q_k \|_{L_\infty(\mathbf{P}_2)}}{|SQ_k|(\underline{z})} \theta_3^{\nu(r_k)}. \tag{14}$$

Proof. We have

$$(fQ_k - P_k)(\underline{z}) = \sum_{\underline{j} \notin \mathscr{I}_k} d_{\underline{j},k} \underline{z}^{\underline{j}}$$

After we multiply this by $S(\underline{z})$, we obtain a series involving different indices. However, each $\underline{j} \notin \mathscr{I}_k$ has $|\underline{j}| \geqslant \nu(r_k)$, and for any $\underline{m} \in \mathbb{N}^l$,

$$|\underline{j} + \underline{m}| = |\underline{j}| + |\underline{m}| \geqslant \nu(r_k).$$

Thus

$$[S(fQ_k - P_k)](\underline{z}) = \sum_{|\underline{j}| \geqslant \nu(r_k)} c_{\underline{j},k} \underline{z}^{\underline{j}}. \tag{15}$$

Here, the usual formula for Maclaurin series coefficients gives

$$|c_{\underline{j},k}| = \left| \left(\frac{1}{2\pi \mathrm{i}}\right)^l \int_{\Delta \mathbf{P}_2} \frac{[S(fQ_k - P_k)](\underline{t})}{\underline{t}^{\underline{j}+\underline{1}}} \mathrm{d}\underline{t} \right|,$$

where $\Delta \mathbf{P}_2 := \{\underline{z}:\ |z_j| = \theta_2 \rho_j,\ 1 \leqslant j \leqslant l\}$ is the boundary of $\mathbf{P}_2$, $\mathrm{d}\underline{t} = \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_l$ and $\underline{1} = (1, 1, \ldots, 1)$. Now for $\underline{j} \notin \mathscr{I}_k$, our condition (5) ensures that the coefficient of $\underline{z}^{\underline{j}}$ in SP_k is 0. Thus

$$|c_{\underline{j},k}| = \left| \left(\frac{1}{2\pi \mathrm{i}}\right)^l \int_{\Delta \mathbf{P}_2} \frac{(SfQ_k)(\underline{t})}{\underline{t}^{\underline{j}+\underline{1}}} \mathrm{d}\underline{t} \right|$$

$$\leqslant C \| Q_k \|_{L_\infty(\mathbf{P}_2)} \Big/ \prod_{\sigma=1}^{l} (\theta_2 \rho_\sigma)^{j_\sigma},$$

where C depends only on Sf and $\mathbf{P}_2$. Then for $\underline{z} \in \mathbf{P}_1$, we obtain from (15) that

$$|f - r_k|(\underline{z}) \leqslant C \frac{\| Q_k \|_{L_\infty(\mathbf{P}_2)}}{|SQ_k|(\underline{z})} \sum,$$

where

$$\sum := \sum_{|\underline{j}| \geqslant \nu(r_k)} (\theta_1/\theta_2)^{|\underline{j}|} = \sum_{\sigma = \nu(r_k)}^{\infty} (\theta_1/\theta_2)^{\sigma} \sum_{\underline{j}: |\underline{j}| = \sigma} 1 \leqslant \sum_{\sigma = \nu(r_k)}^{\infty} (\theta_1/\theta_2)^{\sigma} (\sigma + 1)^{l-1} \leqslant C_1 \theta_3^{\nu(r_k)}$$

with C_1 depending only on $\theta_1, \theta_2, \theta_3$ (recall that $\theta_3 > \theta_1/\theta_2$). □

To estimate the size of the set on which $|SQ_k|$ in (14) is small, we need:

Lemma 2.2. *Let $\rho > 0$ and $Q(\underline{z})$ be a polynomial that is of degree $\leqslant n$ in each of its variables, that is $\partial Q \leqslant n$. Assume that Q is normalized by the condition*

$$\max\{|Q(\underline{z})|: |z_j| \leqslant \rho,\ 1 \leqslant j \leqslant l\} = 1. \tag{16}$$

