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On the Froissart phenomenon in multivariate homogeneous Padé approximation

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In univariate Padé approximation we learn from the Froissart phenomenon that Padé approximants to perturbed Taylor series exhibit almost cancelling pole–zero combinations that are unwanted. The location of these pole–zero doublets was recently characterized for rational functions by the so-called Froissart polynomial. In this paper the occurrence of the Froissart phenomenon is explored for the first time in a multivariate setting. Several obvious questions arise. Which definition of Padé approximant is to be used? Which multivariate rational functions should be investigated? When considering univariate projections of these functions, our analysis confirms the univariate results obtained so far in [13], under the condition that the noise is added after projection. At the same time, it is apparent from section 4 that for the unprojected multivariate Froissart polynomial no conjecture can be formulated yet.

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1. Introduction

The technique of Padé approximation is rather well understood when one is dealing with univariate functions. A lot of information can, for instance, be found in [1]. In the sequel we adopt the notation $[n/m]^f$ for the Padé approximant to the function $f(z)$ computed from the polynomials $p(z)$ and $q(z)$ which are respectively of degree n and m and which satisfy

$$(fq-p)(z) = \sum_{i=n+m+1}^{\infty} e_i z^i.$$

Problems such as

- normality: conditions for a Padé approximant to be unique in the Padé table;

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- consistency: the exact reconstruction of rational functions by the Padé approximation technique;
- continuity: conditions for the Padé operator $\mathcal{P}_{n,m} : f \to [n/m]^f$ to be a continuous operator of f;

have all been studied in the past.

Related to these problems is the study of the effect of noise on the Padé approximation process. Properties such as non-normality and consistency are very noise-sensitive. How the presence of noise in input affects the final output has always been an important issue in numerical procedures. Already by Froissart [2,9,11,14] it was pointed out that the Padé approximants $[n/n]^f(z)$ for the simple rational function $f(z) = 1/(1 - z)$, when computed in floating-point arithmetic, have a stable pole near $z = 1$, a ghost zero way out and $n - 1$ pole–zero doublets that almost cancel. Recently this phenomenon has been described in a more formal way by Gilewicz and Pindor [12,13], first for the simple pole case $f(z) = 1/(1 - z)$ and afterwards for irreducible rational functions with a numerator degree smaller than the denominator degree.

When looking into these problems for multivariate Padé approximants, several questions arise.

- Which definition of multivariate Padé approximant should we consider?
- Can we expect a connection with Froissart's result on some univariate cuts?
- Which multivariate function should we study first?

In order to be able to answer these questions, let us briefly recall the results obtained in [12]. It is well known that in exact arithmetic the Padé approximants $[n/m]^f$ with $n \geqslant 0$ and $m \geqslant 1$ for

$$f(z) = \frac{1}{1 - z} = \sum_{i=0}^{\infty} z^i \tag{1}$$

all equal f and hence form an infinite block in the Padé table for f. When the series (1) is perturbed by uniformly distributed noise εr_i, with $\varepsilon \ll 1$ and r_i uniformly distributed in $(-1, 1)$, to

$$f_\varepsilon(z) = \sum_{i=0}^{\infty} (1 + \varepsilon r_i) z^i,$$

then the infinite block disappears and the Padé approximants inside the block do not equal but rather resemble f in the following sense. The results are again exactly computed results. The Padé approximants $[n - 1/n]^{f_\varepsilon}$ are given by

$$[n - 1/n]^{f_\varepsilon}(z) = \frac{K_{n-1}(z) + \varepsilon P_{n-1}^{(1)}(z) + \varepsilon^2 P_{n-1}^{(2)}(z)}{(1 - z)K_{n-1}(z) + \varepsilon Q_n^{(1)}(z)} \tag{2}$$

with $K_{n-1}(z)$ as specified below. In a similar way, Gilewicz and Pindor [13] showed that for

$$g(z) = \frac{1}{1-z} + \frac{1}{1-z/\alpha}, \quad \alpha \neq 1, \tag{3}$$

the $[n-1/n]^{g_\varepsilon}$ approximants to the disturbed series

$$g_\varepsilon(z) = \sum_{i=0}^{\infty} \left(1 + \frac{1}{\alpha^i} + \varepsilon r_i\right) z^i$$

are given by

$$[n-1/n]^{g_\varepsilon}(z) = \frac{\left(2 - \left(1 + \frac{1}{\alpha}\right)z\right) K_{n-2}^{(\alpha)}(z) + \varepsilon P_{n-1}^{(1)}(z) + \varepsilon^2 P_{n-1}^{(2)}(z) + \varepsilon^3 P_{n-1}^{(3)}(z)}{(1-z)\left(1 - \frac{z}{\alpha}\right) K_{n-2}^{(\alpha)}(z) + \varepsilon Q_n^{(1)}(z) + \varepsilon^2 Q_n^{(2)}(z)}. \tag{4}$$

In both cases $P_{n-1}^{(i)}$ and $Q_n^{(i)}$ are polynomials of degrees $n-1$ and n, respectively. The polynomials $K_{n-1}(z)$ and $K_{n-2}^{(\alpha)}(z)$ are called Froissart polynomials and, because of the continuity of the roots of a polynomial with respect to the coefficients, they indicate the location of the pole–zero doublets that result from the perturbation. Without going into detail, we can write

$$K_{n-1}(z) = \begin{vmatrix} 1 & \dots & z^{n-1} \\ d_{2n-1} & \dots & d_n \\ \vdots & & \vdots \\ d_{n+1} & \dots & d_2 \end{vmatrix} = \sum_{i=0}^{n-1} k_i z^i, \tag{5}$$

where $d_i = r_i - 2r_{i-1} + r_{i-2}$ and

$$K_{n-2}^{(\alpha)}(z) = \begin{vmatrix} 1 & \dots & z^{n-2} \\ \rho_{2n-1}^{(1,\alpha)} & \dots & \rho_{n+1}^{(1,\alpha)} \\ \vdots & & \vdots \\ \rho_{n+2}^{(1,\alpha)} & \dots & \rho_4^{(1,\alpha)} \end{vmatrix} = \sum_{i=0}^{n-2} k_i^{(\alpha)} z^i, \tag{6}$$

where

$$\delta_j^{(1,\alpha)} = r_j - r_{j-1} - \frac{1}{\alpha}(r_{j-1} - r_{j-2}),$$
$$\rho_j^{(1,\alpha)} = \delta_j^{(1,\alpha)} - \delta_{j-1}^{(1,\alpha)} - \frac{1}{\alpha}\left(\delta_{j-1}^{(1,\alpha)} - \delta_{j-2}^{(1,\alpha)}\right).$$

It is conjectured in [12] that for uniformly distributed r_i

$$(z-1)\frac{K_{n-1}(z)}{k_{n-1}} \approx z^n - 1$$

or

$$\frac{1}{k_{n-1}} K_{n-1}(z) \approx \sum_{i=0}^{n-1} z^i.$$

When moving the pole of (1) from $z = 1$ to $z = \beta$, then some β-dependence in (5) can be observed but at the same time, in case the r_i are uniformly distributed, many pole–zero doublets stay in the neighbourhood of the unit circle. For $K_{n-2}^{(\alpha)}(z)$ it was suggested that with uniformly distributed noise

$$(z-1)(z-\alpha)\frac{K_{n-2}^{(\alpha)}(z)}{k_{n-2}^{(\alpha)}} \approx z^n - 1. \tag{7}$$

When studying multivariate Padé approximants, one of the first things that attracts attention is that there is no single unique definition for the notion of multivariate Padé approximant [5]. But among all the possible definitions that of the homogeneous Padé approximant stands out because it is very close to the univariate definition (for other reasons other definitions may of course be preferred). Among the properties shared with the univariate Padé approximant are the consistency and continuity property of the homogeneous Padé operator, similar conditions for normality of an approximant and the square block structure of the homogeneous Padé table [6]. So it is natural that we start off the study of the Froissart phenomenon in multivariate Padé approximation by looking at the homogeneous Padé approximants.

