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Continued fractions for special functions in Maple

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The `numtheory[cfrac]` command in Maple does not adhere to the standard continued fraction terminology and only offers some basic support to handle continued fractions. Therefore the idea was launched to develop a full-fletched continued fraction package.

The all-time scientific best-seller is undoubtedly the *Handbook of Mathematical Functions* by Abramowitz and Stegun [1], which is now being revised in the *Digital Library for Mathematical Functions* project [2]. Yet, the topic of continued fractions is not well represented in [1]. The same is true for most software packages. For instance, Maple only offers some basic support to handle continued fractions through its `numtheory[cfrac]` command. Therefore the idea was launched to develop a continued fraction package for use in the symbolic computing environment Maple.

Basic functionality of this package includes routines for creating continued fractions and computing approximants with the possibility to make use of tail estimates. Furthermore, a continued fraction can be transformed from one form to another and it is also possible to construct the corresponding continued fraction from a given series. In the sequel, we will briefly demonstrate this functionality.

1 Creating a continued fraction

A continued fraction is an expression of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} = b_0 + \frac{a_1}{b_1} \; + \; \frac{a_2}{b_2} \; + \; \frac{a_3}{b_3} \; + \; \ldots = b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m}{b_m} \right) \tag{1}$$

where the partial numerators a_m and partial denominators b_m are all complex numbers with $a_m \neq 0$ for all m. A common name for the ordered pair $[a_m, b_m]$ is element. For example, a continued fraction for $\arctan(z)$ is given by [3]

$$\arctan(z) = \frac{z/(1+z^2)}{1} \; + \; \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-a_m z^2/(1+z^2)}{1} \right), \qquad iz \notin (-\infty, -1) \cup (1, +\infty), \tag{2}$$

where

$$a_{2k} = \frac{2k(2k-1)}{(4k-3)(4k-1)},$$
$$a_{2k+1} = \frac{2k(2k-1)}{(4k-1)(4k+1)}.$$

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Note that, except for the first partial numerator, all the partial numerators are of the form a_{2k} for even indices or of the form a_{2k+1} for odd indices, while the partial denominators are all 1.

Such a continued fraction can easily be constructed using our package with the `create` command. In fact, any continued fraction that can be written in the form

$$b_0 + f\left(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} + \mathop{\mathrm{K}}_{\substack{k=0\\ \ell=n+1+kt}}^{\infty}\left(\frac{c_1(\ell)}{d_1(\ell)} + \frac{c_2(\ell+1)}{d_2(\ell+1)} + \cdots + \frac{c_t(\ell+t-1)}{d_t(\ell+t-1)}\right)\right) \quad (3)$$

can be created. Here, only the last part is obligatory. That is, a continued fraction may have a *front term* b_0, a *factor* f, some non-general *begin elements* a_i/b_i for $i = 1, \ldots, n$, but it is always followed by a repetition of its *general elements* $c_j(m)/d_j(m)$ for $j = 1, \ldots, t$. To define the continued fraction given by (2), the following `create` statement is used:

```
> arctancf := create( 'contfrac',
    begin = [ [z/(1+z^2),1] ],
    general = [ [-(m*(m-1)*z^2)/((2*m-3)*(2*m-1)*(1+z^2)),1],
      [-((m-1)*(m-2)*z^2)/((2*m-3)*(2*m-1)*(1+z^2)),1] ] );
```

$$arctancf := table\Bigg(\bigg[type = contfrac,\ front = 0,\ factor = 1,\ variable = z,\ index = m,$$

$$begin = \left[\left[\frac{z}{1+z^2}, 1\right]\right],$$

$$general = \left[\left[-\frac{m\,(m-1)\,z^2}{(2m-3)\,(2m-1)\,(1+z^2)}, 1\right], \left[-\frac{(m-1)\,(m-2)\,z^2}{(2m-3)\,(2m-1)\,(1+z^2)}, 1\right]\right]\bigg]\Bigg)$$

The `create` command can take several arguments. The first one is compulsory and set to `'contfrac'`. The only other compulsory argument is `'general'`, which must be assigned the list of items of the form $[c_j(m), d_j(m)]$ for the successive general elements of (3). If they occur, the front term can be assigned to the `'front'` argument, the factor to the `'factor'` argument, and the begin elements can be assigned to the `'begin'` argument as a list of items of the form $[a_i, b_i]$. By default, it is assumed that all the general elements are expressions in the index m. Another symbol can be used for this index by explicitly assigning it to the `'index'` argument. Likewise, it is assumed that all the elements can be expressions in the variable z; this can be changed by explicitly reassigning the `'variable'` argument. Note that not all indices that appear at the left hand sides in the table were explicitly specified as arguments to the `create` command; they were added automatically and given their default value.

2 Retrieving information of a continued fraction

Since the `create` command returns a table, the Maple functions `assigned` and `indices` can be used to check which parts from the general form (3) have been specified. The new commands `nthnumer` and `nthdenom` can be used to get the n-th partial numerator and denominator respectively, while the `nthelement` command returns them both within a list. For example, the following statement gives the fifth partial numerator of (2):

```
> nthnumer( arctancf, 5 );
```

$$-\frac{4\,z^2}{21\,(1+z^2)}$$

A list with the first 5 elements of (2) can be retrieved by executing the following statement:

```
> seq( nthelement( arctancf, i ), i=1..5 );
```

$$\left[\frac{z}{1+z^2}, 1\right], \left[-\frac{2\,z^2}{3\,(1+z^2)}, 1\right], \left[-\frac{2\,z^2}{15\,(1+z^2)}, 1\right], \left[-\frac{12\,z^2}{35\,(1+z^2)}, 1\right], \left[-\frac{4\,z^2}{21\,(1+z^2)}, 1\right]$$

After retrieving the elements, their limiting behaviour can be checked, which plays an important role in the further analysis of the continued fraction.

