

Singular rules for a multivariate quotient–difference algorithm

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Using the multivariate *qdg*-algorithm developed in [5], it is possible to compute the partial numerators and denominators of a continued fraction representation associated with a descending staircase in a table of multivariate rational interpolants, more precisely, multivariate Newton–Padé approximants. The algorithm is only applicable if every three successive elements on the staircase are different. If a singularity occurs in the defining system of equations for the multivariate rational interpolant then singular rules must be developed. For the univariate Newton–Padé approximant this was done in [3] by Claessens and Wuytack. The idea to perturb the initial staircase and walk around the block structure in the table in order to avoid the singularity, is explored now in a multivariate setting. Another approach would be to use block bordering methods in combination with reverse bordering [4] in order to solve the rank deficient linear system of interpolation conditions (Newton–Padé approximation conditions) recursively. Since this last technique can also be used for scattered multivariate data exhibiting near-singularity, we describe the second approach in a separate paper [7]. Here we deal only with partially grid-structured data (satisfying the so-called rectangle rule or inclusion property).

1. The multivariate rational interpolation problem

For the sake of notational simplicity we will restrict our description to the bivariate case. Let a bivariate function $f(x, y)$ be known in the data points (x_i, y_j) with $(i, j) \in \mathbb{N}^2$ and let I be a finite subset of \mathbb{N}^2 indexing those data points which will be used as interpolation points. With the data points we construct the polynomial basis functions

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$$B_{ij}(x, y) = \prod_{k=0}^{i-1} (x - x_k) \prod_{l=0}^{j-1} (y - y_l).$$

The problem of interpolating these data by a bivariate rational function was formulated in [5] as follows. Choose finite subsets N (from “Numerator”) and D (from “Denominator”) of \mathbb{N}^2 with $N \subset I$ and compute bivariate polynomials

$$\begin{aligned} p(x, y) &= \sum_{(i,j) \in N} a_{ij} B_{ij}(x, y), \quad \#N = n + 1, \\ q(x, y) &= \sum_{(i,j) \in D} b_{ij} B_{ij}(x, y), \quad \#D = m + 1, \end{aligned} \tag{1a}$$

where we denote $\partial p = N$ and $\partial q = D$, such that

$$(fq - p)(x_i, y_j) = 0, \quad (i, j) \in I, \#I = n + m + 1. \tag{1b}$$

If $q(x_i, y_j) \neq 0$ then this last condition implies that

$$f(x_i, y_j) = \frac{p}{q}(x_i, y_j), \quad (i, j) \in I. \tag{1c}$$

We say that I satisfies the inclusion property if whenever a point (i, j) belongs to I , all the points in the rectangle emanating from the origin with (i, j) as its furthestmost corner belong to I . Condition (1b), for instance, is met if the following two conditions are satisfied by the polynomials given in (1a) [5]:

$$(fq - p)(x, y) = \sum_{(i,j) \in \mathbb{N}^2 \setminus I} d_{ij} B_{ij}(x, y), \tag{2a}$$

$$I \text{ satisfies the inclusion property,} \tag{2b}$$

where the series development (2a) is formal. Condition (2a), with restriction (2b), can also be used if some or all of the interpolation points or their coordinates coincide since it can be replaced by conditions in terms of bivariate divided differences [5]:

$$(fq)[x_0, \dots, x_i][y_0, \dots, y_j] = p[x_0, \dots, x_i][y_0, \dots, y_j], \quad (i, j) \in N, \tag{3a}$$

$$(fq)[x_0, \dots, x_i][y_0, \dots, y_j] = 0, \quad (i, j) \in I \setminus N. \tag{3b}$$

Using a generalization of Leibniz’ theorem [5] we can substitute $(fq)[x_0, \dots, x_i][y_0, \dots, y_j]$ in (3a–b), with the notation $c_{\mu i, \nu j} = f[x_\mu, \dots, x_i][y_\nu, \dots, y_j]$, by:

$$\begin{aligned}
 (fq)[x_0, \dots, x_i][y_0, \dots, y_j] &= \sum_{\mu=0}^i \sum_{\nu=0}^j q[x_0, \dots, x_\mu][y_0, \dots, y_\nu] c_{\mu i, \nu j} \\
 &= \sum_{\mu=0}^i \sum_{\nu=0}^j b_{\mu\nu} c_{\mu i, \nu j} \\
 &= \sum_{(\mu, \nu) \in D} b_{\mu\nu} c_{\mu i, \nu j}.
 \end{aligned}$$

Also

$$p[x_0, \dots, x_i][y_0, \dots, y_j] = a_{ij}, \quad (i, j) \in N.$$

From now on we denote a rational function satisfying (3) by $[N/D]_I$. Numbering the points in the sets N, D and I as:

$$N = \{(i_0, j_0), \dots, (i_n, j_n)\}, \tag{4a}$$

$$D = \{(d_0, e_0), \dots, (d_m, e_m)\}, \tag{4b}$$

$$I = N \cup \{(i_{n+1}, j_{n+1}), \dots, (i_{n+m}, j_{n+m})\}, \tag{4c}$$

condition (3) becomes [5]:

$$\begin{pmatrix} c_{d_0 i_0, e_0 j_0} & \dots & c_{d_m i_0, e_m j_0} \\ \vdots & & \vdots \\ c_{d_0 i_n, e_0 j_n} & \dots & c_{d_m i_n, e_m j_n} \end{pmatrix}
 \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_m e_m} \end{pmatrix} =
 \begin{pmatrix} a_{i_0 j_0} \\ \vdots \\ a_{i_n j_n} \end{pmatrix}, \tag{5a}$$

$$\begin{pmatrix} c_{d_0 i_{n+1}, e_0 j_{n+1}} & \dots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \dots & c_{d_m i_{n+m}, e_m j_{n+m}} \end{pmatrix}
 \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_m e_m} \end{pmatrix} =
 \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5b}$$

If the rank of (5b) is not maximal, we look at $[N/D]_I$ as being a set of rational functions of which the numerator and denominator are given by (1a) and are satisfying (5a) and (5b). Such a solution $[N/D]_I$ is called degenerate.

With the numberings (4a), (4b) and (4c) of the respective indices in N, D and I we can set up descending chains of index sets, defining bivariate polynomials of “lower degree” and bivariate rational interpolation problems of “lower order”:

$$\begin{aligned}
 N &= N_n \supset \dots \supset N_k = \{(i_0, j_0), \dots, (i_k, j_k)\} \supset \dots \supset N_0 = \{(i_0, j_0)\}, \\
 k &= 0, \dots, n,
 \end{aligned} \tag{6a}$$

$$D = D_m \supset \dots \supset D_l = \{(d_0, e_0), \dots, (d_l, e_l)\} \supset \dots \supset D_0 = \{(d_0, e_0)\},$$

$$l = 0, \dots, m, \tag{6b}$$

$$I = I_{n+m} \supset \dots \supset I_{k+l} = \{(i_0, j_0), \dots, (i_{k+l}, j_{k+l})\} \supset \dots \supset I_0 = \{(i_0, j_0)\},$$

$$k + l = 0, \dots, n + m, \tag{6c}$$

$$I_{k+l, k+l} = \{(i_{k+l}, j_{k+l}), \dots, (i_{k+l}, j_{k+l})\}, \quad I \setminus N = I_{n+1, n+m}.$$

If we assume that each set $I_{k+l} \subset I$ satisfies the inclusion property in its turn, we can compute with these subsets the following entries in a “table” of multivariate rational interpolants:

$$\begin{array}{ccc} [N_0/D_0]_{I_0} & \dots & [N_0/D_m]_{I_m} \\ \vdots & & \vdots \\ [N_n/D_0]_{I_n} & \dots & [N_n/D_m]_{I_{n+m}} \end{array} \tag{7}$$

For the calculation of these entries, if they are not degenerate, one can use the recursive E -algorithm [4]. Then $[N/D]_I = [N_n/D_m]_{I_{n+m}} = E_m^{(n)}$ with:

$$B_{ik, jl}(x, y) = B_{kl}(x, y) / B_{ij}(x, y), \quad k \geq i, l \geq j,$$

$$E_0^{(l)} = \sum_{(i, j) \in N_l} c_{d_{0i}, e_{0j}} B_{d_{0i}, e_{0j}}(x, y), \quad l = 0, \dots, n + m, \tag{8a}$$

$$E_k^{(l)} = \frac{E_{k-1}^{(l)} g_{k-1, k}^{(l+1)} - E_{k-1}^{(l+1)} g_{k-1, k}^{(l)}}{g_{k-1, k}^{(l+1)} - g_{k-1, k}^{(l)}}, \quad l = 0, 1, \dots, n, k = 1, 2, \dots, m, \tag{8b}$$

$$t_k(l) = \sum_{(i, j) \in N_l} c_{d_{ki}, e_{kj}} B_{d_{ki}, e_{kj}}(x, y), \quad k = 0, \dots, m, l = 0, \dots, n + m,$$

$$g_{0, k}^{(l)} = t_k(l) - t_{k-1}(l), \quad k = 1, \dots, m, l = 0, \dots, n + m, \tag{9a}$$

$$g_{h, k}^{(l)} = \frac{g_{h-1, k}^{(l)} g_{h-1, h}^{(l+1)} - g_{h-1, k}^{(l+1)} g_{h-1, h}^{(l)}}{g_{h-1, h}^{(l+1)} - g_{h-1, h}^{(l)}}, \quad k = h + 1, h + 2, \dots \tag{9b}$$

If we let n and m increase, infinite chains of index sets as in (6) can be constructed and an infinite table of multivariate rational interpolants results. Of course, in practice, only a finite number of entries will be computed. We have seen in [2] that degeneracy has consequences for neighbouring elements in the table. Let us first resume the multivariate quotient–difference algorithm for the non-degenerate case. The new singular rules for the degenerate case are proven in section 4.