With respect to the second question raised above, whether there is an immediate connection between the multivariate phenomenon and the univariate one, the answer is yes. The homogeneous Padé approximants satisfy a very strong projection property, meaning that they reduce to univariate Padé approximants when projected onto 1-dimensional subspaces. We shall recall this property in section 2. Let us already point out that the projection property should be applied with care when comparing the multivariate results with the univariate ones, as is illustrated in section 4.

With respect to the question of which perturbed rational functions may exhibit a similar behaviour, another property of the homogeneous multivariate Padé approximant plays a role. When the rational function $f(x_1, \ldots, x_t)$ is of the form

$$f(x_1, \ldots, x_t) = (g \circ \ell)(x_1, \ldots, x_t)$$

with $\ell(x_1, \ldots, x_t)$ a linear function of the variables $x_1, \ldots, x_t$ and g a univariate rational function, then the homogeneous Padé approximants to f will turn out to be the univariate Padé approximants $[n/m]^g(z)$ for $z = \ell(x_1, \ldots, x_t)$. Hence functions f of this type are not really interesting because the univariate conclusions apply immediately. This covers all functions $f(x_1, \ldots, x_t)$ of the form

$$f(x_1, \ldots, x_t) = 1/(a + b_1 x_1 + \cdots + b_t x_t).$$

We shall therefore start our discussion by looking at another type of rational function, namely

$$f(x,y) = \frac{1}{\alpha - x} + \frac{1}{\beta - y}. \tag{8}$$

Only for reasons of notational simplicity do we deal with the bivariate case instead of with the general multivariate case. All conclusions will remain valid for the function

$$f(x_1,\dots,x_t) = \frac{1}{\alpha_1 - x_1} + \cdots + \frac{1}{\alpha_t - x_t}.$$

2. Homogeneous multivariate Padé approximants

A lot of information on homogeneous multivariate Padé approximants can be found in [4,5]. We recall the determinant representation of numerator and denominator because these are needed in the sequel.

Given a Taylor series expansion

$$f(x,y) = \sum_{i+j=0}^{\infty} c_{ij} x^i y^j, \tag{9}$$

we introduce the homogeneous expressions

$$C_\ell(x,y) = \sum_{i+j=\ell} c_{ij} x^i y^j, \quad \ell = 0,1,2,\dots.$$

For chosen ν and μ, we also define the homogeneous expressions

$$A_\ell(x,y) = \sum_{i+j=\nu\mu+\ell} a_{ij} x^i y^j, \quad \ell = 0,\dots,\nu,$$
$$B_\ell(x,y) = \sum_{i+j=\nu\mu+\ell} b_{ij} x^i y^j, \quad \ell = 0,\dots,\mu,$$

and the bivariate polynomials

$$p(x,y) = \sum_{\ell=0}^{\nu} A_\ell(x,y),$$
$$q(x,y) = \sum_{\ell=0}^{\mu} B_\ell(x,y).$$

In the homogeneous multivariate Padé approximation problem the values ν and μ play the role of the univariate degrees n and m, and the polynomials $p(x,y)$ and $q(x,y)$ are determined from the conditions

$$(fq - p)(x,y) = \sum_{i+j \geqslant \nu\mu+\nu+\mu+1} e_{ij} x^i y^j, \tag{10}$$

which can be rewritten as

$$\begin{cases} C_0(x,y)B_0(x,y) = A_0(x,y), \\ C_1(x,y)B_0(x,y) + C_0(x,y)B_1(x,y) = A_1(x,y), \\ \quad \vdots \\ C_\nu(x,y)B_0(x,y) + \cdots + C_{\nu-\mu}(x,y)B_\mu(x,y) = A_\nu(x,y), \end{cases} \tag{11a}$$

$$\begin{cases} C_{\nu+1}(x,y)B_0(x,y) + \cdots + C_{\nu+1-\mu}(x,y)B_\mu(x,y) \equiv 0, \\ \quad \vdots \\ C_{\nu+\mu}(x,y)B_0(x,y) + \cdots + C_\nu(x,y)B_\mu(x,y) \equiv 0, \end{cases} \tag{11b}$$

where $C_\ell(x,y) \equiv 0$ if $\ell < 0$. This is exactly the system of defining equations for univariate Padé approximants if the term $c_\ell x^\ell$ in the univariate definition is substituted by

$$C_\ell(x,y) = \sum_{i+j=\ell} c_{ij} x^i y^j, \quad \ell = 0, 1, 2, \ldots.$$

The homogeneous multivariate Padé approximant $[\nu/\mu]_H^f$ for $f(x,y)$ is defined as the unique irreducible form of a solution $p(x,y)/q(x,y)$ of (11). Several suitable normalizations are possible. This unicity of the irreducible form is a distinctive characteristic of the homogeneous approach and is thoroughly discussed in [4,5]. From (11) it is easy to obtain a determinant formula for p and q. We recall it for $\nu = n-1$ and $\mu = n$. The dimension of the determinants below is $(n+1) \times (n+1)$.

$$p(x,y) = \sum_{i+j=(n-1)n}^{n^2-1} a_{ij} x^i y^j = \begin{vmatrix} \sum_{i=0}^{n-1} C_i(x,y) & \sum_{i=0}^{n-2} C_i(x,y) & \ldots & C_0(x,y) & 0 \\ C_n(x,y) & C_{n-1}(x,y) & \ldots & C_1(x,y) & C_0(x,y) \\ \vdots & \vdots & & \vdots & \vdots \\ C_{2n-1}(x,y) & C_{2n-2}(x,y) & \ldots & C_n(x,y) & C_{n-1}(x,y) \end{vmatrix}, \tag{12}$$

$$q(x,y) = \sum_{i+j=(n-1)n}^{n^2} b_{ij} x^i y^j = \begin{vmatrix} 1 & 1 & \ldots & 1 & 1 \\ C_n(x,y) & C_{n-1}(x,y) & \ldots & C_1(x,y) & C_0(x,y) \\ \vdots & \vdots & & \vdots & \vdots \\ C_{2n-1}(x,y) & C_{2n-2}(x,y) & \ldots & C_n(x,y) & C_{n-1}(x,y) \end{vmatrix}. \tag{13}$$

The homogeneous Padé approximants also satisfy the following projection property [3].

Theorem 1. Let $(x, y) = (\lambda_1 z, \lambda_2 z)$ with $\lambda_i \in \mathbb{C}$ for $i = 1, 2$ and let $f_{\lambda_1,\lambda_2}(z) = f(\lambda_1 z, \lambda_2 z)$. Then the irreducible form of the homogeneous multivariate Padé approximant $[\nu/\mu]_H^f(\lambda_1 z, \lambda_2 z)$ restricted to the values $(x, y) = (\lambda_1 z, \lambda_2 z)$ equals the univariate Padé approximant $[\nu/\mu]^{f_{\lambda_1,\lambda_2}}(z)$ for the function $f_{\lambda_1,\lambda_2}(z)$.