Let $\varepsilon \in (0, 1)$. Then the lemniscate

$$\mathbf{L} := \{\underline{z}: |z_j| \leqslant \rho,\ 1 \leqslant j \leqslant l \text{ and } |Q(\underline{z})| \leqslant \varepsilon^n\}$$

has

$$\operatorname{meas}(\mathbf{L}) \leqslant (16\pi\rho^2)^l \varepsilon^2 \max\left\{1, \log_2 \frac{2^{l-1}}{\varepsilon}\right\}^{l-1}; \tag{17}$$

$$\operatorname{cap}^{(l)}(\mathbf{L}) \leqslant C_1 \rho^l \varepsilon \max\left\{1, \log_2 \frac{1}{\varepsilon}\right\}^{l-1}; \tag{18}$$

$$\Gamma_l^{\mathrm{F}}(\mathbf{L}) \leqslant C_2 \rho^l \varepsilon^{1/l} \max\left\{1, \log_2 \frac{1}{\varepsilon}\right\}^{l-1}. \tag{19}$$

Here C_1 and C_2 are independent of ρ, ε, Q, n.

Proof. See Theorems 1.2 and 1.3 in [13]. □

At this stage, one would like to apply the estimates (17) to (19) in (14). Unfortunately, to do this one needs a normalization such as

$$\| SQ_k \|_{L_\infty(\mathbf{P})} = 1$$

for a suitable $\mathbf{P}$, whereas all that (14) naturally permits is

$$\| Q_k \|_{L_\infty(\mathbf{P})} = 1.$$

This means that we have to deal separately with the sets/lemniscates on which Q_k is small and on which S is small. To show that the union of these two sets is small, we need an estimate for meas/$\operatorname{cap}^{(l)}$/$\Gamma_l^{\mathrm{F}}(\mathbf{L}_1 \cup \mathbf{L}_2)$ in terms of meas/$\operatorname{cap}^{(l)}$/$\Gamma_l^{\mathrm{F}}(\mathbf{L}_j)$, $j = 1, 2$. For meas, such estimates are trivial, but we could not find them in the literature for $\operatorname{cap}^{(l)}$ and Γ_l^{F}. So we shall prove a weak estimate, which is, however, sufficient for our purposes. Recall first the subadditivity type property of logarithmic capacity in the plane: Let

$$h(t) := \left(\log \frac{1}{t}\right)^{-1}, \quad t \in (0, 1).$$

Then for sets $\mathbf{L}_1$, $\mathbf{L}_2$ contained in $\{z\colon |z| \leqslant \rho\}$, [16, p. 289]

$$h\left(\frac{\operatorname{cap}(\mathbf{L}_1 \cup \mathbf{L}_2)}{\rho}\right) \leqslant h\left(\frac{\operatorname{cap}(\mathbf{L}_1)}{\rho}\right) + h\left(\frac{\operatorname{cap}(\mathbf{L}_2)}{\rho}\right). \tag{20}$$

Lemma 2.3. *Let* $0 < \alpha < \frac{1}{2}$, $l \geqslant 1$, $\rho > 0$ *and* $\rho^* := \max\{1, \rho\}$. *Then for Borel sets* $\mathbf{L}_1$, $\mathbf{L}_2$ *contained in the polydisc*

$$\mathbf{P} := \{\underline{z}\colon |z_j| \leqslant \rho,\ 1 \leqslant j \leqslant l\},$$

we have

$$\operatorname{cap}^{(l)}(\mathbf{L}_1 \cup \mathbf{L}_2) \leqslant C_1 (\rho^*)^A \sum_{j=1}^{2} (\operatorname{cap}^{(l)}(\mathbf{L}_j))^\alpha; \tag{21}$$

and

$$\Gamma_l^{\mathrm{F}}(\mathbf{L}_1 \cup \mathbf{L}_2) \leqslant C_2 (\rho^*)^A \sum_{j=1}^{2} (\Gamma_l^{\mathrm{F}}(\mathbf{L}_j))^\alpha. \tag{22}$$

Here C_1 *and* C_2 *and* A *depend on* l, *but not on* ρ, $\mathbf{L}_1$ *or* $\mathbf{L}_2$.

