Spherical Orthogonal Polynomials and Symbolic-Numeric Gaussian Cubature Formulas

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Abstract. It is well-known that the classical univariate orthogonal polynomials give rise to highly efficient Gaussian quadrature rules. We show how the classical orthogonal polynomials can be generalized to a multivariate setting and how this generalization leads to Gaussian cubature rules for specific families of multivariate polynomials.
The multivariate homogeneous orthogonal functions that we discuss here satisfy a unique slice projection property: they project to univariate orthogonal polynomials on every one-dimensional subspace spanned by a vector from the unit hypersphere. We therefore call them spherical orthogonal polynomials.

1 Spherical Orthogonal Polynomials

The orthogonal polynomials under discussion were first introduced in [1] in a different form and later in [3] in the current form. Originally they were not termed spherical orthogonal polynomials because of a lack of insight into the mechanism behind the definition.

In dealing with multivariate polynomials and functions we shall often switch between the cartesian and the spherical coordinate system. The cartesian coordinates $X = (x_1, \dots, x_n) \in \mathbb{R}^n$ are then replaced by $X = (x_1, \dots, x_n) = (\xi_1 z, \dots, \xi_n z)$ with $\xi_k, z \in \mathbb{R}$ where the directional vector $\xi = (\xi_1, \dots, \xi_n)$ belongs to the unit sphere $S_n = \{\xi : ||\xi||_p = 1\}$. Here $||\cdot||_p$ denotes one of the usual Minkowski norms. While ξ contains the directional information of X, the variable z contains the (possibly signed) distance information. With the multi-index $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{N}^n$ the notations $X^\kappa, \kappa!$ and $|\kappa|$ respectively denote

$$X^\kappa = x_1^{\kappa_1} \dots x_n^{\kappa_n} \qquad \kappa! = \kappa_1! \dots \kappa_n! \qquad |\kappa| = \kappa_1 + \dots + \kappa_n$$

Since z can be positive as well as negative and hence two directional vectors can generate X, we also introduce a signed distance function

$$\mathrm{sd}(X) = \mathrm{sgn}(x_1)||X||_p$$

For the sequel of the discussion we need some more notation. We denote by $\mathbb{R}[z]$ the linear space of polynomials in the variable z with real coefficients, by

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$\mathbb{R}[\xi] = \mathbb{R}[\xi_1, \ldots, \xi_n]$ the linear space of n-variate polynomials in ξ_k with real coefficients, by $\mathbb{R}(\xi) = \mathbb{R}(\xi_1, \ldots, \xi_n)$ the commutative field of rational functions in ξ_k and with real coefficients, by $\mathbb{R}(\xi)[z]$ the linear space of polynomials in the variable z with coefficients from $\mathbb{R}(\xi)$ and by $\mathbb{R}[\xi][z]$ the linear space of polynomials in the variable z with coefficients from $\mathbb{R}[\xi]$.

Let us introduce the linear functional Γ acting on the variable z as $\Gamma(z^i) = c_i(\xi)$, where $c_i(\xi)$ is a homogeneous expression of degree i in the ξ_k. For our purpose we take

$$c_i(\xi) = \sum_{|\kappa|=i} c_\kappa \xi^\kappa \qquad c_\kappa = \frac{|\kappa|!}{\kappa!} \int \cdots \int_{||X||_p \le 1} w\left(||X||_p\right) X^\kappa \, dX \tag{1}$$

$$\Gamma(z^i) = \int \cdots \int_{||X||_p \le 1} w\left(||X||_p\right) \left(\sum_{k=1}^{n} x_k \xi_k \right)^i dX \qquad dX = dx_1 \ldots dx_n \tag{2}$$

The n-variate polynomials under investigation are of the form

$$V_m(X) = \mathcal{V}_m(z) = \sum_{i=0}^{m} B_{m^2-i}(\xi) z^i \qquad B_{m^2-i}(\xi) = \sum_{|\kappa|=m^2-i} b_\kappa \xi^\kappa \tag{3}$$