2. The qdg-algorithm and singular interpolation problems

Our aim is to consider descending staircases of multivariate rational interpolants in table (7) and to compute the interpolants on the staircase from a continued fraction representation. Let us for the moment suppose that the homogeneous system of eqs. (5b) has maximal rank m . In [3] it is shown that given a descending staircase (10) of different elements,

$$\begin{aligned}
 T_r = \{ & [N_r/D_0]_{I_r} \\
 & [N_{r+1}/D_0]_{I_{r+1}} \quad [N_{r+1}/D_1]_{I_{r+2}} \\
 & [N_{r+2}/D_1]_{I_{r+3}} \quad [N_{r+2}/D_2]_{I_{r+4}} \\
 & \vdots \quad \dots \},
 \end{aligned}
 \tag{10}$$

it is possible to construct a continued fraction $C_r(x, y)$ of the form

$$\begin{aligned}
 C_r(x, y) = & [N_r/D_0]_{I_r} + \cfrac{[N_{r+1}/D_0]_{I_{r+1}} - [N_r/D_0]_{I_r}}{1} + \cfrac{-q_1^{(r+1)}}{1 + q_1^{(r+1)}} + \cfrac{-e_1^{(r+1)}}{1 + e_1^{(r+1)}} \\
 & + \cfrac{-q_2^{(r+1)}}{1 + q_2^{(r+1)}} + \cfrac{-e_2^{(r+1)}}{1 + e_2^{(r+1)}} + \dots,
 \end{aligned}
 \tag{11}$$

of which the successive convergents equal the successive elements on the descending staircase (10). Here

$$\begin{aligned}
 [N_r/D_0]_{I_r} &= \sum_{(i,j) \in N_r} c_{d_0i,e_0j} B_{d_0i,e_0j}(x, y) = E_0^{(r)}, \\
 [N_{r+1}/D_0]_{I_{r+1}} &= \sum_{(i,j) \in N_{r+1}} c_{d_0i,e_0j} B_{d_0i,e_0j}(x, y) = E_0^{(r+1)},
 \end{aligned}$$

and the coefficients $q_l^{(r+1)}$ and $e_l^{(r+1)}$ can be computed using the following rules. Starting values are given by

$$\begin{aligned}
 q_1^{(r+1)} &= \cfrac{E_1^{(r+1)} - E_0^{(r+1)}}{E_0^{(r+1)} - E_0^{(r)}} \\
 &= \cfrac{t_0(r+2) - t_0(r+1)}{t_0(r+1) - t_0(r)} \cfrac{g_{0,1}^{(r+1)}}{g_{0,1}^{(r+1)} - g_{0,1}^{(r+2)}},
 \end{aligned}
 \tag{12}$$

while for $l \geq 2$,

$$q_l^{(r+1)} = \cfrac{E_l^{(r+1)} - E_{l-1}^{(r+1)}}{E_{l-1}^{(r+1)} - E_{l-1}^{(r+1-1)}}
 \tag{13a}$$

$$= \frac{e_{l-1}^{(r+2)} q_{l-1}^{(r+2)} g_{l-2,l-1}^{(r+l-1)} - g_{l-2,l-1}^{(r+l)} g_{l-1,l}^{(r+l)}}{e_{l-1}^{(r+1)} g_{l-2,l-1}^{(r+l-1)} g_{l-1,l}^{(r+l)} - g_{l-1,l}^{(r+l+1)}}, \tag{13b}$$

and for $l \geq 1$,

$$e_l^{(r+1)} + 1 = \frac{E_l^{(r+l+1)} - E_l^{(r+l)}}{E_l^{(r+l)} - E_{l-1}^{(r+l)}} + 1 \tag{14a}$$

$$= \frac{g_{l-1,l}^{(r+l)} - g_{l-1,l}^{(r+l+1)}}{g_{l-1,l}^{(r+l)}} \left(q_l^{(r+2)} + 1 \right). \tag{14b}$$

If we arrange the values $q_i^{(r+1)}$ and $e_i^{(r+1)}$ in a table as follows

$$\begin{array}{cccc}
 & & & q_1^{(1)} \\
 & & & e_1^{(1)} \\
 & q_1^{(2)} & & q_2^{(1)} \\
 & & e_1^{(2)} & e_2^{(1)} \\
 & q_1^{(3)} & & q_2^{(2)} & \ddots \\
 & & e_1^{(3)} & e_2^{(2)} \\
 & q_1^{(4)} & & q_2^{(3)} & \ddots \\
 & \vdots & e_1^{(4)} & \vdots & e_2^{(3)} \\
 & & \vdots & & \vdots
 \end{array}$$

where subscripts indicate columns and superscripts indicate downward sloping diagonals, then (13b) links the elements in the rhombus

$$\begin{array}{ccc}
 & e_{l-1}^{(r+1)} & \\
 q_{l-1}^{(r+2)} & & q_l^{(r+1)} \\
 & e_{l-1}^{(r+2)} &
 \end{array}$$

and (14b) links two elements on an upward sloping diagonal

$$\begin{array}{c}
 e_l^{(r+1)} \\
 q_l^{(r+2)}
 \end{array}$$

If the rank of the homogeneous system (5b) is not maximal, but has a deficiency of s , then we proved in [2] that the table of rational interpolants (7) is, under certain conditions, structured as follows.

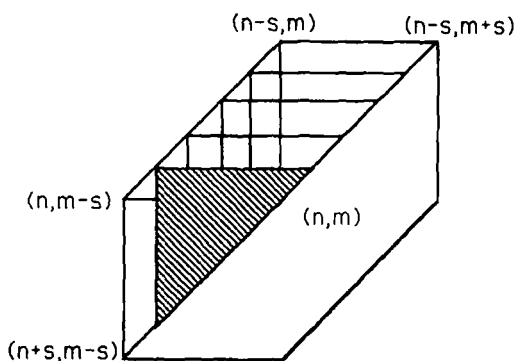
THEOREM 1

Let $p(x, y)$ and $q(x, y)$ be defined by (1), (4) and (5) with I satisfying the inclusion property. Let the rank of the coefficient matrix $C_{n+1, n+m}$ in (5b) be given by $m - s$. Then for each pair (k, l) with $0 \leq k \leq s, 0 \leq l \leq s, k + l = s$ and the rank of $C_{n-k+1, n+m-s}$ equal to its maximal rank $m - l$, the following holds.

(a) If the solution $[N_{n-k}/D_{m-l}]_{I_{n+m-s}} = (p/q)(x, y)$ is such that $\partial p = N_{n-k}, \partial q = D_{m-l}$ and

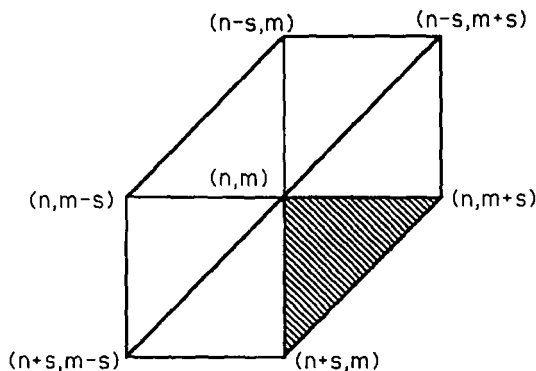
$$(fq - p)(x, y) = \sum_{(i,j) \in \mathbb{N}^2 \setminus I_{n+m}} d_{ij} B_{ij}(x, y),$$

with $d_{i_{n+m+1}j_{n+m+1}} \neq 0$ then $[N_{n-k}/D_{m-l}]_{I_{n+m-s}} \in [N_i/D_j]_{i_{i+j}}$ if and only if (i, j) belongs to the triangle with corner elements $(n - k, m - l), (n + l, m - l)$ and $(n - k, m + k)$.



(b) For $s_1 \geq 0, s_2 \geq 0$ and $s_1 + s_2 \leq s$:

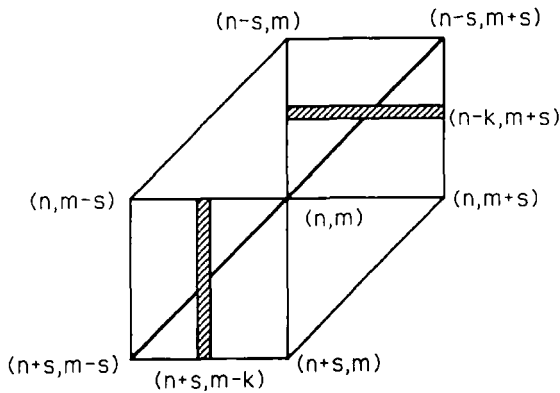
$$\bigcap_{(s_1, s_2)} [N_{n+s+1}/D_{m+s_2}]_{I_{n+m+s_1+s_2}} \neq \emptyset.$$



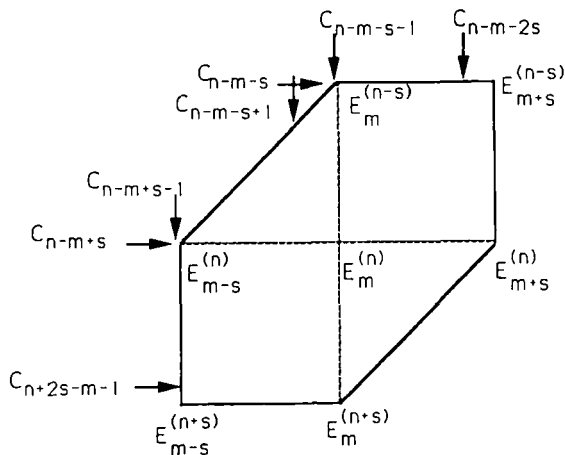
(c) Let $0 \leq k \leq s$ and the rank of $C_{n-k+1, n+m-s}$ be equal to $m - s + k$:

$$\bigcap_{j=0}^s [N_{n-k}/D_{m+j}]_{I_{n-k+m+j}} \neq \emptyset,$$

$$\bigcap_{i=0}^s [N_{n+i}/D_{m-k}]_{I_{n+i+m-k}} \neq \emptyset.$$



It is clear that in this case, with the conditions of theorem 1(a–c) fulfilled, some of the continued fractions $C_r(x, y)$ are perturbed because the elements of the staircase T_r may not all be different. For a rank deficiency of s in $C_{n+1, n+m}$ it concerns the continued fractions $C_r(x, y)$ for $r = n - m - 2s, \dots, n - m + 2s - 1$.