Proof. We shall first prove (21) by induction on l. Note that the function h satisfies

$$2h(t) = h(t^{1/2}).$$

Hence (20) gives for $\mathbf{L}_1, \mathbf{L}_2$ contained in $\{z\colon |z| \leqslant \rho\}$

$$h\left(\frac{\operatorname{cap}(\mathbf{L}_1 \cup \mathbf{L}_2)}{\rho}\right) \leqslant 2h\left(\max_j \frac{\operatorname{cap}(\mathbf{L}_j)}{\rho}\right) = h\left(\left[\max_j \frac{\operatorname{cap}(\mathbf{L}_j)}{\rho}\right]^{1/2}\right).$$

Using monotonicity of h then gives

$$\operatorname{cap}(\mathbf{L}_1 \cup \mathbf{L}_2) \leqslant \sqrt{\rho} \sum_{j=1}^{2} (\operatorname{cap} \mathbf{L}_j)^{1/2}. \tag{23}$$

This is essentially the case $l = 1$ of (21): Recalling that $\operatorname{cap} \mathbf{L}_j \leqslant \rho$, we obtain

$$\operatorname{cap}(\mathbf{L}_1 \cup \mathbf{L}_2) \leqslant \rho^{1-\alpha} \sum_{j=1}^{2} (\operatorname{cap} \mathbf{L}_j)^\alpha.$$

Next, as an induction hypothesis, assume that we have proved (21) for $l - 1$, so that for Borel sets $\mathbf{L}_1$, $\mathbf{L}_2$ contained in $\{(z_1, z_2, \ldots, z_{l-1})\colon |z_j| \leqslant \rho,\ 1 \leqslant j \leqslant l-1\}$,

$$\operatorname{cap}^{(l-1)}(\mathbf{L}_1 \cup \mathbf{L}_2) \leqslant C(\rho^*)^B \sum_{j=1}^{2} (\operatorname{cap}^{(l-1)}(\mathbf{L}_j))^\alpha,$$

for suitable constants C and B. We proceed to prove (21) for l. Recall that

$$\operatorname{cap}^{(l)}(\mathbf{L}_1 \cup \mathbf{L}_2) = \int_0^\infty \operatorname{cap}\{z_1\colon \operatorname{cap}^{(l-1)}\{\underline{z}'\colon \underline{z} \in \mathbf{L}_1 \cup \mathbf{L}_2\} > s\}\,\mathrm{d}s,$$

where if

$$\underline{z} = (z_1, z_2, \ldots, z_l) \quad \text{then} \quad \underline{z}' = (z_2, z_3, \ldots, z_l).$$

Then, using our induction hypothesis,

$$s < \operatorname{cap}^{(l-1)}\{\underline{z}'\colon \underline{z} \in \mathbf{L}_1 \cup \mathbf{L}_2\} \;\Rightarrow\; \frac{s}{C\rho^{*B}} < \sum_{j=1}^{2} (\operatorname{cap}^{(l-1)}\{\underline{z}'\colon \underline{z} \in \mathbf{L}_j\})^\alpha$$

$$\Rightarrow\; \operatorname{cap}^{(l-1)}\{\underline{z}'\colon \underline{z} \in \mathbf{L}_j\} > \left(\frac{s}{2C\rho^{*B}}\right)^{1/\alpha}$$

for either $j = 1$ or $j = 2$. Then using (23), we obtain

$$\operatorname{cap}\{z_1\colon \operatorname{cap}^{(l-1)}\{\underline{z}'\colon \underline{z} \in \mathbf{L}_1 \cup \mathbf{L}_2\} > s\}$$

$$\leqslant \sqrt{\rho} \sum_{j=1}^{2} \operatorname{cap}\left\{z_1\colon \operatorname{cap}^{(l-1)}\{\underline{z}'\colon \underline{z} \in \mathbf{L}_j\} > \left(\frac{s}{2C\rho^{*B}}\right)^{1/\alpha}\right\}^{1/2}.$$