The function $V_m(X)$ is a polynomial of degree m in z with polynomial coefficients from $\mathbb{R}[\xi]$. The coefficients $B_{m(m-1)}(\xi), \ldots, B_{m^2}(\xi)$ are homogeneous polynomials in the parameters ξ_k. The function $V_m(X)$ does itself not belong to $\mathbb{R}[X]$ but since $V_m(X) = \mathcal{V}_m(z)$, it belongs to $\mathbb{R}[\xi][z]$. Therefore the function $V_m(X)$ is given the name spherical polynomial: with every $\xi \in S_n$ the function $V_m(X) = \mathcal{V}_m(z)$ is associated which is a polynomial of degree m in the variable $z = \mathrm{sd}(X)$. Imposing the orthogonality conditions

$$\Gamma(z^i \mathcal{V}_m(z)) = 0 \qquad i = 0, \ldots, m-1 \tag{4}$$

signifies for $\mathcal{V}_m(z)$ and for $i = 0, \ldots, m-1$:

$$\Gamma(z^i \mathcal{V}_m(z)) = \int \cdots \int_{||X||_p \le 1} w(||X||_p) \left(\sum_{k=1}^{n} x_k \xi_k \right)^i \mathcal{V}_m \left(\sum_{k=1}^{n} x_k \xi_k \right) dX = 0$$

With the $c_i(\xi)$ we define the polynomial Hankel determinants

$$H_m(\xi) = \begin{vmatrix} c_0(\xi) & \cdots & c_{m-1}(\xi) \\ \vdots & \iddots & c_m(\xi) \\ & & \vdots \\ c_{m-1}(\xi) & \cdots & c_{2m-2}(\xi) \end{vmatrix} \qquad H_0(\xi) = 1$$

and call the functional Γ definite if

$$H_m(\xi) \not\equiv 0 \qquad m \ge 0$$

In the sequel of the text we assume that $\mathcal{V}_m(z)$ satisfies (4) and that Γ is a definite functional. Also we use both the notation $V_m(X)$ and $\mathcal{V}_m(z)$ interchangeably to refer to (3).

2 Connection with Classical Orthogonal Polynomials

With $w(||X||_p) = 1$ in (4) so-called spherical Legendre polynomials are obtained and with $w(||X||_p) = 1/\sqrt{1 - ||X||_p^2}$ spherical Tchebyshev polynomials. Let us now fix $\xi = \xi^*$ and take a look at the projected polynomials

$$\mathcal{V}_{m,\xi^*}(z) = V_m(\xi_1^* z, \dots, \xi_n^* z)$$

which are polynomials of degree m in z. Are these projected polynomials themselves orthogonal? If so, what is their relationship to the univariate Legendre and Tchebyshev polynomials?

We introduce the univariate linear functional c^* acting on the variable z, by

$$c^*(z^i) = c_i(\xi^*) = \Gamma(z^i)\mid_{\xi=\xi^*} \tag{5}$$

In what follows we use the notation $V_m(z)$ to denote the univariate polynomials of degree m orthogonal with respect to the linear

Theorem 1. *Let the monic univariate polynomials $V_m(z)$ satisfy the orthogonality conditions $c^*(z^i V_m(z)) = 0$ with c^* given by (5) and $i = 0, \dots, m-1$, and let the multivariate functions $V_m(X) = \mathcal{V}_m(z)$ satisfy the orthogonality conditions (4). Then for $X^* = (\xi_1^* z, \dots, \xi_n^* z)$,*

$$H_m(\xi^*) V_m(z) = p_m(\xi^*) \mathcal{V}_{m,\xi^*}(z) = p_m(\xi^*) V_m(X^*)$$

In words, Theorem 2 says that, up to a normalizing factor $p_m(\xi^*)/H_m(\xi^*)$, the orthogonality conditions and the projection operator commute.

With respect to the projection property it is important to point out that $c^*(z^i)$ does not coincide with the one-dimensional version of c_κ given by (2), meaning (2) for $n = 1$ and $\kappa = i$. While in the one-dimensional situation, the linear functional

$$c(z^i) = c_i = \int_{-1}^{1} w(|x|) x^i \, dx \tag{6}$$

gives rise to the classical orthogonal polynomials, we do not immediately retrieve these classical polynomials from the projection, because the projected functional c^* given by (5) does not coincide with the functional c given by (6).