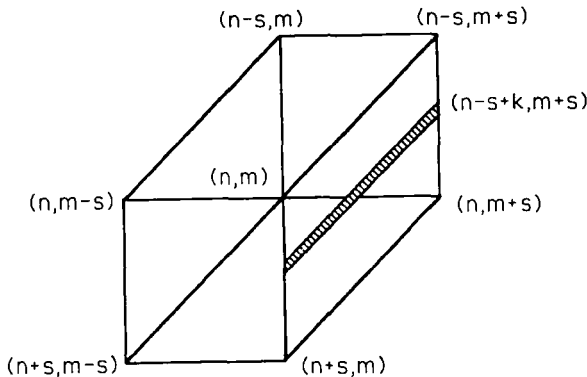
It is essential for the sequel of the discussion that we do not use indefinite E -values with zero denominator determinants. So we shall now list several well-defined E -values solving a number of interpolation problems. Using theorem 1a it is easy to fill the left upper half of the hexagonal block with nonsingular E -values, namely by copying the nondegenerate solutions from the leftmost upward sloping diagonal over the small triangles that fill this half of the hexagon. Using theorem 1b the triangle emanating from (n, m) can be filled with the regular values from the rightmost upward sloping diagonal. Moreover, in the case of an isolated hexagonal block it was proved in [1] that all its entries $[N_{n+s-i}/D_{m+i}]_{I_{n+m+s}}$ for $i = 0, \dots, s$ are equal. Using theorem 1c it was also proved in [1] that the right lower half of the structure can be filled with regular E -values using the following ratios of determinants. Let $E_{k,s}^{(l,u)}$ with numerator and denominator respectively of degree l and k , denote

$$E_{k,s}^{(l,u)} = \frac{\begin{vmatrix} t_0(l) & \dots & t_0(u-s) & t_0(u+1) & \dots & t_0(l+k+s) \\ \delta t_0(l) & \dots & & & & \\ \vdots & & & & & \\ \delta t_{k-1}(l) & \dots & & & & \end{vmatrix}}{\begin{vmatrix} 1 & \dots & 1 \\ \delta t_0(l) & \dots & \\ \vdots & & \\ \delta t_{k-1}(l) & \dots & \end{vmatrix}},$$

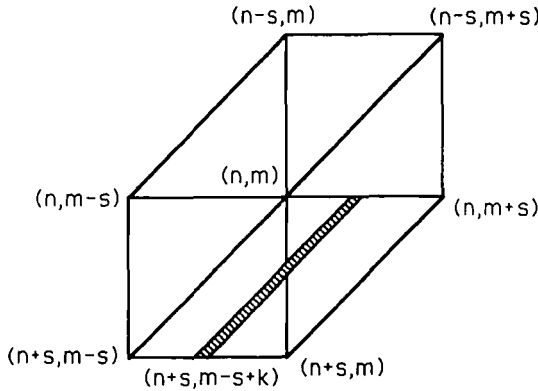
with

$$\delta t_j(i) = t_{j+1}(i) - t_j(i), \quad j \geq 0, \quad t_j(i) = 0, \quad i < 0.$$

Then, in the case of an isolated singular block [1], $E_{m,s}^{(n-s+k,n+m)}$ is well-defined and belongs to $[N_{n-s+k+i}/D_{m+s-i}]_{I_{n+m+k}}$ with $k = 1, \dots, s-1$ and $i = 0, \dots, s$, meaning that the solution $E_{m,s}^{(n-s+k,n+m)}$ of $[N_{n-s+k}/D_{m+s}]_{I_{n+m+k}}$ can be shifted downwards in the hexagonal block, in the direction of the upward sloping diagonals, because it also solves the interpolation problem posed in $[N_{n-s+k+i}/D_{m+s-i}]_{I_{n+m_k}}$.



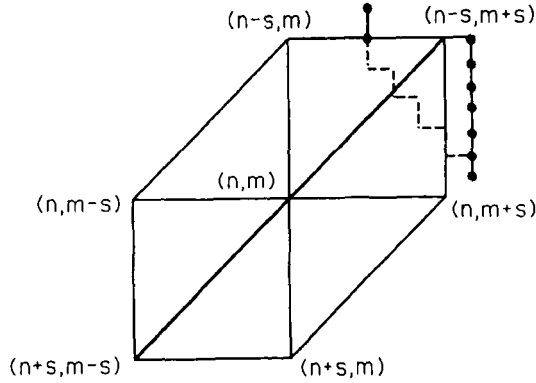
In [1] we also find that $E_{m-s+k,s}^{(n,n+m)}$ is well-defined and belongs to $[N_{n+s-i}/D_{m-s+k+i}]_{I_{n+m+k}}$ with $k = 1, \dots, s-1$ and $i = 0, \dots, s$, meaning that in the case of an isolated block the solution $E_{m-s+k,s}^{(n,n+m)}$ of $[N_{n+s}/D_{m-s+k}]_{I_{n+m+k}}$ can be shifted upwards, because it also solves $[N_{n+s-i}/D_{m-s+k+i}]_{I_{n+m+k}}$.



These new ratios of determinants which strongly resemble the E -values (the classical values are obtained for $s = 0$ and for $u \geq k + l + s$) are calculated recursively like the E -values [1] but now using help-entries

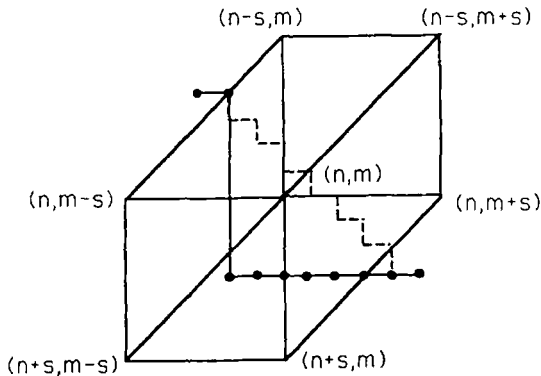
$$g_{h,k,s}^{(l,u)} = \frac{\begin{vmatrix} \delta t_k(l) & \dots & \delta t_k(u-s) & \delta t_k(u+1) & \dots & \delta t_k(l+h+s) \\ \delta t_0(l) & \dots & & & & \\ \vdots & & & & & \\ \delta t_{h-1}(l) & \dots & & & & \end{vmatrix}}{\begin{vmatrix} 1 & \dots & 1 \\ \delta t_0(l) & \dots & \\ \vdots & & \\ \delta t_{h-1}(l) & \dots & \end{vmatrix}}.$$

In this presentation the rank-deficiency s is fixed because we focus our attention on a particular singular interpolation problem, namely $[N_n/D_m]_{I_{n+m}}$. Hence we shall drop the index s in the notation of both $E_{k,s}^{(l,u)}$ and $g_{h,k,s}^{(l,u)}$. From now on when we refer to a particular element $[N_l/D_k]_{I_{l+k}}$ in the table, we only take the regular solutions in consideration. This means that for the left upper half of the hexagonal block we work with theorem 1a (i.e. copies of the nondegenerate elements on the leftmost upward sloping diagonal) and that for the right lower half of the block we work with theorem 1c (i.e. copies of the rightmost column or copies of the bottom line as in the figures above). Intermediate values $E_{m-s+k}^{(n-s+l,n+m)}$ solving $[N_{n-s+l}/D_{m+k}]_{I_{n+m-s+l+k}}$ and $[N_{n+l}/D_{m-s+k}]_{I_{n+m-s+l+k}}$ with $1 \leq l \leq s, 1 \leq k \leq s, l+k > s$



$$T_r^* = \left\{ \begin{array}{l} \left[\begin{array}{l} r \\ 0 \end{array} \right] \\ \left[\begin{array}{l} r+1 \\ 0 \end{array} \right] \dots \\ \vdots \\ \left[\begin{array}{l} n-s \\ n-r-s-1 \end{array} \right] \quad \left[\begin{array}{l} n-s \\ m+s+1 \end{array} \right] \\ \left[\begin{array}{l} n-s+1 \\ m+s+1 \end{array} \right] \\ \vdots \\ \left[\begin{array}{l} m+s+r+1 \\ m+s+1 \end{array} \right] \\ \left[\begin{array}{l} m+s+r+2 \\ m+s+1 \end{array} \right] \quad \left[\begin{array}{l} m+s+r+2 \\ m+s+2 \end{array} \right] \\ \vdots \end{array} \right\}. \tag{16}$$

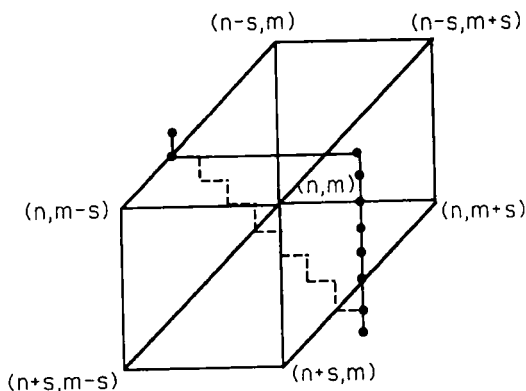
The second case is where T_r enters the hexagonal block by crossing the upward sloping diagonal, in other words, when r ranges from $n - m - s$ to $n - m + s - 1$. If we write $r = n - m - s + t$ then for even $t = 2u, u = 0, \dots, s - 1$,



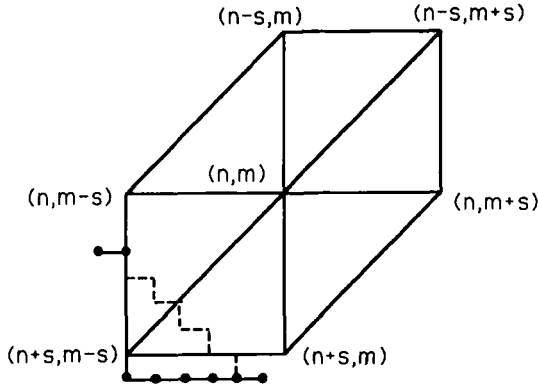
$$\begin{aligned}
 T_r^* = \left\{ \begin{array}{l} \left[\begin{array}{c} r \\ 0 \end{array} \right] \\ \left[\begin{array}{c} r+1 \\ 0 \end{array} \right] \dots \\ \vdots \\ \left[\begin{array}{c} n-s+u \\ m-u \end{array} \right] \\ \left[\begin{array}{c} n+u+1 \\ m-u \end{array} \right] \quad \left[\begin{array}{c} n+u+1 \\ m-u+1 \end{array} \right] \quad \dots \quad \left[\begin{array}{c} n+u+1 \\ m+s-u \end{array} \right] \quad \left[\begin{array}{c} n+u+1 \\ m+s-u+1 \end{array} \right] \\ \left[\begin{array}{c} n+u+2 \\ m+s-u+1 \end{array} \right] \quad \dots \\ \vdots \end{array} \right\}, \tag{17a}
 \end{aligned}$$

and for odd $t = 2u + 1, u = 0, \dots, s - 1,$

$$\begin{aligned}
 T_r^* = \left\{ \begin{array}{l} \left[\begin{array}{c} r \\ 0 \end{array} \right] \\ \left[\begin{array}{c} r+1 \\ 0 \end{array} \right] \dots \\ \vdots \\ \left[\begin{array}{c} n-s+u+1 \\ m-u-1 \end{array} \right] \quad \left[\begin{array}{c} n-s+u+1 \\ m+s-u \end{array} \right] \\ \left[\begin{array}{c} n-s+u+2 \\ m+s-u \end{array} \right] \\ \vdots \\ \left[\begin{array}{c} n+u+1 \\ m+s-u \end{array} \right] \\ \left[\begin{array}{c} n+u+2 \\ m+s-u \end{array} \right] \quad \left[\begin{array}{c} n+u+2 \\ m+s-u+1 \end{array} \right] \\ \vdots \\ \dots \end{array} \right\}. \tag{17b}
 \end{aligned}$$



The third case is where T_r enters the hexagonal block through the square in the left bottom corner of the hexagon, in other words when r ranges from $n - m + s$ to $n - m + 2s - 1$.



$$\begin{aligned}
 T_r^* = & \left\{ \begin{array}{l} \left[\begin{array}{c} r \\ 0 \end{array} \right] \\ \left[\begin{array}{c} r+1 \\ 0 \end{array} \right] \dots \\ \vdots \\ \left[\begin{array}{c} m-s+r \\ m-s \end{array} \right] \\ \left[\begin{array}{c} n+s+1 \\ m-s \end{array} \right] \quad \left[\begin{array}{c} n+s+1 \\ m-s+1 \end{array} \right] \quad \dots \quad \left[\begin{array}{c} n+s+1 \\ n+s-r \end{array} \right] \quad \left[\begin{array}{c} n+s+1 \\ n+s-r+1 \end{array} \right] \\ \left[\begin{array}{c} n+s+2 \\ n+s-r+1 \end{array} \right] \quad \dots \end{array} \right\} \tag{18}
 \end{aligned}$$

If l new elements are introduced by working with T_r^* , $l + 2$ coefficients in $C_r(x, y)$ are perturbed and new rules must be given for the partial numerators and denominators in the new continued fraction $C_r^*(x, y)$ associated with T_r^* . Before proceeding to the continued fraction representation $C_r^*(x, y)$, we introduce some new quantities. We define

$$v_{m+1}^{(n-m)} = \frac{E_m^{(n+1)} - E_m^{(n)}}{E_m^{(n)} - E_m^{(n-1)}}, \tag{19}$$

which links 3 consecutive elements in a column of the E -table, and

$$h_m^{(n-m+1)} = \frac{E_m^{(n)} - E_{m-1}^{(n)}}{E_{m-1}^{(n)} - E_{m-2}^{(n)}}, \tag{20}$$

which links 3 consecutive elements in a row of the E -table.