Hence

$$\operatorname{cap}^{(l)}(\mathbf{L}_1 \cup \mathbf{L}_2) = \int_0^\infty \operatorname{cap}\{z_1\colon \operatorname{cap}^{(l-1)}\{\underline{z}'\colon \underline{z} \in \mathbf{L}_1 \cup \mathbf{L}_2\} > s\}\,\mathrm{d}s$$

$$\leqslant \sqrt{\rho} \sum_{j=1}^{2} \int_0^\infty \operatorname{cap}\left\{z_1\colon \operatorname{cap}^{(l-1)}\{\underline{z}'\colon \underline{z} \in \mathbf{L}_j\} > \left(\frac{s}{2C\rho^{*B}}\right)^{1/\alpha}\right\}^{1/2} \mathrm{d}s$$

$$= \sqrt{\rho} \sum_{j=1}^{2} \int_0^\infty \operatorname{cap}\{z_1\colon \operatorname{cap}^{(l-1)}\{\underline{z}'\colon \underline{z} \in \mathbf{L}_j\} > t\}^{1/2}\, 2\alpha C\rho^{*B} t^{\alpha-1}\,\mathrm{d}t. \tag{24}$$

Now if $\eta > 0$, Hölder's inequality and the fact that $|z_1| \leqslant \rho$ for $\underline{z} \in \mathbf{L}_j$ give

$$I_j := \int_0^\infty \operatorname{cap}\{z_1\colon \operatorname{cap}^{(l-1)}\{\underline{z}'\colon \underline{z} \in \mathbf{L}_j\} > t\}^{1/2} t^{\alpha-1}\,\mathrm{d}t$$

$$\leqslant \sqrt{\rho} \int_0^\eta t^{\alpha-1}\,\mathrm{d}t + \left[\int_\eta^\infty \operatorname{cap}\{z_1\colon \operatorname{cap}^{(l-1)}\{\underline{z}'\colon \underline{z} \in \mathbf{L}_j\} > t\}\,\mathrm{d}t\right]^{1/2} \times \left[\int_\eta^\infty t^{2\alpha-2}\,\mathrm{d}t\right]^{1/2}$$

$$\leqslant \sqrt{\rho}\,\frac{\eta^\alpha}{\alpha} + [\operatorname{cap}^{(l)}(\mathbf{L}_j)]^{1/2} \times \left[\frac{\eta^{-1+2\alpha}}{1-2\alpha}\right]^{1/2}.$$

(It is here that we use $\alpha < \frac{1}{2}$.) Choosing $\eta := \operatorname{cap}^{(l)}(\mathbf{L}_j)$, we obtain

$$I_j \leqslant C_1 \rho^{*1/2} (\operatorname{cap}^{(l)}(\mathbf{L}_j))^\alpha,$$

for some C_1 depending only on α. Substituting into (24) gives (21) for l with suitable C_1 and A.

We proceed to prove (22). Recall first that

$$\Gamma_l^{\mathrm{F}}(\mathbf{L}) = \sup\{\mathrm{cap}^{(l)}(A(\mathbf{L})) \colon A \text{ unitary}\}.$$

Also, if A is unitary, and $\|\cdot\|$ denotes the usual Euclidean norm, then (see [25, p. 74])

$$\|A\underline{z}\| = \|\underline{z}\|.$$

In particular, as $\mathbf{L}_j$, $j = 1, 2$, is contained in the ball $\{\underline{z} \colon \|\underline{z}\| \leqslant \sqrt{l}\rho\}$, so is $A(\mathbf{L}_j)$. Thus

$$\underline{z} \in A(\mathbf{L}_j) \Rightarrow |z_j| \leqslant \sqrt{l}\rho, \quad 1 \leqslant j \leqslant l.$$