The application of this property to the problem under investigation will be discussed in section 4.

In the sequel we focus on $f(x, y)$ given by (8), which we perturb by adding uniformly distributed noise, to obtain

$$f_\varepsilon(x, y) = \sum_{i=0}^{\infty} \frac{x^i}{\alpha^{i+1}} + \sum_{j=0}^{\infty} \frac{y^j}{\beta^{j+1}} + \varepsilon \sum_{i,j=0}^{\infty} r_{ij} x^i y^j, \quad r_{ij} \in (-1, 1). \tag{14}$$

3. Calculating the multivariate Froissart polynomial

The main result of this section is not merely a numerical experiment in the style of Froissart's original investigation, to see whether the so-called Froissart phenomenon does or does not occur for homogeneous Padé approximants. We actually want to establish two things.

First, the occurrence of the Froissart phenomenon in homogeneous Padé approximation is confirmed by computing explicit formulas for $[n-1/n]_H^{f_\varepsilon}$, in a similar way as was done in (2) and (4). The reader can see that a so-called bivariate Froissart polynomial associated with $[n-1/n]_H^{f_\varepsilon}$ is obtained by transforming the determinant representations (12) and (13). The transformations carried out on $p(x, y)$ are very similar to those carried out on $q(x, y)$.

Second, the analogy between the univariate and multivariate situation, especially with respect to the so-called univariate and multivariate Froissart polynomials containing the almost cancelling pole–zero combinations, once more confirms our intuition about homogeneous Padé approximants. They are indeed very close to the classical univariate Padé approximants, with the origin (or the point around which the series development (9) is given) being an exceptional point. That $(0, 0)$ plays a special role was confirmed in a recent convergence theorem of type de Montessus de Ballore [8]. It is also obvious from the fact that the approximants $[n-1/n]_H^{f_\varepsilon}$ are singular for $(x, y) = (0, 0)$. In an earlier study Werner [17] already indicated that when evaluating the homogeneous Padé approximant, one should avoid some critical directions while approaching the origin.

In the future one could investigate whether the natural boundary, established by the Froissart polynomial, may have any effect on the convergence of the homogeneous multivariate Padé approximants $[n-1/n]_H^{f_\varepsilon}$. This problem should also be seen in the light of the recent convergence results in measure and capacity obtained in [7]. In section 4 we illustrate that the $[n-1/n]_H^{f_\varepsilon}$ possibly still converge beyond the boundary of the Froissart polynomial. In the univariate case this behaviour was discussed by Gammel [10].

Before proceeding, we introduce some notations.

3.1. Notations

With the noise r_{ij} that is being added to (8) as in (14) and that is uniformly distributed in $(-1,1)$, we define

$$\begin{aligned}
R_k(x,y) &= \sum_{i+j=k} r_{ij}x^iy^j,\\
\Delta R_k(x,y) &= R_k(x,y) - R_{k-1}(x,y),\\
\Theta R_k(x,y) &= R_k(x,y) - \left(\frac{x}{\alpha}+\frac{y}{\beta}\right)R_{k-1}(x,y) + \frac{x}{\alpha}\frac{y}{\beta}R_{k-2}(x,y),\\
\nabla R_k(x,y) &= \Delta R_k(x,y) - \left(\frac{x}{\alpha}+\frac{y}{\beta}\right)\Delta R_{k-1}(x,y) + \frac{x}{\alpha}\frac{y}{\beta}\Delta R_{k-2}(x,y),\\
\Phi R_k(x,y) &= \nabla R_k(x,y) - \left(\frac{x}{\alpha}+\frac{y}{\beta}\right)\nabla R_{k-1}(x,y) + \frac{x}{\alpha}\frac{y}{\beta}\nabla R_{k-2}(x,y).
\end{aligned}$$

At the same time we use the fact that for (8)

$$\begin{aligned}
C_k(x,y) &= \frac{x^k}{\alpha^{k+1}} + \frac{y^k}{\beta^{k+1}} + \varepsilon R_k(x,y),\\
\Delta C_k(x,y) &= C_k(x,y) - C_{k-1}(x,y)\\
&= \frac{x^{k-1}}{\alpha^k}\left(\frac{x}{\alpha}-1\right) + \frac{y^{k-1}}{\beta^k}\left(\frac{y}{\beta}-1\right) + \varepsilon\Delta R_k(x,y).
\end{aligned}$$

Since there is no doubt that all expressions with R_k and C_k are functions of x and y, and to simplify notations, we will omit writing (x,y) in the sequel. The homogeneous order, denoted by ω, and the homogeneous degree, denoted by ∂, of the functions listed above are respectively given by

$$\begin{aligned}
\partial(\Delta R_k) &= k, & \omega(\Delta R_k) &= k-1,\\
\partial(\Theta R_k) &= k, & \omega(\Theta R_k) &= k,\\
\partial(\nabla R_k) &= k, & \omega(\nabla R_k) &= k-1,\\
\partial(\Phi R_k) &= k, & \omega(\Phi R_k) &= k-1.
\end{aligned}$$

3.2. *Rewriting the denominator of* $[n-1/n]_H^{f_\varepsilon}$

After subtracting column $i+1$ from column i for $i = 1, \ldots, n$, the denominator (13) looks like

$$q(x,y) = (-1)^n \begin{vmatrix} \Delta C_n & \Delta C_{n-1} & \ldots & \Delta C_1 \\ \vdots & \vdots & & \vdots \\ \Delta C_{2n-1} & \Delta C_{2n-2} & \ldots & \Delta C_n \end{vmatrix}.$$

Subtracting column $i+1$ multiplied by $x/\alpha + y/\beta$ from column i and adding column $i+2$ multiplied by $(x/\alpha)(y/\beta)$ to column i for $i = 1, \ldots, n-2$, yields for $q(x,y)$

$$(-1)^n \begin{vmatrix} \varepsilon\nabla R_n & \ldots & \varepsilon\nabla R_3 & \Delta C_2 & \Delta C_1 \\ \varepsilon\nabla R_{n+1} & \ldots & \varepsilon\nabla R_4 & \Delta C_3 & \Delta C_2 \\ \vdots & & \vdots & \vdots & \vdots \\ \varepsilon\nabla R_{2n-1} & \ldots & \varepsilon\nabla R_{n+2} & \Delta C_{n+1} & \Delta C_n \end{vmatrix}$$

or, reformulated,

$$\begin{aligned} &(-1)^n\varepsilon^{n-2} \\ &\times \begin{vmatrix} \nabla R_n & \ldots & \nabla R_3 & \frac{x}{\alpha^2}\left(\frac{x}{\alpha}-1\right) + \frac{y}{\beta^2}\left(\frac{y}{\beta}-1\right) & \frac{1}{\alpha}\left(\frac{x}{\alpha}-1\right) + \frac{1}{\beta}\left(\frac{y}{\beta}-1\right) \\ \nabla R_{n+1} & \ldots & \nabla R_4 & \frac{x^2}{\alpha^3}\left(\frac{x}{\alpha}-1\right) + \frac{y^2}{\beta^3}\left(\frac{y}{\beta}-1\right) & \frac{x}{\alpha^2}\left(\frac{x}{\alpha}-1\right) + \frac{y}{\beta^2}\left(\frac{y}{\beta}-1\right) \\ \vdots & & \vdots & \vdots & \vdots \\ \nabla R_{2n-1} & \ldots & \nabla R_{n+2} & \frac{x^n}{\alpha^{n+1}}\left(\frac{x}{\alpha}-1\right) + \frac{y^n}{\beta^{n+1}}\left(\frac{y}{\beta}-1\right) & \frac{x^{n-1}}{\alpha^n}\left(\frac{x}{\alpha}-1\right) + \frac{y^{n-1}}{\beta^n}\left(\frac{y}{\beta}-1\right) \end{vmatrix} \\ &+ \varepsilon^{n-1}Q_n^{(1)} + \varepsilon^n Q_n^{(2)}. \end{aligned}$$