LEMMA 1

(a)

$$v_{m+1}^{(n-m)} = \frac{g_{m-1,m}^{(n-1)} - g_{m-1,m}^{(n)} e_m^{(n-m+1)}}{g_{m-1,m}^{(n-1)} e_m^{(n-m)}} q_m^{(n-m+1)}.$$

(b)

$$h_m^{(n-m+1)} = \frac{g_{m-1,m}^{(n)}}{g_{m-1,m}^{(n)} - g_{m-1,m}^{(n+1)}} e_{m-1}^{(n-m+2)}.$$

Proof

(a)

$$\begin{aligned} v_{m+1}^{(n-m)} &= \frac{E_m^{(n+1)} - E_m^{(n)}}{E_m^{(n)} - E_m^{(n-1)}} \\ &= e_m^{(n-m+1)} \frac{E_m^{(n)} - E_{m-1}^{(n)}}{E_m^{(n)} - E_m^{(n-1)}} \\ &= e_m^{(n-m+1)} \frac{1}{e_m^{(n-m)}} \frac{E_m^{(n)} - E_{m-1}^{(n)}}{E_m^{(n-1)} - E_{m-1}^{(n-1)}} \\ &= \frac{g_{m-1,m}^{(n-1)} - g_{m-1,m}^{(n)} e_m^{(n-m+1)}}{g_{m-1,m}^{(n-1)} e_m^{(n-m)}} q_m^{(n-m+1)}. \end{aligned}$$

The proof of (b) is straightforward. □

From the above rules it is easy to see that $v_{m+1}^{(n-m)}$ is computed from a rhombus rule similar to that for $q_{m+1}^{(n-m)}$:

$$\begin{array}{ccc} & e_m^{(n-m)} & \\ q_m^{(n-m+1)} & & v_{m+1}^{(n-m)} \\ & e_m^{(n-m+1)} & \end{array}$$

and that $h_m^{(n-m+1)}$ can be stored at $q_m^{(n-m+1)}$:

$$\begin{array}{ccc} & h_m^{(n-m+1)} & \\ e_{m-1}^{(n-m+2)} & & \end{array}$$

In analogy with (13a) and (14a) we define for $n - s + 1 \leq r + l \leq n$, $m - s + 1 \leq l \leq m$, $n + m - s + 1 \leq r + 2l$,

$$q_l^{*(r+1)} = \frac{E_l^{(r+l,n+m)} - E_{l-1}^{(r+l,n+m)}}{E_{l-1}^{(r+l,n+m)} - E_{l-1}^{(r+l-1,n+m)}}, \tag{21}$$

and for $n - s + 1 \leq r + l + 1 \leq n, m - s + 1 \leq l \leq m, n + m - s + 1 \leq r + 2l + 1,$

$$e_l^{*-(r+1)} = \frac{E_l^{(r+l+1,n+m)} - E_l^{(r+l,n+m)}}{E_l^{(r+l,n+m)} - E_{l-1}^{(r+l,n+m)}}. \tag{22}$$

In analogy with (19) and (20) we define for $n - s + 1 \leq r + l \leq n - 1, m - s \leq l \leq m, r + 2l - 1 \geq n - m - s,$

$$v_{l+1}^{*(r)} = \frac{E_l^{(r+l+1,n+m)} - E_l^{(r+l,n+m)}}{E_l^{(r+l,n+m)} - E_l^{(r+l-1,n+m)}}, \tag{23}$$

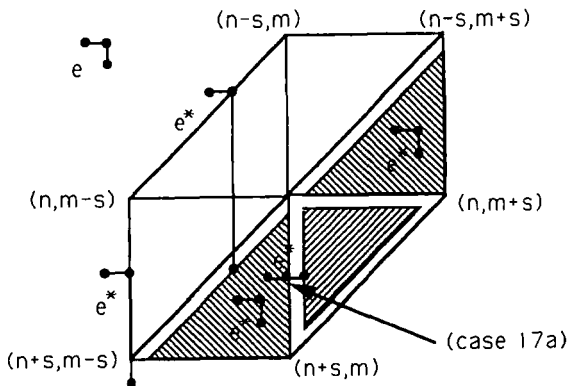
and for $n - s \leq r + l \leq n, m - s + 2 \leq l \leq m, r + 2l - 2 \geq n + m - s,$

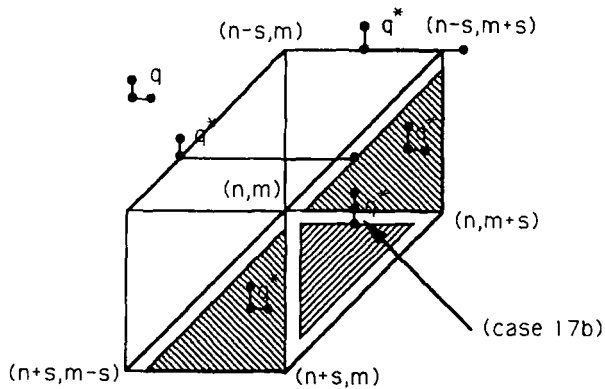
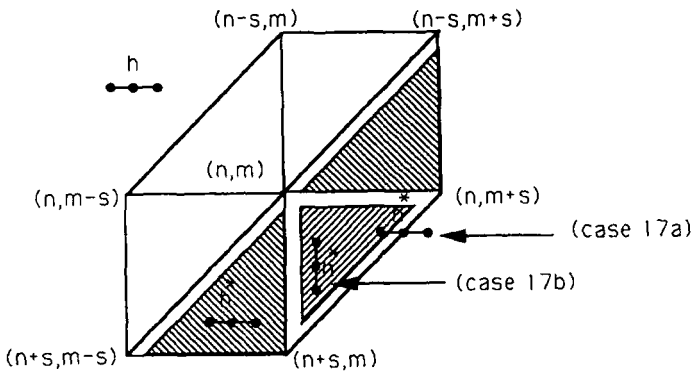
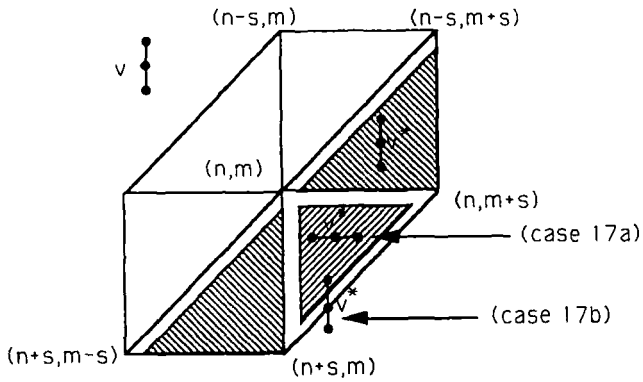
$$h_l^{*(r+1)} = \frac{E_l^{(r+l,n+m)} - E_{l-1}^{(r+l,n+m)}}{E_{l-1}^{(r+l,n+m)} - E_{l-2}^{(r+l,n+m)}}. \tag{24}$$

It is clear that for definitions linking 3 elements from the E -table, certain transition rules apply when entering and leaving the hexagonal singular block. For $q_l^{*(r+1)}, e_l^{*(r+1)}, v_l^{*(r+1)}$ and $h_l^{*(r+1)}$ this transition is described by the following special rules. All these special rules are summarized in the following figures. When entering the singular block, we use

$$q_l^{*(n+m-s-2l+2)} = \frac{E_l^{(n+m-s-l+1,n+m)} - E_{l-1}^{(n+m-s-l+1)}}{E_{l-1}^{(n+m-s-l+1)} - E_{l-1}^{(n+m-s-l)}}, \quad m - s + 1 \leq l \leq m + 1,$$

$$q_l^{*(n-s-l+1)} = \frac{E_{m+s+1}^{(n-s)} - E_{l-1}^{(n-s)}}{E_{l-1}^{(n-s)} - E_{l-1}^{(n-s-1)}}, \quad m + 1 \leq l \leq m + s, \tag{25}$$





and

$$\begin{aligned}
 e_l^{*(n+m-s-2l+1)} &= \frac{E_l^{(n+m-s-l+1, n+m)} - E_l^{(n+m-s-l)}}{E_l^{(n+m-s-l+1)} - E_{l-1}^{(n+m-s-l)}}, & m-s+1 \leq l \leq m, \\
 e_{m-s}^{*(r+1)} &= \frac{E_{m+s}^{(n+s+1)} - E_{m-s}^{(r+m-s)}}{E_{m-s}^{(r+m-s)} - E_{m-s-1}^{(r+m-s)}}, & n-m+s \leq r \leq n-m+2s-1.
 \end{aligned}
 \tag{26}$$

When leaving the singular block we use for staircase (16)

$$q_l^{*(n-s-l+1)} = \frac{E_{m+s+1}^{(n-s)} - E_{l-1}^{(n-s)}}{E_{l-1}^{(n-s)} - E_{l-1}^{(n-s-1)}}, \quad m+1 \leq l \leq m+s; \tag{27}$$

for staircase (18)

$$e_{m-s}^{*(r+1)} = \frac{E_{m-s}^{(n+s+1)} - E_{m-s}^{(r+m-s)}}{E_{m-s}^{(r+m-s)} - E_{m-s-1}^{(r+m-s)}}, \quad n-m+s \leq r \leq n-m+2s-1; \tag{28}$$

for staircase (17a)

$$h_l^{*(n+m+s-2l+2)} = \frac{E_l^{(n+m+s-l+1)} - E_{l-1}^{(n+m+s-l+1)}}{E_{l-1}^{(n+m+s-l+1)} - E_m^{(n-1, n+m)}}, \quad m+2 \leq l \leq m+s, \tag{29}$$

with $E_m^{(n-1, n+m)} = E_{m+s}^{(n-1)}$, and for staircase (17b)

$$v_{l+1}^{*(n+m+s-2l)} = \frac{E_l^{(n+m+s-l+1)} - E_l^{(n+m+s-l)}}{E_l^{(n+m+s-l)} - E_{m-1}^{(n, n+m)}}, \quad m+1 \leq l \leq m+s-1, \tag{30}$$

with $E_{m-1}^{(n, n+m)} = E_{m-1}^{(n+s)}$.