Hence applying the inequality (21) to $A(\mathbf{L}_j)$, $j = 1, 2$, we obtain for some C_3 depending on l, but not on $\mathbf{L}_j$, $j = 1, 2$, or ρ,

$$\begin{aligned}\mathrm{cap}^{(l)}(A(\mathbf{L}_1 \cup \mathbf{L}_2)) &= \mathrm{cap}^{(l)}(A(\mathbf{L}_1) \cup A(\mathbf{L}_2)) \\ &\leqslant C_3 \rho^{*A} \sum_{j=1}^{2} (\mathrm{cap}^{(l)}(A(\mathbf{L}_j)))^{\alpha} \\ &\leqslant C_3 \rho^{*A} \sum_{j=1}^{2} (\Gamma_l^{\mathrm{F}}(\mathbf{L}_j))^{\alpha}.\end{aligned}$$

Taking sup's over unitary A gives (22). □

Proof of Theorem 1.2. Let $\mathbf{K}$ be a compact subset of $\mathbf{P}$. We can find $0 < \theta_1 < \theta_2 < 1$ such that with $\mathbf{P}_1$ defined by (13), we have $\mathbf{K} \subset \mathbf{P}_1$. Set

$$\rho := \max_{1 \leqslant j \leqslant l} \rho_j$$

and normalize Q_k, the denominator in r_k, so that it satisfies (16). We may also normalize S so that it satisfies (16). Let

$$\mathbf{E}_k := \{\underline{z} \colon |z_j| \leqslant \rho \ \forall j, \ |Q_k|(\underline{z}) \leqslant \varepsilon^{\partial Q_k}\};$$

$$\mathbf{F} := \{\underline{z} \colon |z_j| \leqslant \rho \ \forall j, \ |S|(\underline{z}) \leqslant \varepsilon^{\partial S}\}.$$

We obtain from (14) that for $\underline{z} \in \mathbf{K} \subset \mathbf{P}_1$, $\underline{z} \notin \mathbf{E}_k \cup \mathbf{F}$,

$$|f - r_k|(\underline{z}) \leqslant C_1 \varepsilon^{-\partial Q_k - \partial S} \theta_3^{\nu(r_k)} < \theta^{\nu(r_k)}$$

if $1 > \theta > \theta_3 > \theta_1/\theta_2$ and k is large enough. Here θ may be made independent of ε, in view of our hypothesis (6). Recall also that $\partial Q_k \leqslant \partial \mathscr{D}_k$. Together, Lemmas 2.2 and 2.3 show that $\mathbf{E}_k \cup \mathbf{F}$ has small meas/$\mathrm{cap}^{(l)}/\Gamma_l^{\mathrm{F}}$. □

Unfortunately, the method of proof of Theorem 1.2 does not yield the conclusion of Theorem 1.3. The problem is the power of ρ appearing in the estimates in Lemma 2.2. So we use the well-known approach based on errors of best approximation. Recall that

$$\mathscr{R}_k := \{\underline{j} = (j_1, j_2, \ldots, j_l) \colon 0 \leqslant j_i \leqslant k, \ 1 \leqslant i \leqslant l\}$$

is the "box" or hypercube index set. Given a compact set $\mathbf{K}$ on which f is analytic, we set

$$E_k(f;\mathbf{K}) := \min\{\|f - r_k\|_{L_\infty(\mathbf{K})}: r_k \text{ of type } \mathscr{R}_k/\mathscr{R}_k\},$$

the error in approximation of f on $\mathbf{K}$ by rational functions of type $\mathscr{R}_k/\mathscr{R}_k$.