Here $Q_n^{(i)}$ are polynomials of degree n^2 and order $(n-1)n$. In the next transformations we concentrate on the first determinant in this sum. After subtracting x/α times the last column from the one but last column, we obtain for the O(ε^{n-2}) contribution

$$\begin{aligned} &(-1)^n\varepsilon^{n-2}\frac{1}{\beta}\left(\frac{y}{\beta}-1\right)\left(\frac{y}{\beta}-\frac{x}{\alpha}\right) \\ &\times \begin{vmatrix} \nabla R_n & \ldots & \nabla R_3 & 1 & \frac{1}{\alpha}\left(\frac{x}{\alpha}-1\right) + \frac{1}{\beta}\left(\frac{y}{\beta}-1\right) \\ \nabla R_{n+1} & \ldots & \nabla R_4 & \frac{y}{\beta} & \frac{x}{\alpha^2}\left(\frac{x}{\alpha}-1\right) + \frac{y}{\beta^2}\left(\frac{y}{\beta}-1\right) \\ \vdots & & \vdots & \vdots & \vdots \\ \nabla R_{2n-1} & \ldots & \nabla R_{n+2} & \frac{y^{n-1}}{\beta^{n-1}} & \frac{x^{n-1}}{\alpha^n}\left(\frac{x}{\alpha}-1\right) + \frac{y^{n-1}}{\beta^n}\left(\frac{y}{\beta}-1\right) \end{vmatrix}. \end{aligned}$$

Subtracting row $i-1$ multiplied by $x/\alpha + y/\beta$ from row i and adding row $i-2$ multiplied by $(x/\alpha)(y/\beta)$ to row i for $i = n, \ldots, 3$, results in

$$(-1)^n \varepsilon^{n-2} \frac{1}{\beta}\left(\frac{y}{\beta}-1\right)\left(\frac{y}{\beta}-\frac{x}{\alpha}\right)$$
$$\times \begin{vmatrix} \nabla R_n & \ldots & \nabla R_3 & 1 & \frac{1}{\alpha}\left(\frac{x}{\alpha}-1\right)+\frac{1}{\beta}\left(\frac{y}{\beta}-1\right) \\ \nabla R_{n+1} & \ldots & \nabla R_4 & \frac{y}{\beta} & \frac{x}{\alpha^2}\left(\frac{x}{\alpha}-1\right)+\frac{y}{\beta^2}\left(\frac{y}{\beta}-1\right) \\ \Phi R_{n+2} & \ldots & \Phi R_5 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ \Phi R_{2n-1} & \ldots & \Phi R_{n+2} & 0 & 0 \end{vmatrix}.$$

The final step of the rewriting consists in placing the last two columns in front and using the rule of the Schur complement. In this way, we get for $q(x,y)$

$$(-1)^{3n-4} \varepsilon^{n-2} \frac{1}{\beta}\left(\frac{y}{\beta}-1\right)\left(\frac{y}{\beta}-\frac{x}{\alpha}\right)$$
$$\times \begin{vmatrix} 1 & \frac{1}{\alpha}\left(\frac{x}{\alpha}-1\right)+\frac{1}{\beta}\left(\frac{y}{\beta}-1\right) & \nabla R_n & \ldots & \nabla R_3 \\ \frac{y}{\beta} & \frac{x}{\alpha^2}\left(\frac{x}{\alpha}-1\right)+\frac{y}{\beta^2}\left(\frac{y}{\beta}-1\right) & \nabla R_{n+1} & \ldots & \nabla R_4 \\ 0 & 0 & \Phi R_{n+2} & \ldots & \Phi R_5 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \Phi R_{2n-1} & \ldots & \Phi R_{n+2} \end{vmatrix}$$
$$+ \varepsilon^{n-1} Q_n^{(1)} + \varepsilon^n Q_n^{(2)}$$
$$= (-1)^{3n-5} \varepsilon^{n-2} \frac{1}{\alpha}\frac{1}{\beta}\left(\frac{x}{\alpha}-1\right)\left(\frac{y}{\beta}-1\right)\left(\frac{y}{\beta}-\frac{x}{\alpha}\right)^2 K_{n-2}^{(\alpha,\beta)}(x,y)$$
$$+ \varepsilon^{n-1} Q_n^{(1)} + \varepsilon^n Q_n^{(2)}. \tag{15}$$

We will call $K_{n-2}^{(\alpha,\beta)}(x,y)$, which is defined by

$$K_{n-2}^{(\alpha,\beta)}(x,y)=\begin{vmatrix}\Phi R_{n+2} & \dots & \Phi R_5\\ \vdots & & \vdots\\ \Phi R_{2n-1} & \dots & \Phi R_{n+2}\end{vmatrix}=\sum_{i+j=(n-2)(n+1)}^{(n-2)(n+2)} k_{ij}x^iy^j, \tag{16}$$

the bivariate Froissart polynomial.

3.3. *Rewriting the numerator of* $[n-1/n]_H^{f_\varepsilon}$

After subtracting column $i+1$ from column i for $i=1,\dots,n$, the numerator (12) looks like

$$p(x,y)=\begin{vmatrix}C_{n-1} & C_{n-2} & \dots & C_0 & 0\\ \Delta C_n & \Delta C_{n-1} & \dots & \Delta C_1 & C_0\\ \vdots & \vdots & & \vdots & \vdots\\ \Delta C_{2n-1} & \Delta C_{2n-2} & \dots & \Delta C_n & C_{n-1}\end{vmatrix}.$$

Subtracting column $i+1$ multiplied by $x/\alpha+y/\beta$ from column i and adding column $i+2$ multiplied by $(x/\alpha)(y/\beta)$ to column i for $i=1,\dots,n-2$, yields for $p(x,y)$

$$\begin{vmatrix}\varepsilon\Theta R_{n-1} & \dots & \varepsilon\Theta R_2 & C_1 & C_0 & 0\\ \varepsilon\nabla R_n & \dots & \varepsilon\nabla R_3 & \Delta C_2 & \Delta C_1 & C_0\\ \varepsilon\nabla R_{n+1} & \dots & \varepsilon\nabla R_4 & \Delta C_3 & \Delta C_2 & C_1\\ \vdots & & \vdots & \vdots & \vdots & \vdots\\ \varepsilon\nabla R_{2n-1} & \dots & \varepsilon\nabla R_{n+2} & \Delta C_{n+1} & \Delta C_n & C_{n-1}\end{vmatrix}$$