THEOREM 2

The continued fraction representation $C_r^*(x, y)$ associated with T_r^* as in (16), (17) or (18) is respectively given by the following formulas. The first and last line of the expression for C_r^* coincide with that for C_r , while the middle part deals with the discrepancy between T_r^* and T_r .

(a) For (16) containing $m-n+2s+r+1$ new elements in T_r^* :

$$\begin{aligned}
 C_r^*(x, y) &= E_0^{(r)} + \frac{E_0^{(r+1)} - E_0^{(r)}}{1} + \sum_{i=1}^{n-s-r-1} \left(\frac{-q_i^{(r+1)}}{1 + q_i^{(r+1)}} + \frac{-e_i^{(r+1)}}{1 + e_i^{(r+1)}} \right) \\
 &\quad + \frac{-q_{n-s-r}^{*(r+1)}}{1 + q_{n-s-r}^{*(r+1)}} + \frac{-e_{m+s+1}^{(n-m-2s)}}{1 + e_{m+s+1}^{(n-m-2s)}} + \sum_{i=1}^{m-n+2s+r+1} \frac{-v_{m+s+2}^{(n-m-2s+i-1)}}{1 + v_{m+s+2}^{(n-m-2s+i-1)}}
 \end{aligned}$$

$$+ \sum_{i=m+s+2}^{\infty} \left(\frac{-q_i^{(r+1)}}{1+q_i^{(r+1)}} + \frac{-e_i^{(r+1)}}{1+e_i^{(r+1)}} \right). \tag{31}$$

(b) For (17a) with $u = 0$, containing $s - 1$ new elements in T_r^* :

$$\begin{aligned} C_r^*(x, y) = & E_0^{(r)} + \frac{E_0^{(r+1)} - E_0^{(r)}}{1} + \sum_{i=1}^{m-1} \left(\frac{-q_i^{(r+1)}}{1+q_i^{(r+1)}} + \frac{-e_i^{(r+1)}}{1+e_i^{(r+1)}} \right) + \frac{-q_m^{(r+1)}}{1+q_m^{(r+1)}} \\ & + \frac{-e_m^{*(r+1)}}{1+e_m^{*(r+1)}} + \sum_{i=0}^{s-2} \frac{-v_{m+1}^{*(n-m-s+i+1)}}{1+v_{m+1}^{*(n-m-s+i+1)}} + \frac{-h_{m+s}^{*(n-m-s+2)}}{1+h_{m+s}^{*(n-m-s+2)}} \\ & + \sum_{i=m+s+1}^{\infty} \left(\frac{-q_i^{(r+1)}}{1+q_i^{(r+1)}} + \frac{-e_i^{(r+1)}}{1+e_i^{(r+1)}} \right). \end{aligned} \tag{32}$$

(c) For (17a) with $u \neq 0$, containing $s - 1$ new elements in T_r^* :

$$\begin{aligned} C_r^*(x, y) = & E_0^{(r)} + \frac{E_0^{(r+1)} - E_0^{(r)}}{1} + \sum_{i=1}^{m-u-1} \left(\frac{-q_i^{(r+1)}}{1+q_i^{(r+1)}} + \frac{-e_i^{(r+1)}}{1+e_i^{(r+1)}} \right) + \frac{-q_{m-u}^{(r+1)}}{1+q_{m-u}^{(r+1)}} \\ & + \frac{-e_{m-u}^{*(r+1)}}{1+e_{m-u}^{*(r+1)}} + \frac{-q_{m-u+1}^{*(r+1)}}{1+q_{m-u+1}^{*(r+1)}} + \sum_{i=1}^{u-1} \frac{-h_{m-u+i+1}^{*(r+1-i)}}{1+h_{m-u+i+1}^{*(r+1-i)}} \\ & + \frac{-e_m^{*(n-m-s+u+2)}}{1+e_m^{*(n-m-s+u+2)}} + \sum_{i=1}^{s-u-2} \frac{-v_{m+1}^{*(n-m-s+u+i+1)}}{1+v_{m+1}^{*(n-m-s+u+i+1)}} + \frac{-h_{m+s-u}^{*(r+2)}}{1+h_{m+s-u}^{*(r+2)}} \\ & + \sum_{i=m+s-u+1}^{\infty} \left(\frac{-q_i^{(r+1)}}{1+q_i^{(r+1)}} + \frac{-e_i^{(r+1)}}{1+e_i^{(r+1)}} \right). \end{aligned} \tag{33}$$

For $u = s - 1$ the term $\frac{-e_m^{*(n-m-s+u+2)}}{1+e_m^{*(n-m-s+u+2)}}$ does not appear and $\sum_{i=1}^{s-u-2}$ is void.

(d) For (17b) containing $s - 1$ new elements in T_r^* :

$$\begin{aligned} C_r^*(x, y) = & E_0^{(r)} + \frac{E_0^{(r+1)} - E_0^{(r)}}{1} + \sum_{i=1}^{m-u-1} \left(\frac{-q_i^{(r+1)}}{1+q_i^{(r+1)}} + \frac{-e_i^{(r+1)}}{1+e_i^{(r+1)}} \right) \\ & + \frac{-q_{m-u}^{*(r+1)}}{1+q_{m-u}^{*(r+1)}} + \frac{-e_{m-u}^{*(r+1)}}{1+e_{m-u}^{*(r+1)}} + \sum_{i=1}^{s-u-2} \frac{-v_{m-u+1}^{*(r+i)}}{1+v_{m-u+1}^{*(r+i)}} \\ & + \frac{-q_{m-u+1}^{*(n-m+u)}}{1+q_{m-u+1}^{*(n-m+u)}} + \sum_{i=1}^{u-1} \frac{-h_{m-u+i+1}^{*(n-m+u-i)}}{1+h_{m-u+i+1}^{*(n-m+u-i)}} + \frac{-v_{m+s-u+1}^{*(r-1)}}{1+v_{m+s-u+1}^{*(r-1)}} \\ & + \frac{-e_{m+s-u}^{(r+1)}}{1+e_{m+s-u}^{(r+1)}} + \sum_{i=m-u+s+1}^{\infty} \left(\frac{-q_i^{(r+1)}}{1+q_i^{(r+1)}} + \frac{-e_i^{(r+1)}}{1+e_i^{(r+1)}} \right). \end{aligned} \tag{34}$$

For $u = 0$ the term $\frac{-q_{m-u+1}^{*(n-m+u)}}{1+q_{m-u+1}^{*(n-m+u)}}$ does not appear and $\sum_{i=1}^{u-1}$ is void.

(e) For (17b) with $u = s - 1$, containing $s - 1$ new elements in T_r^* :

$$\begin{aligned}
 C_r^*(x, y) = & E_0^{(r)} + \frac{E_0^{(r+1)} - E_0^{(r)}}{1} + \sum_{i=1}^{m-s} \left(\frac{-q_i^{(r+1)}}{1+q_i^{(r+1)}} + \frac{-e_i^{(r+1)}}{1+e_i^{(r+1)}} \right) \\
 & + \frac{-q_{m-s+1}^{*(r+1)}}{1+q_{m-s+1}^{*(r+1)}} + \sum_{i=0}^{s-2} \frac{-h_{m-s+i+2}^{*(n-m+s-i-1)}}{1+h_{m-s+i+2}^{*(n-m+s-i-1)}} + \frac{-v_{m+2}^{*(r-1)}}{1+v_{m+2}^{*(r-1)}} \\
 & + \frac{-e_{m+s-u}^{(r+1)}}{1+e_{m+s-u}^{(r+1)}} + \sum_{i=m-u+s+1}^{\infty} \left(\frac{-q_i^{(r+1)}}{1+q_i^{(r+1)}} + \frac{-e_i^{(r+1)}}{1+e_i^{(r+1)}} \right). \tag{35}
 \end{aligned}$$

(f) For (18) containing $n - m + 2s - r$ new elements in T_r^* :

$$\begin{aligned}
 C_r^*(x, y) = & E_0^{(r)} + \frac{E_0^{(r+1)} - E_0^{(r)}}{1} + \sum_{i=1}^{m-s-1} \left(\frac{-q_i^{(r+1)}}{1+q_i^{(r+1)}} + \frac{-e_i^{(r+1)}}{1+e_i^{(r+1)}} \right) + \frac{-q_{m-s}^{(r+1)}}{1+q_{m-s}^{(r+1)}} \\
 & + \frac{-e_{m-s}^{*(r+1)}}{1+e_{m-s}^{*(r+1)}} + \frac{-q_{m-s+1}^{(n-m+2s+1)}}{1+q_{m-s+1}^{(n-m+2s+1)}} + \sum_{i=1}^{n-m+2s-r} \frac{-h_{m-s+i+1}^{(n-m+2s-i+1)}}{1+h_{m-s+i+1}^{(n-m+2s-i+1)}} \\
 & + \frac{-e_{n+s-r+1}^{(r+1)}}{1+e_{n+s-r+1}^{(r+1)}} + \sum_{i=n+s-r+2}^{\infty} \left(\frac{-q_i^{(r+1)}}{1+q_i^{(r+1)}} + \frac{-e_i^{(r+1)}}{1+e_i^{(r+1)}} \right). \tag{36}
 \end{aligned}$$

Proof

Since the staircases T_r and T_r^* coincide outside the hexagonal block, we must not concentrate on the partial numerators and denominators outside that block. So we focus only on the new contributions.

Case (a):

This case treats staircase T_r^* given by (16) with $n - m - 2s \leq r \leq n - m - s - 1$. So we enter the block via row $n - s$ and jump out immediately in order to descend along column $m + s + 1$. This gives first

$$q_{n-s-r}^{*(r+1)} = \frac{E_{m+s-1}^{(n-s)} - E_{n-s-r-1}^{(n-s)}}{E_{n-s-r-1}^{(n-s)} - E_{n-s-r-1}^{(n-s-1)}},$$

and then

$$e_{m+s+1}^{(n-m-2s)} = \frac{E_{m+s+1}^{(n-s+1)} - E_{m+s+1}^{(n-s)}}{E_{m+s+1}^{(n-s)} - E_{m+s}^{(n-s)}},$$

with $E_{m+s}^{(n-s)}$ copied from $E_m^{(n-s)}$. By descending along column $m + s + 1$ we introduce the coefficients $v_{m+s+2}^{(n-m-2s+i-1)}$ with $i = 1, \dots, m - n + 2s + r + 1$.