3 Gaussian Cubature Formulas

If the functional Γ is positive definite, in other words if $H_m(\xi) > 0$ for $m \geq 0$, then the zeroes $z_i^{(m)}(\xi^*)$ of $\mathcal{V}_{m,\xi^*}(z)$ are real and simple. In a neighbourhood of $z_i^{(m)}(\xi^*)$ holds that $z_i^{(m)}(\xi^*) = \phi_i^{(m)}(\xi^*)$ for a unique holomorphic function $\phi_i^{(m)}(\xi^*)$. Let us denote

$$\mathcal{W}_{m-1}(u) = \Gamma\left(\frac{\mathcal{V}_m(z) - \mathcal{V}_m(u)}{z - u}\right)$$

$$A_i^{(m)}(\xi) = \frac{\mathcal{W}_{m-1,\xi}(z_i^{(m)})}{\mathcal{V}'_{m,\xi}(z_i^{(m)})} = \frac{\mathcal{W}_{m-1}(\phi_i^{(m)}(\xi))}{\mathcal{V}'_m(\phi_i^{(m)}(\xi))} \tag{7}$$

Here the functions $\mathcal{W}_{m-1}(z)$ are also spherical polynomials, now of degree $m-1$ in z. Then the following cubature formula can rightfully be called a Gaussian cubature formula [2].

Theorem 2. *Let $\mathcal{P}(z)$ be a polynomial of degree $2m-1$ belonging to $\mathbb{R}(\xi)[z]$. Let the functions $\phi_i^{(m)}(\xi)$ be mutually distinct. Then*

$$\int \cdots \int_{||X||_p \leq 1} w(||X||_p)\, \mathcal{P}(\textstyle\sum_{k=1}^n \xi_k x_k)\, dX = \sum_{i=1}^m A_i^{(m)}(\xi)\mathcal{P}(\phi_i^{(m)}(\xi))$$

The m-point Gaussian cubature formula from Theorem 3, with its parametrized nodes and weights, in fact exactly integrates an entire parametrized family of polynomials, over a domain in $\mathbb{R}^n$. We illustrate Theorem 3 with a bivariate example. Consider again the ℓ_2-norm and take

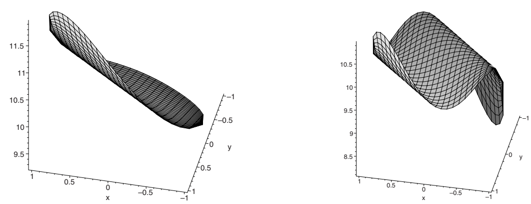
$$\mathcal{P}(z) = \frac{\xi_1}{\xi_2 + 1} z^3 + \frac{\xi_2}{\xi_1^2 + 1} z^2 + z + 10$$

Then

$$\phi_1^{(2)}(\xi) = \frac{1}{2}\sqrt{\xi_1^2 + \xi_2^2} \qquad \phi_2^{(2)}(\xi) = -\frac{1}{2}\sqrt{\xi_1^2 + \xi_2^2} \qquad A_1^{(2)}(\xi) = A_2^{(2)}(\xi) = \frac{\pi}{2}$$

$$\begin{aligned}\iint_{||(x,y)|| \leq 1} \mathcal{P}(\xi_1 x + \xi_2 y) dx\, dy &= \frac{\pi\left(\xi_2^3 + \xi_2\xi_1^2 + 40\xi_1^2 + 40\right)}{4\left(\xi_1^2 + 1\right)} \\ &= A_1^{(2)}\mathcal{P}(\phi_1^{(2)}(\xi)) + A_2^{(2)}\mathcal{P}(\phi_2^{(2)}(\xi))\end{aligned}$$

Two members of this family that are exactly integrated over the unit disk are for instance $\mathcal{P}_1(x,y)$ and $\mathcal{P}_2(x,y)$ which are obtained by choosing $(\xi_1, \xi_2) = (3/5, 4/5)$ and $(\xi_1, \xi_2) = (-\sqrt{2}/2, -\sqrt{2}/2)$ respectively.



$\mathcal{P}_1(x,y)$ $\mathcal{P}_2(x,y)$

Fig. 1.

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