or, reformulated,

$$\varepsilon^{n-2}\times\begin{vmatrix}\Theta R_{n-1} & \dots & \Theta R_2 & \frac{x}{\alpha^2}+\frac{y}{\beta^2} & \frac{1}{\alpha}+\frac{1}{\beta} & 0\\ \nabla R_n & \dots & \nabla R_3 & \frac{x}{\alpha^2}\left(\frac{x}{\alpha}-1\right)+\frac{y}{\beta^2}\left(\frac{y}{\beta}-1\right) & \frac{1}{\alpha}\left(\frac{x}{\alpha}-1\right)+\frac{1}{\beta}\left(\frac{y}{\beta}-1\right) & \frac{1}{\alpha}+\frac{1}{\beta}\\ \nabla R_{n+1} & \dots & \nabla R_4 & \frac{x^2}{\alpha^3}\left(\frac{x}{\alpha}-1\right)+\frac{y^2}{\beta^3}\left(\frac{y}{\beta}-1\right) & \frac{x}{\alpha^2}\left(\frac{x}{\alpha}-1\right)+\frac{y}{\beta^2}\left(\frac{y}{\beta}-1\right) & \frac{x}{\alpha^2}+\frac{y}{\beta^2}\\ \vdots & & \vdots & \vdots & \vdots & \vdots\\ \nabla R_{2n-1} & \dots & \nabla R_{n+2} & \frac{x^n}{\alpha^{n+1}}\left(\frac{x}{\alpha}-1\right)+\frac{y^n}{\beta^{n+1}}\left(\frac{y}{\beta}-1\right) & \frac{x^{n-1}}{\alpha^n}\left(\frac{x}{\alpha}-1\right)+\frac{y^{n-1}}{\beta^n}\left(\frac{y}{\beta}-1\right) & \frac{x^{n-1}}{\alpha^n}+\frac{y^{n-1}}{\beta^n}\end{vmatrix}$$
$$+\,\varepsilon^{n-1}P_{n-1}^{(1)}+\varepsilon^nP_{n-1}^{(2)}+\varepsilon^{n+1}P_{n-1}^{(3)}.$$

Here $P_{n-1}^{(i)}$ are polynomials of degree n^2-1 and order $(n-1)n$. In the next transformations we concentrate on the first determinant again. After subtracting x/α times the one but last column from column $n-1$, we obtain for the $\mathrm{O}(\varepsilon^{n-2})$ contribution

$$\varepsilon^{n-2}\frac{1}{\beta}\left(\frac{y}{\beta}-\frac{x}{\alpha}\right)$$

$$\times\begin{vmatrix} \Theta R_{n-1} & \dots & \Theta R_2 & 1 & \frac{1}{\alpha}+\frac{1}{\beta} & 0 \\ \nabla R_n & \dots & \nabla R_3 & \frac{y}{\beta}-1 & \frac{1}{\alpha}\left(\frac{x}{\alpha}-1\right)+\frac{1}{\beta}\left(\frac{y}{\beta}-1\right) & \frac{1}{\alpha}+\frac{1}{\beta} \\ \nabla R_{n+1} & \dots & \nabla R_4 & \frac{y}{\beta}\left(\frac{y}{\beta}-1\right) & \frac{x}{\alpha^2}\left(\frac{x}{\alpha}-1\right)+\frac{y}{\beta^2}\left(\frac{y}{\beta}-1\right) & \frac{x}{\alpha^2}+\frac{y}{\beta^2} \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ \nabla R_{2n-1} & \dots & \nabla R_{n+2} & \frac{y^{n-1}}{\beta^{n-1}}\left(\frac{y}{\beta}-1\right) & \frac{x^{n-1}}{\alpha^n}\left(\frac{x}{\alpha}-1\right)+\frac{y^{n-1}}{\beta^n}\left(\frac{y}{\beta}-1\right) & \frac{x^{n-1}}{\alpha^n}+\frac{y^{n-1}}{\beta^n} \end{vmatrix}.$$

Subtracting row $i-1$ multiplied by $x/\alpha + y/\beta$ from row i and adding row $i-2$ multiplied by $(x/\alpha)(y/\beta)$ to row i for $i = n, \dots, 4$, results in

$$\varepsilon^{n-2}\frac{1}{\beta}\left(\frac{y}{\beta}-\frac{x}{\alpha}\right)$$

$$\times\begin{vmatrix} \Theta R_{n-1} & \dots & \Theta R_2 & 1 & \frac{1}{\alpha}+\frac{1}{\beta} & 0 \\ \nabla R_n & \dots & \nabla R_3 & \frac{y}{\beta}-1 & \frac{1}{\alpha}\left(\frac{x}{\alpha}-1\right)+\frac{1}{\beta}\left(\frac{y}{\beta}-1\right) & \frac{1}{\alpha}+\frac{1}{\beta} \\ \nabla R_{n+1} & \dots & \nabla R_4 & \frac{y}{\beta}\left(\frac{y}{\beta}-1\right) & \frac{x}{\alpha^2}\left(\frac{x}{\alpha}-1\right)+\frac{y}{\beta^2}\left(\frac{y}{\beta}-1\right) & \frac{x}{\alpha^2}+\frac{y}{\beta^2} \\ \Phi R_{n+2} & \dots & \Phi R_5 & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ \Phi R_{2n-1} & \dots & \Phi R_{n+2} & 0 & 0 & 0 \end{vmatrix}.$$

The final step of the rewriting consists in placing the last three columns in front and using the rule of the Schur complement. In this way we get for $p(x, y)$

$$\varepsilon^{n-2}\frac{1}{\beta}\left(\frac{y}{\beta}-\frac{x}{\alpha}\right)(-1)^{3(n-2)}$$

$$\times\begin{vmatrix} 1 & \frac{1}{\alpha}+\frac{1}{\beta} & 0 & \Theta R_{n-1} & \dots & \Theta R_2 \\ \frac{y}{\beta}-1 & \left(\frac{x}{\alpha}-1\right)+\left(\frac{y}{\beta}-1\right) & \frac{1}{\alpha}+\frac{1}{\beta} & \nabla R_n & \dots & \nabla R_3 \\ \frac{y}{\beta}\left(\frac{y}{\beta}-1\right) & \frac{x}{\alpha^2}\left(\frac{x}{\alpha}-1\right)+\frac{y}{\beta^2}\left(\frac{y}{\beta}-1\right) & \frac{x}{\alpha^2}+\frac{y}{\beta^2} & \nabla R_{n+1} & \dots & \nabla R_4 \\ 0 & 0 & 0 & \Phi R_{n+2} & \dots & \Phi R_5 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \Phi R_{2n-1} & \dots & \Phi R_{n+2} \end{vmatrix}$$

$$+\,\varepsilon^{n-1}P^{(1)}_{n-1}+\varepsilon^{n}P^{(2)}_{n-1}+\varepsilon^{n+1}P^{(3)}_{n-1}$$

$$= (-1)^{3n-7}\varepsilon^{n-2}\frac{1}{\alpha}\frac{1}{\beta}\left(\frac{1}{\alpha}+\frac{1}{\beta}-\frac{x}{\alpha\beta}-\frac{y}{\alpha\beta}\right)\left(\frac{y}{\beta}-\frac{x}{\alpha}\right)^2 K^{(\alpha,\beta)}_{n-2}(x,y)$$

$$+\,\varepsilon^{n-1}P^{(1)}_{n-1}+\varepsilon^{n}P^{(2)}_{n-1}+\varepsilon^{n+1}P^{(3)}_{n-1}. \tag{17}$$