Case (b, c):

This case treats staircase T_r^* given by (17a) with $r = n - m - s + 2u$ for $u = 0, \dots, s - 1$. We enter the block horizontally in $E_{m-u}^{(n-s+u)}$ and we jump down to $\lfloor \frac{n+u+1}{m-u} \rfloor$ which does not contain $E_{m-u}^{(n-s+u)}$ as a solution. Since $E_{m-u}^{(n-s+u+1, n+m)}$ solves $\lfloor \frac{n+u+1}{m-u} \rfloor$, our first new partial numerator is

$$e_{m-u}^{*(r+1)} = \frac{E_{m-u}^{(n-s+u+1, n+m)} - E_{m-u}^{(n-s+u)}}{E_{m-u}^{(n-s+u)} - E_{m-u-1}^{(n-s+u)}}.$$

Since $E_{m-u}^{(n-i, n+m)}$ solves $\lfloor \frac{n+s-i}{m-u} \rfloor$ for $i = 0, \dots, s - u - 1$, we proceed from column $m - u$ to column $m - u + 1$ for $u \neq 0$ by

$$q_{m-u+1}^{*(r+1)} = \frac{E_{m-u+1}^{(n-s+u+1, n+m)} - E_{m-u}^{(n-s+u+1, n+m)}}{E_{m-u}^{(n-s+u+1, n+m)} - E_{m-u}^{(n-s+u, n+m)}},$$

and we continue along this horizontal until column m . The case $u = 0$ is different because there is no horizontal movement then. Since $E_m^{(n-s+u+2, n+m)}$ solves $\lfloor \frac{n+u+1}{m+1} \rfloor$ for $u = 1, \dots, s - 2$ (this does not apply to the case $u = s - 1$), the transition from column m to column $m + 1$ involves

$$e_m^{*(n-m-s+u+2)} = \frac{E_m^{(n-s+u+2, n+m)} - E_m^{(n-s+u+1, n+m)}}{E_m^{(n-s+u+1, n+m)} - E_{m-1}^{(n-s+u+1, n+m)}},$$

except for the case $u = 0$ where this transition involves

$$v_{m+1}^{*(n-m-s+1)} = \frac{E_m^{(n-s+2, n+m)} - E_m^{(n-s+1, n+m)}}{E_m^{(n-s+1, n+m)} - E_m^{(n-s, n+m)}},$$

with $E_m^{(n-s, n+m)} = E_m^{(n-s)}$ because of its definition. Since we shift $E_m^{(n+i+u-s, n+m)}$ into $\lfloor \frac{n+i}{m+u} \rfloor$ for $u = 1, \dots, s - 2$ and $i = 1, \dots, s - u - 1$, we continue our walk horizontally but introducing

$$v_{m+1}^{*(n-m-s+u+i+1)} = \frac{E_m^{(n-s+u+i+2, n+m)} - E_m^{(n-s+u+i+1, n+m)}}{E_m^{(n-s+u+i+1, n+m)} - E_m^{(n-s+u+i, n+m)}}.$$

We leave the block by means of

$$h_{m+s-u}^{*(r+2)} = \frac{E_{m+s-u}^{(n+u+1)} - E_{m+s-u-1}^{(n+u+1)}}{E_{m+s-u-1}^{(n+u+1)} - E_m^{(n-1, n+m)}} = \frac{E_{m+s-u}^{(n+u+1)} - E_{m+s-u-1}^{(n+u+1)}}{E_{m+s-u}^{(+)} - E_{m+s}^{(n-1)}},$$

and continue horizontally to retrieve the old staircase T_r with

$$q_{m+s-u+1}^{(r+1)} = \frac{E_{m+s-u+1}^{(n+u+1)} - E_{m+s-u}^{(n+u+1)}}{E_{m+s-u}^{(n+u+1)} - E_{m+s-u-1}^{(n+u+1)}},$$

using the fact that $E_{m+s-u-1}^{(n+u-1)} = E_{m+s-u}^{(n+u)}$.

Case (d, e):

Here we deal with staircase T_r^* given by (17b), in other words, with $r = n - m - s + 2u + 1$ with $u = 0, \dots, s - 1$. We enter the hexagonal block vertically through $E_{m-u-1}^{(n-s+u+1)}$ and we jump to $[\frac{n-s+u+1}{m+s-u}]$, which contains $E_{m-u}^{(n-s+u+1, n+m)}$. By shifting $E_{m-u}^{(n-s+i+1, n+m)}$ into $[\frac{n-s+i+1}{m+s-u}]$ for $i = 0, \dots, s - u - 1$, we obtain $q_{m-u}^{*(r+1)}$ as a partial numerator and the next partial numerators by continuing along column $m + s - u$. The transition from row $n - s + u + 1$ to row $n - s + u + 2$ generates $e_{m-u}^{*(r+1)}$ because $E_{m-u-1}^{(r+m-u)} = E_{m-u-1}^{(r+m-u, n+m)}$. We continue vertically until row n , involving for $i = 1, \dots, s - u - 2$

$$v_{m-u-1}^{*(r+1)} = \frac{E_{m-u}^{(n-s+u+i+2, n+m)} - E_{m-u}^{(n-s+u+i+1, n+m)}}{E_{m-u}^{(n-s+u+i+1, n+m)} - E_{m-u}^{(n-s+u+i, n+m)}}.$$

The case $u = s - 1$ is different because there is no vertical movement there. The transition from row n to row $n + 1$ involves for $u = 1, \dots, s - 2$ (this does not apply to $u = 0$)

$$q_{m-u+1}^{*(n-m+u)} = \frac{E_{m-u+1}^{(n, n+m)} - E_{m-u}^{(n, n+m)}}{E_{m-u}^{(n, n+m)} - E_{m-u}^{(n-1, n+m)}},$$

except for $u = s - 1$ where it involves

$$h_{m-s+2}^{*(n-m+s-1)} = \frac{E_{m-s+2}^{(n, n+m)} - E_{m-s+1}^{(n, n+m)}}{E_{m-s+1}^{(n, n+m)} - E_{m-s}^{(n, n+m)}},$$

with $E_{m-s}^{(n, n+m)} = E_{m-s}^{(n)}$. By shifting $E_{m-u+i}^{(n, n+m)}$ into $[\frac{n+i}{m+s-u}]$ for $u = 2, \dots, s - 1$ and $i = 1, \dots, u - 1$, we can continue our vertical movement, but obtaining

$$h_{m-u+i+1}^{*(r-i+1)} = \frac{E_{m-u+i+1}^{(n, n+m)} - E_{m-u+i}^{(n, n+m)}}{E_{m-u+i}^{(n, n+m)} - E_{m-u+i-1}^{(n, n+m)}}.$$

We leave the block through

$$v_{m+s-u+1}^{*(r-1)} = \frac{E_{m+s-u}^{(n+u+1)} - E_{m+s-u}^{(n+u)}}{E_{m+s-u}^{(n+u)} - E_{m-1}^{(n, n+m)}},$$

and we continue along the same column to retrieve T_r with

$$\begin{aligned} e_{m+s-u}^{(r+1)} &= \frac{E_{m+s-u}^{(n+u+2)} - E_{m+s-u}^{(n+u+1)}}{E_{m+s-u}^{(n+u+1)} - E_{m+s-u-1}^{(n+u+1)}} \\ &= \frac{E_{m+s-u}^{(n+u+2)} - E_{m+s-u}^{(n+u+1)}}{E_{m+s-u}^{(n+u+1)} - E_{m+s-u}^{(n+u)}}. \end{aligned}$$

Case (f):

In the case of staircase (18) we enter the hexagonal block horizontally through column $m - s$ and we immediately jump out of the block with

$$e_{m-s}^{*(r+1)} = \frac{E_{m-s}^{(n+s+1)} - E_{m-s}^{(r+m-s)}}{E_{m-s}^{(r+m-s)} - E_{m-s-1}^{(r+m-s)}}.$$

When proceeding horizontally on row $n + s + 1$ in order to retrieve the old staircase, the next partial numerator is

$$\begin{aligned} q_{m-s+1}^{(n-m+2s+1)} &= \frac{E_{m-s+1}^{(n+s+1)} - E_{m-s}^{(n+s+1)}}{E_{m-s}^{(n+s+1)} - E_{m-s}^{(r+m-s)}} \\ &= \frac{E_{m-s+1}^{(n+s+1)} - E_{m-s}^{(n+s+1)}}{E_{m-s}^{(n+s+1)} - E_{m-s}^{(n+s)}} \end{aligned}$$

followed by $h_{m-s+i+1}^{(n-m+2s-i+1)}$ for $i = 1, \dots, n - m + 2s - r$. □

From the above theorem it is clear which $v_i^{(r+1)}$ and $h_i^{(r+1)}$ have to be computed in order to deal successfully with the singular hexagonal structure in the table of rational interpolants. When we want to be able to walk around the entire block we have to fill the triangular tables

$$\begin{array}{cc} v_{m+1}^{*(n-m-s+1)} & v_{m+s+2}^{(n-m-2s)} \\ v_{m+1}^{*(n-m-s+2)} & \vdots \\ \ddots & \vdots \\ v_{m-s+4}^{*(n-m+s-4)} & \dots & v_{m+1}^{*(n-m-1)} \\ & & v_{m+s+2}^{(n-m-s-1)} \end{array}$$

and

$$\begin{array}{cccc} & & & h_m^{*(n-m-s+4)} \\ & & & \vdots \\ & & \ddots & \\ h_{m-s+2}^{*(n-m+s-1)} & h_{m-s+3}^{*(n-m+s-2)} & \dots & h_m^{*(n-m+1)} \\ h_{m-s+2}^{(n-m+2s)} & \dots & \dots & h_{m+1}^{(n-m+s+1)} \end{array}$$

and the diagonals

$$\begin{array}{c}
 v_{m+s+1}^{*(n-m-s)} \\
 \vdots \\
 v_{m+3}^{*(n-m+s-4)} \\
 \vdots \\
 v_{m+2}^{*(n-m+s-2)}
 \end{array} \tag{37}$$

and

$$\begin{array}{c}
 h_{m+s}^{*(n-m-s+2)} \\
 \vdots \\
 h_{m+2}^{*(n-m+s-2)} \\
 h_{m+1}^{*(n-m+s)}
 \end{array} \tag{38}$$

4. Singular rules for the qdg-algorithm

To be able to use the continued fraction representations obtained in the previous section, we must find a coherent computation scheme for its partial numerators and denominators. Let us first introduce the following notations:

$$G_l^{(r)} = \frac{g_{l,l+1}^{(r)} - g_{l,l+1}^{(r+1)}}{g_{l,l+1}^{(r)}}$$

and

$$G_l^{(r,n+m)} = \frac{g_{l,l+1}^{(r,n+m)} - g_{l,l+1}^{(r+1,n+m)}}{g_{l,l+1}^{(r,n+m)}} .$$

The next lemma deals with the recursive computation of the values of the diagonals (37) and (38).