Lemma 2.4. *Assume the hypotheses of Theorem* 1.3. *Let* $\rho > 0$, *and* S *be a polynomial such that* fS *is analytic in*

$$\mathbf{P} := \{\underline{z}: |z_j| \leqslant \rho,\ 1 \leqslant j \leqslant l\}. \tag{25}$$

Then

$$\lim_{k\to\infty} E_k(fS;\mathbf{P})^{1/k} = 0. \tag{26}$$

Proof. The hypotheses of Theorem 1.3 guarantee that fS is meromorphic in $\mathbb{C}^l$ in the usual sense of several complex variables. See, for example, [23, p. 231]. Consequently, there exist entire functions g and h such that $fS = g/h$. See, for example, [23, p. 262]. By taking the partial sums of g and h (which approximate g and h faster than geometrically on compact sets), we obtain rational functions that approximate fS faster than geometrically on compact sets on which h does not vanish. The solubility of the second Cousin problem on $\mathbb{C}^l$ allow us to ensure that h does not vanish on $\mathbf{P}$. See [23, pp. 253ff.]. □

For more on multivariate functions satisfying (26), see [7, 15].

Proof of Theorem 1.3. Let $0 < \delta < \min\{\frac{1}{2}, \eta\}$, where η is as in (11). Let $\rho > \lambda > 0$ and S be a polynomial such that fS is analytic in the polydisc $\mathbf{P}$ given by (25). Let

$$\mathbf{K} := \{\underline{z}: |z_j| \leqslant \lambda,\ 1 \leqslant j \leqslant l\}$$

and

$$\mathscr{R}_k^{\#} := \mathscr{R}_{\langle \delta L_k \rangle}.$$

Let $r_k^* = P_k^*/Q_k^*$ be a best rational function of type $\mathscr{R}_k^{\#}/\mathscr{R}_k^{\#}$ to fS on $\mathbf{P}$, so that

$$\|fS - r_k^*\|_{L_\infty(\mathbf{P})} = E_k(fS;\mathbf{P}).$$

Now $r_k = P_k/Q_k$ satisfies

$$(fQ_k - P_k)(\underline{z}) = \sum_{\underline{j}\notin \mathscr{I}_k} c_{\underline{j},k}\,\underline{z}^{\underline{j}}.$$

We claim that

$$[SQ_k^*(fQ_k - P_k)](\underline{z}) = \sum_{\underline{j}\notin \mathscr{I}_k} d_{\underline{j},k}\,\underline{z}^{\underline{j}}.$$

This follows as $\mathcal{I}_k$ satisfies the inclusion rule: For $\underline{j} \notin \mathcal{I}_k$ and $\underline{m} \in \mathbb{N}^l$, $\underline{j} + \underline{m} \notin \mathcal{I}_k$. Here, the usual formula for Maclaurin series coefficients gives

$$d_{\underline{j},k} = \left(\frac{1}{2\pi \mathrm{i}}\right)^l \int_{\Delta \mathbf{P}} \frac{[SQ_k^*(fQ_k - P_k)](\underline{t})}{\underline{t}^{\underline{j}+\underline{1}}} \, \mathrm{d}\underline{t},$$

where $\Delta \mathbf{P} := \{\underline{z}\colon |z_j| = \rho,\ 1 \leqslant j \leqslant l\}$, $\mathrm{d}\underline{t} = \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_l$ and $\underline{1} = (1, 1, \ldots, 1)$. Then

$$d_{\underline{j},k} = \left(\frac{1}{2\pi \mathrm{i}}\right)^l \int_{\Delta \mathbf{P}} \frac{[Q_k^* Q_k(Sf - r_k^*)](\underline{t})}{\underline{t}^{\underline{j}+\underline{1}}} \, \mathrm{d}\underline{t} + a_{\underline{j}},$$

where

$$a_{\underline{j}} := \left(\frac{1}{2\pi \mathrm{i}}\right)^l \int_{\Delta \mathbf{P}} \frac{[P_k^* Q_k - SP_k Q_k^*](\underline{t})}{\underline{t}^{\underline{j}+\underline{1}}} \, \mathrm{d}\underline{t}.$$