3.4. Formula for $[n-1/n]_H^{f_\varepsilon}$

Combining expressions (15) and (17) we obtain for $[n-1/n]_H^{f_\varepsilon}$ the following fraction, after dividing numerator and denominator by ε^{n-2},

$$\frac{(-1)^{3n-7}\frac{1}{\alpha}\frac{1}{\beta}\left(\frac{1}{\alpha}+\frac{1}{\beta}-\frac{x}{\alpha\beta}-\frac{y}{\alpha\beta}\right)\left(\frac{y}{\beta}-\frac{x}{\alpha}\right)^2 K_{n-2}^{(\alpha,\beta)}(x,y)+\varepsilon P_{n-1}^{(1)}+\varepsilon^2 P_{n-1}^{(2)}+\varepsilon^3 P_{n-1}^{(3)}}{(-1)^{3n-5}\frac{1}{\alpha}\frac{1}{\beta}\left(\frac{x}{\alpha}-1\right)\left(\frac{y}{\beta}-1\right)\left(\frac{y}{\beta}-\frac{x}{\alpha}\right)^2 K_{n-2}^{(\alpha,\beta)}(x,y)+\varepsilon Q_n^{(1)}+\varepsilon^2 Q_n^{(2)}}. \tag{18}$$

It is important that we point out the great similarity with the univariate formula (4). Putting $\varepsilon = 0$ confirms the consistency of the univariate and homogeneous Padé operators since both deliver the unperturbed rational function as the result of the Padé approximation process:

$$[n-1/n]^{g_\varepsilon}(z)_{|\varepsilon=0} = \frac{2-z-z/\alpha}{(1-z)(1-z/\alpha)}, \qquad [n-1/n]_H^{f_\varepsilon}(x,y)_{|\varepsilon=0} = \frac{\alpha+\beta-x-y}{(\alpha-x)(\beta-y)}. \tag{19}$$

Also, in both situations, for small nonzero ε, almost cancelling zeros and poles creep in, of which the location is given by the so-called Froissart polynomials $K_{n-2}^{(\alpha)}(z)$ and $K_{n-2}^{(\alpha,\beta)}(x,y)$, respectively. In the next section we want to investigate the link between the univariate and homogeneous multivariate result somewhat deeper. As we will see, one must be careful with the application of the projection property given in theorem 1 and the correct interpretation of projected Padé approximants for $f_\varepsilon(x,y)$. As a consequence of these conclusions it will be clear that in the multivariate situation no conjecture about the location of the zeros of $K_{n-2}^{(\alpha,\beta)}(x,y)$ can be formulated yet.

4. Numerical illustrations

In order to illustrate the near-cancellation of the additional zeros and poles due to $K_{n-2}^{(\alpha,\beta)}(x,y)$, we depict the zeros of numerator and denominator of $[n-1/n]_H^{f_\varepsilon}$ for $\alpha = 1 = \beta,\ n = 4$ and $\varepsilon = 10^{-6}$. These can be found in figures 1 and 2, respectively. The zeros of the polynomial $K_{n-2}^{(\alpha,\beta)}(x,y)$ are graphed separately in figure 3. In order to be sure that our function evaluations and pictures are correct and not perturbed a second time by floating-point round-off errors, the results were computed as follows.

The uniformly distributed r_{ij} were computed using the random number generator ran2 of [15]. These floating-point values were from thereon treated as exact numbers: they were output in hexadecimal form, in order to avoid the binary to decimal conversion in traditional output, and afterwards exactly converted to rational numbers for use in Mathematica. The computer algebra system Mathematica was then used for the computation of (12) and (13). In this way (16) and (18) could be obtained without floating-point round-off errors. The contour plots $p(x,y) = 0$, $q(x,y) = 0$ and $K_2^{(1,1)}(x,y) = 0$ were obtained using the program GrafEq developed by Tupper [16]

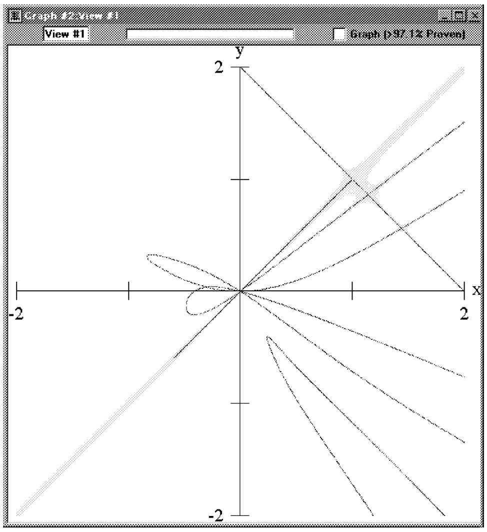

Figure 1. Zeros of the numerator of $[3/4]_H^{f_\varepsilon}$.

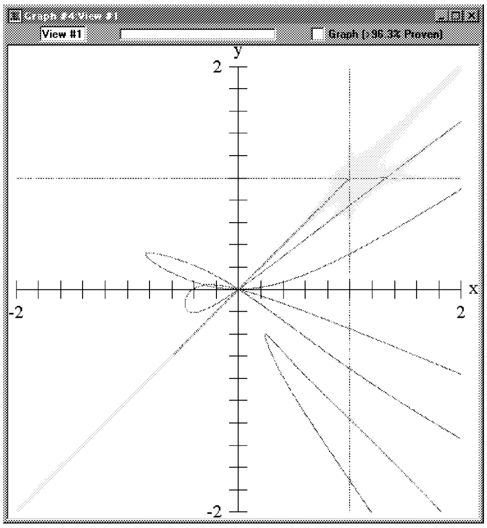

Figure 2. Zeros of the denominator of $[3/4]_H^{f_\varepsilon}$.

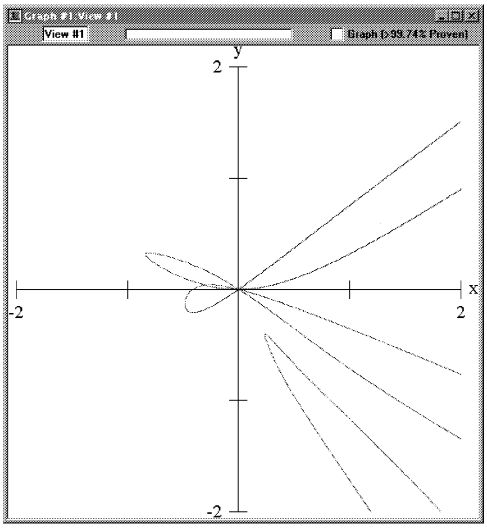

Figure 3. Zeros of factor $K_2^{(1,1)}$ in $[3/4]_H^{f_\varepsilon}$.

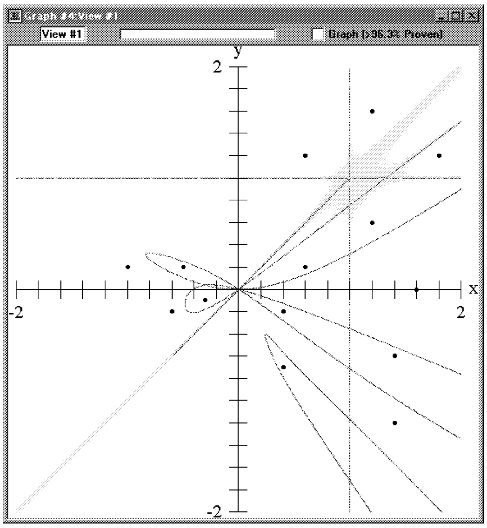

Figure 4. Evaluation points for $[n-1/n]_H^{f_\varepsilon}$.

that produces validated pictures using interval arithmetic. So these pictures are precise and reliable, of course at the cost of computation time. The black pixels indicate verified zero locations, the white area is guaranteed to be free of zeros, the gray pixels represent the unproven part of the graph, which we have kept below 4% in our figures. For the numerator and denominator of (18) computation time gets unreasonable if one wants a 100% proven graph. It should be clear to the reader that the zeros/poles displayed in figures 1 and 2 are approximately the Froissart zeros displayed in figure 3 together with the zeros/poles of (19) with $\alpha = \beta = 1$.