3 Computing approximants

Evaluating a continued fraction means truncating it after a certain number of elements and evaluating the resulting approximant. For this purpose, the `nthapprox` command can be used. This command takes at least two arguments, namely a continued fraction that was previously created with the `create` command as its first argument and a positive number that specifies which approximant to compute. The following statement computes the fifth approximant of (2) symbolically:

```
> nthapprox( arctancf, 5 );
```

$$\frac{\left(315 + 420\,z^2 + 113\,z^4\right) z}{15\,\left(21 + 14\,z^2 + z^4\right)\left(1 + z^2\right)}$$

To do a numerical evaluation, a value for z must be provided as the third argument to `nthapprox`. For example, we can compute the value of the tenth approximant of (2) with $z = 0.567$ as follows:

```
> nthapprox( arctancf, 10, 0.567 );
```

$$0.5158012842495719216894167653770274021221$$

Here, the `Digits` environment variable of Maple has been set to 40. This means that all calculations will be done using 40 decimal figures.

Truncating a continued fraction means replacing its tail with 0. However tails of a continued fraction, which are in themselves again continued fractions, do not need to converge to zero. This indicates that better results may be obtained if the tail is replaced with some other value w, called a *modification*. Such w can be assigned to the `'modification'` argument of the `nthapprox` command.

Looking at (2), since we know that the partial numerators have the limit

$$\lim_{m\to\infty} a_m = a = -\frac{1}{4}\frac{z^2}{1+z^2},$$

a better approximation can be obtained if the following modification is used [3, 4] :

$$w = \frac{a}{1} \; + \; \cdots \; + \; \frac{a}{1} = \frac{\sqrt{4\,a+1}-1}{2} = \frac{1}{2}\left(\sqrt{1-\frac{z^2}{1+z^2}}-1\right).$$

Adding this modification to the above statement leads to:

```
> nthapprox( arctancf, 10, 0.567,
    modification = 1/2*(sqrt(1-z^2/(1+z^2))-1) );
```

$$0.5158012842519658005138841796487955244524$$

which improves the first result by providing two more correct digits, since we know that $\arctan(0.567) = 0.5158012842517300154525 2\ldots$.

4 Contractions and equivalence transformations

An even (respectively odd) contraction of a continued fraction is a new continued fraction of which the approximants of the latter are equal to the even (respectively odd) approximants of the former. Such a contraction can be created with the `transform` command by supplying the original continued fraction as its first argument and `'even_contraction'` (respectively `'odd_contraction'`) as its second argument. The even contraction of (2) can be obtained by executing the following statement:

```
> arctancfeven := transform( arctancf, even_contraction );
```

$$arctancfeven := table\Bigg(\Big[type = contfrac,\ front = 0,\ variable = z,\ index = m,$$
$$begin = \left[\left[\frac{z}{1+z^2}, \frac{z^2+3}{3\,(1+z^2)}\right]\right],\ general = \left[\left[\frac{4\,(m-1)^2\,(2m-3)^2\,z^4}{(4m-7)\,(4m-5)^2\,(4m-3)\,(1+z^2)^2},\right.\right.$$
$$\left.\left.\frac{8\,m^2\,z^2+16\,m^2-24\,m-12\,m\,z^2+5+3\,z^2}{(4m-5)\,(4m-1)\,(1+z^2)}\right]\right]\Big]\Bigg)$$

Evaluating the fifth approximant of this continued fraction for $z = 0.567$ gives

```
> nthapprox( arctancfeven, 5, 0.567 );
```

$$0.5158012842495719216894167653770274021221$$

as expected.

Besides computing contractions, we can also apply an equivalence transformation. Such a transformation has the property that the approximants are not changed. For instance, for each continued fraction for which all partial denominators are different from zero, an equivalent continued fraction exists which has all denominators equal to 1. Using `transform` this can be computed by specifying `'simregular'` as its second argument. Take for instance the following continued fraction expansion for $\arctan(z)$ [5]

$$\arctan(z) = \frac{z}{1+z^2} - \frac{2z^2}{3} - \frac{2z^2}{5(1+z^2)} - \frac{12z^2}{7} - \frac{12z^2}{9(1+z^2)} - \ \dots .$$

If we create this continued fraction and then apply the `'simregular'` transformation, we get:

```
> arctancfwolfram := create( 'contfrac', begin = [ [z,1+z^2] ],
    general = [ [ -m*(m-1)*z^2, 2*m-1 ],
                [ -(m-1)*(m-2)*z^2, (2*m-1)*(1+z^2) ] ] ):
> transform( arctancfwolfram, 'simregular' );
```

$$table\Bigg(\Big[type = contfrac,\ front = 0,\ variable = z,\ index = m,$$
$$begin = \left[\left[\frac{z}{1+z^2}, 1\right], \left[\frac{-2\,z^2}{3\,(1+z^2)}, 1\right]\right],$$
$$general = \left[\left[-\frac{(m-1)\,(m-2)\,z^2}{(2m-3)\,(2m-1)\,(1+z^2)}, 1\right], \left[-\frac{m\,(m-1)\,z^2}{(2m-3)\,(2m-1)\,(1+z^2)}, 1\right]\right]\Big]\Bigg)$$

which clearly equals the continued fraction given in (2).

Finally, this package can also be used to transform series representations and continued fractions from one into the other.

References

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