LEMMA 2

(a)

$$\begin{aligned}
 h_{m+1}^{*(n-m+s)} &= \frac{1}{G_m^{(n+s)}} e_m^{(n-m+s+1)}, \\
 h_{m+2}^{*(n-m+s-2)} &= \frac{G_{m-1}^{(n-1,n+m)} h_{m+1}^{*(n-m+s)} q_m^{*(n-m+1)}}{G_{m+1}^{(n+s-1)} e_m^{*(n-m)}}, \\
 h_{m+s-u}^{*(n-m-s+2u+2)} &= \frac{1}{G_{m+s-u-1}^{(n+u+1)}} h_{m+s-u-1}^{*(n-m-s+2u+4)}, \quad u = 0, \dots, s-3.
 \end{aligned}$$

(b)

$$\begin{aligned}
 v_{m+2}^{*(n-m+s-2)} &= \frac{1}{G_m^{(n+s)}} e_m^{(n-m+s+1)}, \\
 v_{m+s-u+1}^{*(n-m-s+2u)} &= \frac{1}{G_{m+s-u-1}^{(n+u+1)}} v_{m+s-u}^{*(n-m-s+2u+2)}, \quad u = s-2, \dots, 1, \\
 v_{m+s+1}^{*(n-m-s)} &= \frac{G_{m-1}^{(n-1, n+m)}}{G_{m+s-1}^{(n+1)}} \frac{v_{m+s}^{*(n-m-s+2)} q_m^{*(n-m+1)}}{e_m^{*(n-m)}}.
 \end{aligned}$$

Proof

(a) The rule for the bottom value $h_{m+1}^{*(n-m+s)}$ of the diagonal is straightforward from its definition. For the next value,

$$\begin{aligned}
 h_{m+2}^{*(n-m+s-2)} &= \frac{E_{m+2}^{(n+s-1)} - E_{m+1}^{(n+s-1)}}{E_{m+1}^{(n+s-1)} - E_m^{(n-1, n+m)}} \\
 &= \frac{1}{G_{m+1}^{(n+s-1)}} \frac{E_{m+1}^{(n+s)} - E_{m+1}^{(n+s-1)}}{E_{m+1}^{(n+s-1)} - E_m^{(n-1, n+m)}},
 \end{aligned}$$

we use the equality $E_m^{(n+s)} = E_{m+1}^{(n+s-1)}$ and the definition of $h_{m+1}^{*(n-m+s)}$, $q_m^{*(n-m+1)}$ and $e_m^{*(n-m)}$ to derive the computation rule. The other entries on the upward sloping diagonal are given by

$$\begin{aligned}
 h_{m+s-u}^{*(n-m-s+2u+2)} &= \frac{E_{m+s-u}^{(n+u+1)} - E_{m+s-u-1}^{(n+u+1)}}{E_{m+s-u-1}^{(n+u+1)} - E_m^{(n-1, n+m)}} \\
 &= \frac{1}{G_{m+s-u-1}^{(n+u+1)}} \frac{E_{m+s-u-1}^{(n+u+2)} - E_{m+s-u-1}^{(n+u+1)}}{E_{m+s-u-1}^{(n+u+1)} - E_m^{(n-1, n+m)}},
 \end{aligned}$$

with $E_{m+s-u-1}^{(n+u+1)} = E_{m+s-u-2}^{(n+u+2)}$.

(b) The proof of (b) is similar, except that now the top element of the upward sloping diagonal involves $q_m^{*(n-m+1)}$ and $e_m^{*(n-m)}$ and the other elements are computed bottom-up from one another. □

For the other values $v_{l+1}^{*(r)}$ and $h_l^{*(r+1)}$ the recursive computation is completely similar as that of $v_{l+1}^{(r)}$ and $h_l^{(r+1)}$ in lemma 1.

LEMMA 3

(a)

$$v_{l+1}^{*(r)} = G_{l-1}^{(r+l, n+m)} \frac{e_l^{*(r+1)}}{e_l^{*(r)}} q_l^{*(r+1)}.$$

(b) For $r + l < n$:

$$h_l^{*(r+1)} = \frac{1}{G_{l-1}^{(r+l, n+m)}} e_{l-1}^{*(r+2)}.$$

(c) For $r + l = n$:

$$h_l^{*(r+1)} = \frac{1}{G_{l-1}^{(n+s)}} e_{l-1}^{(n+s-l+2)}.$$

Proof

The proof is left to the reader. □

We shall now concentrate on the octagonal gap in the qd-table due to the hexagonal block in the table of rational Hermite interpolants. In what follows the singular rules for the partial numerators and denominators in the new continued fraction representations $C_r^*(x, y)$ are derived in the same order as they have to be implemented computationally. The first of them fills the leftmost column of the octagonal singular block (15) in the qd-table in a bottom-up way.

RULE 1

For $k = 1, \dots, s$,

$$e_{m-s}^{*(n-m+s+k)} = G_{m-s-1}^{(n+k-1)} q_{m-s}^{(n-m+s+k+1)} e_{m-s}^{*(n-m+s+k+1)},$$

with $e_{m-s}^{*(n-m+2s+1)} = e_{m-s}^{(n-m+2s+1)}$, and where by chaining the above rule

$$e_{m-s}^{*(n-m+s+1)} = \prod_{i=0}^{s-1} G_{m-s-1}^{(n+i)} e_{m-s}^{(n-m+2s+1)} \prod_{i=1}^s q_{m-s}^{(n-m+s+i+1)}.$$

Proof

From (26b) we have

$$\begin{aligned} e_{m-s}^{*(n-m+s+k)} &= \frac{E_{m-s}^{(n+s+1)} - E_{m-s}^{(n+k-1)}}{E_{m-s}^{(n+k-1)} - E_{m-s-1}^{(n+k-1)}} \\ &= \frac{E_{m-s}^{(n+s+1)} - E_{m-s}^{(n+k-1)}}{E_{m-s}^{(n+k)} - E_{m-s-1}^{(n+k)}} \frac{E_{m-s}^{(n+k)} - E_{m-s-1}^{(n+k)}}{E_{m-s}^{(n+k-1)} - E_{m-s-1}^{(n+k-1)}} \\ &= \frac{E_{m-s}^{(n+s+1)} - E_{m-s}^{(n+k-1)}}{E_{m-s}^{(n+k)} - E_{m-s-1}^{(n+k)}} q_{m-s}^{(n-m+s+k+1)} G_{m-s-1}^{(n+k-1)}. \end{aligned}$$

Now because of the singular block, $E_{m-s}^{(n+k-1)} = E_{m-s}^{(n+k)}$ and hence the first ratio of E -values equals $e_{m-s}^{*(n-m+s+k+1)}$. □

Let us now concentrate on the leftmost upward sloping diagonal of (15).

RULE 2

For $k = s, \dots, 1$

$$q_{m-k+1}^{*(n-m-s+2k)} = \frac{G_{m-k-1}^{(n-s+k-1)} q_{m-k}^{(n-m-s+2k+1)} e_{m-k}^{*(n-m-s+2k+1)}}{G_{m-k}^{(n-s+k, n+m)} e_{m-k}^{(n-m-s+2k)}},$$

and for $k = s - 1, \dots, 0$

$$e_{m-k}^{*(n-m-s+2k+1)} + 1 = G_{m-k-1}^{(n-s+k)} \left(q_{m-k}^{*(n-m-s+2k+2)} + 1 \right),$$

with

$$q_{m+1}^{*(n-m-s)} = \frac{G_m^{(n-s-1)} q_m^{(n-m-s+1)} e_m^{*(n-m-s+1)}}{G_{m+s}^{(n-s)} e_m^{(n-m-s)}}.$$

Proof

By definition we have from (21)

$$q_{m-k+1}^{*(n-m-s+2k)} = \frac{E_{m-k+1}^{(n-s+k, n+m)} - E_{m-k}^{(n-s+k, n+m)}}{E_{m-k}^{(n-s+k, n+m)} - E_{m-k}^{(n-s+k-1, n+m)}}.$$

This formula differs from the regular expression for $q_{m-k+1}^{(n-m-s+2k)}$ only by the addition of $(n+m)$ in the superscript. Hence the proof is completely analogous to that for (13b) in [3]. From (22) we know that

$$e_{m-k}^{*(n-m-s+2k+1)} = \frac{E_{m-k}^{(n-s+k+1, n+m)} - E_{m-k}^{(n-s+k, n+m)}}{E_{m-k}^{(n-s+k, n+m)} - E_{m-k-1}^{(n-s+k, n+m)}},$$

for which the proof is completely analogous to that for (14b). A special value is $q_{m+1}^{*(n-m-s)}$, which is given by (27) and contains fewer additions of $(n+m)$. Nevertheless, the proof can be constructed in the same way. □

We continue our walk around the octagonal block and focus on the top row of (15), together with special rules for the row immediately above the block since this uses “degenerate” values.

RULE 3

For $k = 2, \dots, s + 1$

$$q_{m+k}^{*(n-m-s-k+1)} = \frac{-g_{m+k-1, m+k}^{(n-s-1)}}{g_{m+k-1, m+k}^{(n-s)}} G_{m+k-2}^{(n-s-1)} q_{m+k-1}^{*(n-m-s-k+2)},$$

with $q_{m+s+1}^{*(n-m-2s)} = q_{m+s+1}^{(n-m-2s)}$, and for $k = 1, \dots, s$

$$e_{m+k}^{(n-m-s-k)} = \frac{-g_{m+k-1,m+k}^{(n-s)}}{g_{m+k-1,m+k}^{(n-s-1)}}.$$

Proof

The first row is computed left to right, where $q_{m+1}^{*(n-m-s)}$ is given by the previous rules. When leaving the singular block as in (16) we have

$$q_{m+k}^{*(n-m-s-k+1)} = \frac{E_{m+s+1}^{(n-s)} - E_{m+k-1}^{(n-s,n+m)}}{E_{m+k-1}^{(n-s,n+m)} - E_{m+k-1}^{(n-s-1,n+m)}},$$

where $E_{m+k-1}^{(n-s,n+m)} = E_{m+k-2}^{(n-s,n+m)}$ and where the superscript $(,n+m)$ can be dropped because of the definition of these E -values. Hence

$$q_{m+k}^{*(n-m-s-k+1)} = \frac{g_{m+k-2,m+k-1}^{(n-s)} - g_{m+k-2,m+k-1}^{(n-s-1)}}{g_{m+k-2,m+k-1}^{(n-s)}} \frac{E_{m+s+1}^{(n-s)} - E_{m+k-2}^{(n-s)}}{E_{m+k-2}^{(n-s)} - E_{m+k-2}^{(n-s-1)}},$$

which completes the computation rule for $q_{m+k}^{*(n-m-s-k+1)}$. By chaining the above rule, an explicit formula can be given for $q_{m+s+1}^{*(n-m-2s)}$ as was done in rule 1 for $e_{m-s}^{*(n-m+s+1)}$. The value $e_{m+k}^{(n-m-s-k)}$ is given by

$$e_{m+k}^{(n-m-s-k)} = \frac{E_{m+k}^{(n-s)} - E_{m+k}^{(n-s-1)}}{E_{m+k}^{(n-s-1)} - E_{m+k-1}^{(n-s-1)}},$$

with $E_{m+k}^{(n-s)}$ copied from $E_{m+k-1}^{(n-s)}$. Now this quotient can be rewritten as

$$e_{m+k}^{(n-m-s-k)} = \frac{-g_{m+k-1,m+k}^{(n-s)}}{g_{m+k-1,m+k}^{(n-s)} - g_{m+k-1,m+k}^{(n-s-1)}} (-G_{m+k-1}^{(n-s-1)}),$$

which completes the proof. □

Let us proceed. Column $m + s + 2$ of q -values is the first to reappear in the continued fraction representations $C_r^*(x, y)$. It can be computed from the q -values and e -values with column index $m + s + 1$ using the well-known non-singular rules. Column $m + s + 1$ of e -values depends solely on column $m + s + 1$ of q -values, so we focus on this last one.