It is well known that in Padé approximation the poles of f do not form a natural boundary for the convergence of the approximants, neither in the univariate [1, pp. 305–315] nor in the multivariate case [7]. Let us now investigate whether the additional zeros of the Froissart polynomial have any effect on the convergence of the $[n-1/n]_H^{f_\varepsilon}$. To this end we choose 14 points in $[-2,2] \times [-2,2] \subseteq \mathbb{R}^2$ and we evaluate both $f(x,y)$ and $[n-1/n]_H^{f_\varepsilon}(x,y)$ in these points for $n = 3,\ldots,8$. The evaluation points are depicted in figure 4 where the real zeros of the denominator of $[3/4]_H^{f_\varepsilon}$ are added to the picture as reference curves. From table 1 we can see that in none of the 84 evaluations the behaviour of the approximant is disturbed.

We return to the bivariate Froissart polynomial to see whether we can obtain any additional conclusions using the projection property given in theorem 1 and the conjectures formulated in [13] for the univariate case. Here we must clearly distinguish between applying the projection property before or after the perturbation by the added noise. If the projection property is applied to the perturbed function $f_\varepsilon(x,y)$ with $(\lambda_1, \lambda_2) = (1, \ell)$, then we are computing Padé approximants to the perturbed univariate series developments

$$\begin{aligned} f_\varepsilon(z,\ell z) &= \sum_{k=0}^{\infty} \left(1 + \ell^k + \varepsilon \sum_{i+j=k} r_{ij}\ell^j\right) z^k \\ &= f(z,\ell z) + \varepsilon \sum_{k=0}^{\infty} \left(\sum_{i+j=k} r_{ij}\ell^j\right) z^k \\ &= \frac{1}{1-z} + \frac{1}{1-\ell z} + \varepsilon \sum_{k=0}^{\infty} \left(\sum_{i+j=k} r_{ij}\ell^j\right) z^k. \end{aligned} \tag{20}$$

It is clear that with uniformly distributed r_{ij}, the noise contributions

$$\sum_{i+j=k} r_{ij}\ell^j, \quad \ell \in \mathbb{R},\ k = 0,1,\ldots,$$

are no longer necessarily uniformly distributed over $(-1,1)$. Hence the conjecture formulated in [13] for the Froissart polynomial associated with

$$\frac{1}{1-z} + \frac{1}{1-\ell z} + \varepsilon \sum_{i=0}^{\infty} r_i z^i$$

does not apply to $f_\varepsilon(z,\ell z)$. Moreover, for the situation arising in (20), no conjecture has been formulated yet. In order to be dealing with the setting considered in [13], one must apply the projection property to the unperturbed $f(x,y)$ and afterwards add uniformly distributed noise terms to the univariate resulting function $f(z,\ell z)$. We illustrate this in figures 5 and 6 that will be explained next.

Table 1
Evaluation of $f(x,y)$ and $[n-1/n]_H^{f_\varepsilon}$ with $\varepsilon = 10^{-6}$.

n	(x,y)	$f(x,y)$	$[n-1/n](x,y)$	(x,y)	$f(x,y)$	$[n-1/n](x,y)$
3	$(-1/2, 1/5)$	1.91666666666	1.91666637898	$(-1, 1/5)$	1.75	1.74999986206
4		1.91666666666	1.91666639635		1.75	1.74999988254
5		1.91666666666	1.91666639319		1.75	1.74999653092
6		1.91666666666	1.91666639537		1.75	1.75000111398
7		1.91666666666	1.91666639139		1.75	1.75000105706
8		1.91666666666	1.91666639103		1.75	1.75000855279
3	$(-3/5, -1/5)$	1.45833333333	1.45833349464	$(6/5, 3/5)$	−2.5	−2.50002013520
4		1.45833333333	1.45833337424		−2.5	−2.49962314033
5		1.45833333333	1.45833361006		−2.5	−2.49994302543
6		1.45833333333	1.45833359115		−2.5	−2.50004962015
7		1.45833333333	1.45833359447		−2.5	−2.49998418565
8		1.45833333333	1.45833359343		−2.5	−2.50002222630
3	$(6/5, 8/5)$	−6.66666666666	−6.67172612455	$(9/5, 6/5)$	−6.25	−6.24934857649
4		−6.66666666666	−6.66662332074		−6.25	−6.24927991704
5		−6.66666666666	−6.66669208105		−6.25	−6.24989791437
6		−6.66666666666	−6.66663811330		−6.25	−6.24838158080
7		−6.66666666666	−6.66645288761		−6.25	−6.24996264045
8		−6.66666666666	−6.66669589454		−6.25	−6.25032940654
3	$(-3/10, -1/10)$	1.67832167832	1.67832151238	$(3/5, 1/5)$	3.75	3.74999880665
4		1.67832167832	1.67832151459		3.75	3.74999966054
5		1.67832167832	1.67832151390		3.75	3.74999923730
6		1.67832167832	1.67832151389		3.75	3.74999922231
7		1.67832167832	1.67832151389		3.75	3.74999922511
8		1.67832167832	1.67832151389		3.75	3.74999922367
3	$(7/5, -6/5)$	−2.04545454545	−2.04546938025	$(7/5, -3/5)$	−1.875	−1.87499723023
4		−2.04545454545	−2.04545069177		−1.875	−1.87499535906
5		−2.04545454545	−2.04545423859		−1.875	−1.87499843901
6		−2.04545454545	−2.04544942515		−1.875	−1.87497173357
7		−2.04545454545	−2.04547439550		−1.875	−1.87499804318
8		−2.04545454545	−2.04546017081		−1.875	−1.87502349134
3	$(2/5, -7/10)$	2.25490196078	2.25490074326	$(2/5, -1/5)$	2.5	2.49999944645
4		2.25490196078	2.25490050872		2.5	2.49999944834
5		2.25490196078	2.25490090406		2.5	2.49999944756
6		2.25490196078	2.25490098789		2.5	2.49999944736
7		2.25490196078	2.25490100471		2.5	2.49999944742
8		2.25490196078	2.25490099571		2.5	2.49999944742
3	$(8/5, 0)$	−0.666666666666	−0.666670516107	$(3/5, 6/5)$	−2.5	−2.50054513475
4		−0.666666666666	−0.666670516617		−2.5	−2.49998241114
5		−0.666666666666	−0.666662930051		−2.5	−2.50003540571
6		−0.666666666666	−0.666665522907		−2.5	−2.49998575627
7		−0.666666666666	−0.666664410270		−2.5	−2.49997084941
8		−0.666666666666	−0.666663566976		−2.5	−2.50001195516