Let $\mathcal{S}$ be the index set associated with S. Now since $\eta > \delta$, it is easy to see from (11) that for large enough k,

$$\mathcal{R}_k^{\#} * \mathcal{D}_k = \mathcal{R}_{\langle \delta L_k \rangle} * \mathcal{D}_k \subseteq \mathcal{I}_k,$$

$$\mathcal{S} * \mathcal{R}_k^{\#} * \mathcal{N}_k = \mathcal{S} * \mathcal{R}_{\langle \delta L_k \rangle} * \mathcal{N}_k \subseteq \mathcal{I}_k.$$

Hence for $\underline{j} \notin \mathcal{I}_k$, the coefficient $a_{\underline{j}}$ of $\underline{z}^{\underline{j}}$ in $(P_k^* Q_k - SP_k Q_k^*)(\underline{z})$ is 0. So

$$|d_{\underline{j},k}| = \left| \left(\frac{1}{2\pi \mathrm{i}}\right)^l \int_{\Delta \mathbf{P}} \frac{[Q_k^* Q_k(Sf - r_k^*)](\underline{t})}{\underline{t}^{\underline{j}+\underline{1}}} \, \mathrm{d}\underline{t} \right|$$

$$\leqslant \| Q_k^* Q_k \|_{L_\infty(\mathbf{P})} E_k(Sf; \mathbf{P}) / \rho^{|\underline{j}|}.$$

Hence for $\underline{z} \in \mathbf{K}$,

$$|f - r_k|(\underline{z}) \leqslant \frac{\| Q_k^* Q_k \|_{L_\infty(\mathbf{P})}}{|SQ_k^* Q|(\underline{z})} E_k(Sf; \mathbf{P}) \textstyle\sum,$$

where

$$\sum := \sum_{\underline{j} \notin \mathcal{I}_k} \left(\frac{\lambda}{\rho}\right)^{|\underline{j}|} \leqslant \sum_{\underline{j} \in \mathbb{N}^l} \left(\frac{\lambda}{\rho}\right)^{|\underline{j}|} =: C_1.$$

Let us normalize $Q_k^* Q_k$ and S so that

$$\| Q_k^* Q_k \|_{L_\infty(\mathbf{P})} = \| S \|_{L_\infty(\mathbf{P})} = 1.$$

Given $\varepsilon \in (0, 1)$, set

$$\mathbf{E}_k := \{\underline{z}\colon |z_j| \leqslant \rho\ \forall j,\ |Q_k^* Q_k|(\underline{z}) \leqslant \varepsilon^{\partial(Q_k^* Q_k)}\};$$

$$\mathbf{F} := \{\underline{z}\colon |z_j| \leqslant \rho\ \forall j,\ |S|(\underline{z}) \leqslant \varepsilon^{\partial S}\}.$$

By our Lemma 2.4, for large enough k,

$$E_k(Sf; \mathbf{P}) \leqslant \varepsilon^{3L_k}.$$

Hence for $z \in \mathbf{K} \backslash (\mathbf{E}_k \cup \mathbf{F})$,

$$|f - r_k|(z) \leqslant C_2 \varepsilon^{-[\partial(Q_k^* Q_k) + \partial S] + 3L_k} \leqslant \varepsilon^{L_k},$$

for large enough k. Here we have used the fact that for large enough k,

$$[\partial(Q_k^* Q_k) + \partial S] \leqslant \delta L_k + L_k + \partial S \leqslant \tfrac{3}{2} L_k.$$

Finally Lemmas 2.2 and 2.3 show that $\mathbf{E}_k \cup \mathbf{F}$ has small meas/cap$^{(l)}/\Gamma_l^{\mathrm{F}}$. In applying those lemmas, recall that ε is independent of ρ. □

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