RULE 4

For $k = 1, \dots, s - 1$

$$q_{m+s+1}^{(n-m-2s+k)} = \frac{G_{m-1}^{(n-s+k-1,n+m)}}{G_{m+s}^{(n-s+k)}} \frac{e_m^{*(n-m-s+k+1)} q_m^{*(n-m-s+k+1)}}{e_m^{*(n-m-s+k)}},$$

where a band of q^* -values and e^* -values is filled using rules constructed from the classical ones:

$$q_l^{*(r+1)} = \frac{G_{l-2}^{(r+l-1, n+m)} e_{l-1}^{*(r+2)} q_{l-1}^{*(r+2)}}{G_{l-1}^{(r+l, n+m)} e_{l-1}^{*(r+1)}},$$

$$l = m - s + 1, \dots, m, \quad r = n + m - s - 2l + 3, \dots, n + m - 2l + 1,$$

and

$$e_l^{*(r+1)} + 1 = G_{l-1}^{(r+l, n+m)} (q_l^{*(r+2)} + 1),$$

$$l = m - s + 1, \dots, m, \quad r = n + m - s - 2l + 2, \dots, n + m - 2l.$$

For $k = s$

$$q_{m+s+1}^{(n-m-s)} = \frac{G_{m-1}^{(n-1, n+m)}}{G_{m+s}^{(n)}} \prod_{i=1}^s \frac{1}{G_{m+s-i}^{(n+i)}} \frac{e_m^{(n-m+s+1)} q_m^{*(n-m+1)}}{e_m^{(n-m)}}.$$

Proof

For the first $s - 1$ elements of column $m + s + 1$ of q -values we have

$$\begin{aligned} q_{m+s+1}^{(n-m-2s+k)} &= \frac{E_{m+s+1}^{(n-s+k)} - E_{m+s}^{(n-s+k)}}{E_{m+s}^{(n-s+k)} - E_{m+s}^{(n-s+k-1)}} \\ &= \frac{1}{G_{m+s}^{(n-s+k)}} \frac{E_{m+s}^{(n-s+k+1)} - E_{m+s}^{(n-s+k)}}{E_{m+s}^{(n-s+k)} - E_{m+s}^{(n-s+k-1)}} \\ &= \frac{1}{G_{m+s}^{(n-s+k)}} \frac{E_{m+s}^{(n-s+k+1)} - E_{m+s}^{(n-s+k)}}{E_{m+s}^{(n-s+k)} - E_{m-1}^{(n-s+k, n+m)}} \frac{E_{m+s}^{(n-s+k-1)} - E_{m-1}^{(n-s+k-1, n+m)}}{E_{m+s}^{(n-s+k)} - E_{m+s}^{(n-s+k-1)}} \\ &\quad \times \frac{E_{m+s}^{(n-s+k)} - E_{m-1}^{(n-s+k, n+m)}}{E_{m+s}^{(n-s+k-1)} - E_{m-1}^{(n-s+k-1, n+m)}} \\ &= \frac{1}{G_{m+s}^{(n-s+k)}} \frac{e_m^{*(n-m-s+k+1)}}{e_m^{*(n-m-s+k)}} \frac{E_m^{(n-s+k, n+m)} - E_{m-1}^{(n-s+k, n+m)}}{E_m^{(n-s+k-1, n+m)} - E_{m-1}^{(n-s+k-1, n+m)}} \\ &= \frac{G_{m-1}^{(n-s+k-1, n+m)}}{G_{m+s}^{(n-s+k)}} \frac{e_m^{*(n-m-s+k+1)} q_m^{*(n-m-s+k+1)}}{e_m^{*(n-m-s+k)}}. \end{aligned}$$

For the bottom value, $k = s$, we can prove

$$\begin{aligned}
 q_{m+s+1}^{(n-m-s)} &= \frac{E_{m+s+1}^{(n)} - E_{m+s}^{(n)}}{E_{m+s}^{(n)} - E_{m+s}^{(n-1)}} \\
 &= \frac{1}{G_{m+s}^{(n)}} \frac{E_{m+s}^{(n+1)} - E_{m+s}^{(n)}}{E_{m+s}^{(n)} - E_{m+s}^{(n-1)}} \\
 &= \frac{1}{G_{m+s}^{(n)}} \frac{E_{m+s}^{(n+1)} - E_{m+s}^{(n)}}{E_{m+s}^{(n)} - E_{m+s}^{(n-1)}} \frac{E_{m+s}^{(n-1)} - E_{m-1}^{(n-1, n+m)}}{E_{m+s}^{(n)} - E_{m+s}^{(n-1)}} \frac{E_{m+s}^{(n)} - E_{m-1}^{(n, n+m)}}{E_{m+s}^{(n-1)} - E_{m-1}^{(n-1, n+m)}}.
 \end{aligned}$$

Using the fact that $E_{m+s}^{(n)} = E_{m+s-1}^{(n+1)} = E_{m+s-2}^{(n+2)}$, we obtain

$$q_{m+s+1}^{(n-m-s)} = \frac{G_{m-1}^{(n-1, n+m)}}{G_{m+s}^{(n)}} \frac{q_m^{*(n-m+1)}}{e_m^{(n-m)}} \frac{E_{m+s}^{(n+1)} - E_{m+s}^{(n)}}{E_{m+s}^{(n)} - E_{m-1}^{(n, n+m)}}.$$

If the above procedure is reapplied to this last quotient of E -values, then finally

$$\frac{E_{m+s}^{(n+1)} - E_{m+s}^{(n)}}{E_{m+s}^{(n)} - E_{m-1}^{(n, n+m)}} = \left(\prod_{i=1}^s \frac{1}{G_{m+s-i}^{(n+i)}} \right) \frac{E_m^{(n+s+1)} - E_m^{(n+s)}}{E_m^{(n+s)} - E_{m-1}^{(n, n+m)}},$$

which completes the proof. □

To complete our walk around the singularity and to close the gap in the qd-table we now calculate the remaining elements.

RULE 5

For $k = s, \dots, 1$

$$q_{m+s-k+2}^{(n-m-s+2k-1)} = \frac{1}{G_{m+s-k+1}^{(n+k)}} e_{m+s-k+1}^{(n-m-s+2k)},$$

and inbetween

$$e_{m+s-k+2}^{(n-m-s+2k-2)} + 1 = G_{m+s-k+1}^{(n+k-1)} \left(q_{m+s-k+2}^{(n-m-s+2k-1)} + 1 \right).$$

On the bottom line we have for $k = 1, \dots, s$

$$e_{m-s+k}^{(n-m+2s-k+1)} + 1 = G_{m-s+k-1}^{(n+s)} \left(q_{m-s+k}^{(n-m+2s-k+2)} + 1 \right).$$

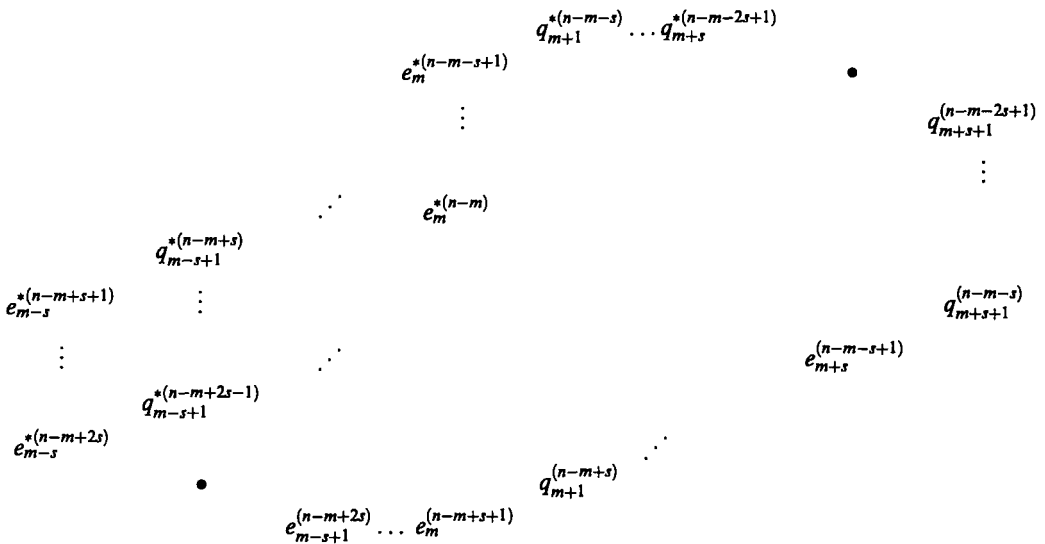
Proof

The computation of $e_{m+s-k+2}^{(n-m-s+2k-2)}$ follows the old rules, while for $q_{m+s-k+2}^{(n-m-s+2k-1)}$ we have

$$\begin{aligned}
 q_{m+s-k+2}^{(n-m-s+2k-1)} &= \frac{E_{m+s-k+2}^{(n+k)} - E_{m+s-k+1}^{(n+k)}}{E_{m+s-k+1}^{(n+k)} - E_{m+s-k+1}^{(n+k-1)}} \\
 &= \frac{E_{m+s-k+2}^{(n+k)} - E_{m+s-k+1}^{(n+k)}}{E_{m+s-k+1}^{(n+k)} - E_{m+s-k}^{(n+k)}} \\
 &= \frac{1}{G_{m+s-k+1}^{(n+k)}} \frac{E_{m+s-k+1}^{(n+k+1)} - E_{m+s-k+1}^{(n+k)}}{E_{m+s-k+1}^{(n+k)} - E_{m+s-k}^{(n+k)}},
 \end{aligned}$$

which completes the proof. The e -values follow the normal rules. □

Using rules 1–5, the gap bordered in (15) involves the computation of the elements listed in the octagon below.



To be able to use the continued fraction representations from theorem 2 we must also calculate the necessary v - and h -values.

References

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