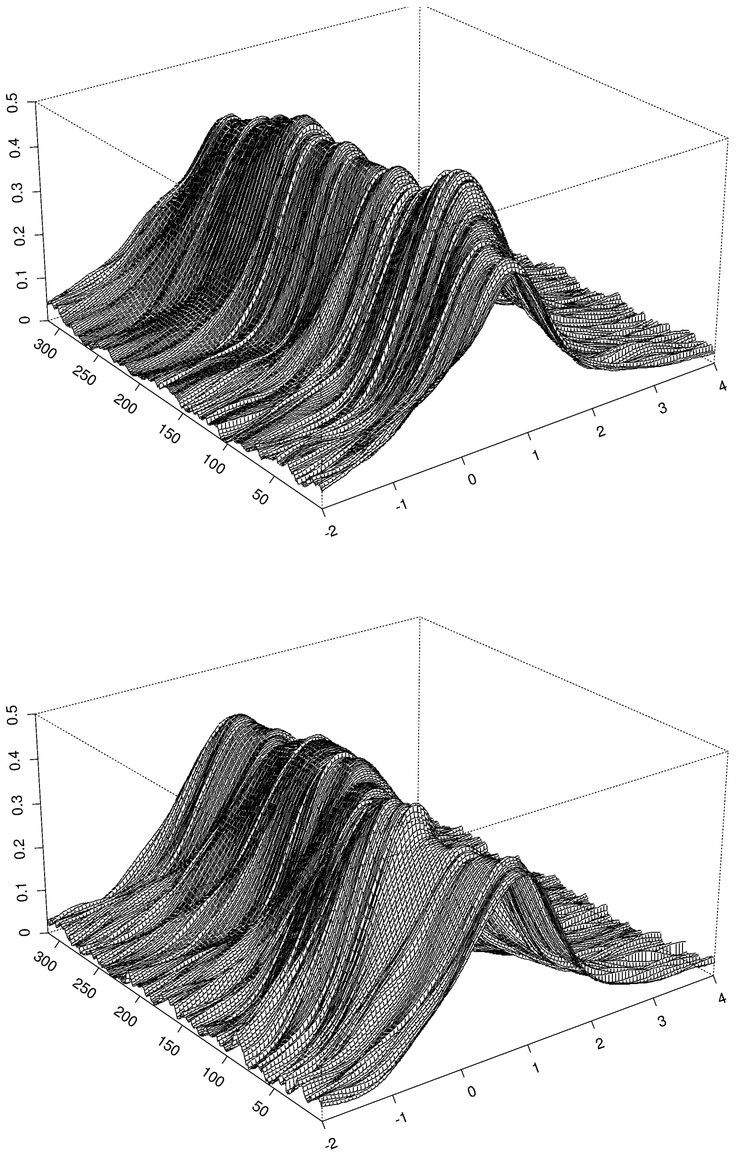

Figure 5. Distribution of the coefficients of z^0 and z^1 (noise after projection).

In figure 5, the numerical experiments performed in [12,13] are backed in the following way. If we choose $\alpha = \alpha_j = -\text{cotan}(1.57 - j/100)$ in (3) for $j = 0, \ldots, 314$, we construct 315 univariate projections (which look like $g(z)$ in (3))

$$f(z, z/\alpha_j) = \frac{1}{1-z} + \frac{1}{1-z/\alpha_j}, \quad j = 0, \ldots, 314,$$

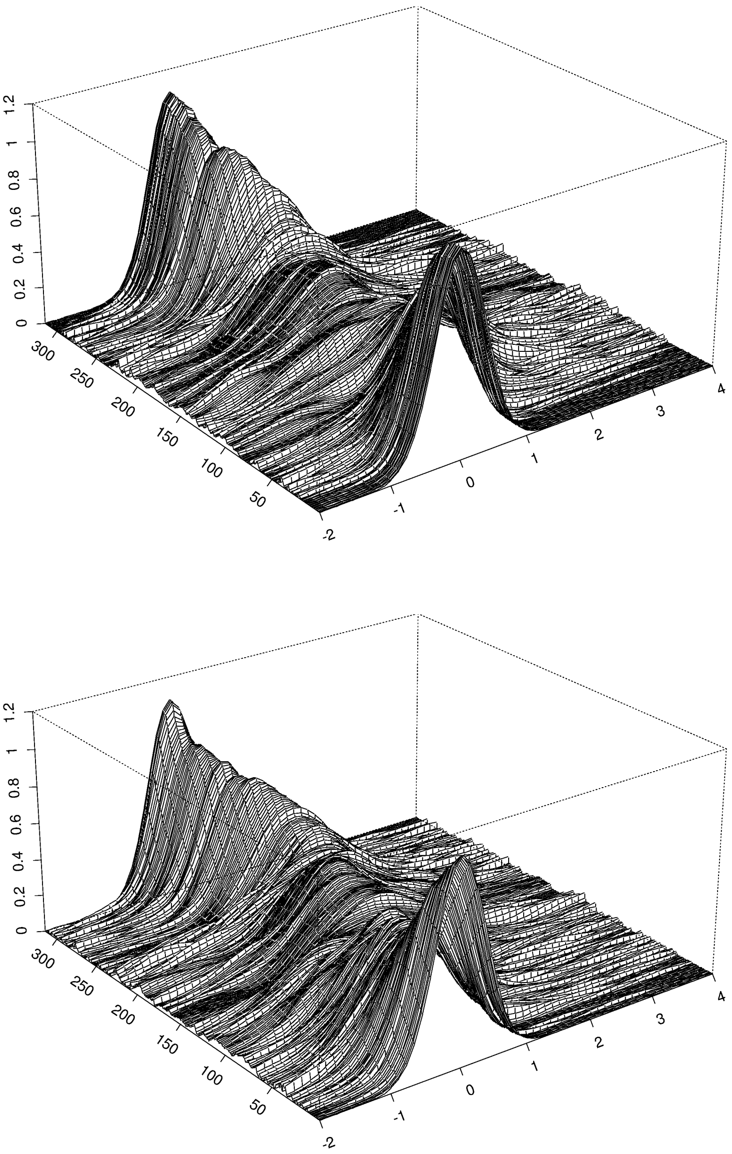

Figure 6. Distribution of the coefficients of z^0 and z^1 (projection after noise).

of the function

$$f(x, y) = \frac{1}{1-x} + \frac{1}{1-y}.$$

From (6) and (7) we know that in each Froissart polynomial associated with the Padé approximant $[n-1/n]^{f_\varepsilon(z,z/\alpha_j)}$ for the perturbed projection $f_\varepsilon(z, z/\alpha_j)$, the coefficients $k_i^{(\alpha_j)}/k_{n-2}^{(\alpha_j)}$ are distributed around 1. Therefore we have plotted for $n = 4$ the distribution of the coefficient $k_0^{(\alpha_j)}/k_2^{(\alpha_j)}$ of z^0 and $k_1^{(\alpha_j)}/k_2^{(\alpha_j)}$ of z^1 in (7). To this end the Froissart polynomial associated with each $[3/4]^{f_\varepsilon(z,z/\alpha_j)}$ is computed for 111 different sets of uniformly distributed random univariate noise. This gives us 111 values for $k_0^{(\alpha_j)}/k_2^{(\alpha_j)}$ and $k_1^{(\alpha_j)}/k_2^{(\alpha_j)}$, which appear to be distributed around 1 in figure 5. Each of the 315 distributions is normalized such that the area under the curve equals 1. The distributions are numbered from 0 up to 314 and are glued together into a tunnel-like appearance.

Figure 6 is different because here we consider projections of the type given in (20). Remember that for these functions the noise contributions are no longer uniformly distributed over $(-1, 1)$, although the r_{ij} are. Let us choose $\ell = \ell_j = \tan(-1.57 + j/100)$ in (20) for $j = 0, \ldots, 314$ and $n = 4$ in (18). For each function 54 different random sets are now considered. The distributions are normalized as above and glued together in figure 6. From the overall picture it is clear that for large $|\ell_j|$ the coefficients of z^0 and z^1 are rather distributed around 0 instead of around 1. These conclusions for (20) show that too little is known to formulate a conjecture about the location of the zeros of the bivariate Froissart polynomial (16).

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