Stable multi-dimensional model reduction and IIR filter design

J. Abouir

Faculté des Sciences et Techniques,
Department de Mathématiques,
Université Hassan II,
B.P. 146 Mohammedia, Morocco
E-mail: Abouir@uh2m.ac.ma

A. Cuyt*

Department of Mathematics and Computer Science,
Universiteit Antwerpen,
Middelheimlaan 1, B-2020 Antwerpen, Belgium
E-mail: Annie.Cuyt@ua.ac.be
*Corresponding author

Abstract: It is well-known that model reduction techniques applied to stable multi-dimensional Linear Shift-Invariant (LSI) systems with Infinite-extent Impulse Response (IIR) do not necessarily guarantee a stable reduced system. Several conditions exist to check stability a posteriori. In this paper we outline a new technique that guarantees, a priori, that the system or filter is stable. In Section 1 we establish the necessary notation and definitions to deal with multi-dimensional systems and filters. Section 2 introduces the technique of multivariate Padé-type approximation and deals with stability. In Section 3 we illustrate the use of the newly proposed technique in filter design.

Keywords: model order reduction; rational transfer function; IIR system; digital filter; stability.

Reference to this paper should be made as follows: Abouir, J. and Cuyt, A. (2007) 'Stable multi-dimensional model reduction and IIR filter design', *Int. J. Computing Science and Mathematics*, Vol. 1, No. 1, pp.16–27.

Biographical notes: Jalil Abouir has a permanent position as Full Professor at the Université Hassan II at Mohammedia (Morocco). He obtained his Diplôme d'Etudes Approfondies (Applied Mathematics) from the Université de Lille I in 1987 and defended his PhD Thesis at the same university in 1992. He obtained the equivalent of the French Thèse d'Etat in 1994 from the University of La Laguna. Since then he has been involved in several research projects with a variety of European and Canadian colleagues.

Annie Cuyt is a Full Professor at the Department of Mathematics and Computer Science of the University of Antwerp. She received her Doctor Scientiae Degree in 1982 from the same university, summa cum laude and with the felicitations of the jury. Subsequently, she was a Research Fellow with the Alexander von Humboldt Foundation (Germany) and she was honoured with a Masuda Research Grant (Japan). She is the author of more than 100 publications in international journals and conference proceedings, the author or editor of several books and the Organiser of a number of international events. Her current interests are in rational approximation theory and its applications in scientific computing.

1 Introduction

Multi-dimensional systems arise in problems like computer-aided tomography, scene analysis, image deblurring, weather prediction, seismology, sonar and radar applications and many other problems. Noises can be filtered from spoken messages or picture images. Systems can transform a message to a form recognisable by a computer. A multi-dimensional discrete signal is represented by a multi-dimensional array

$$x(n_1,\ldots,n_d).$$

For simplicity of notation we restrict our presentation to $d = 2$. An important example of discrete signals is the unit-impulse $\delta(n_1, n_2)$ defined by $\delta(n_1, n_2) = 1$ for $n_1 = n_2 = 0$ and $\delta(n_1, n_2) = 0$ elsewhere. When talking about 2-dimensional LSI systems we always refer to recursive systems or systems with IIR, which transform an input signal $x(n_1, n_2)$ into an output signal $y(n_1, n_2)$ such that $y(n_1, n_2)$ can be described by a difference equation of the form

$$y(n_1,n_2) = \sum_{\substack{(k_1,k_2)\in N \\ N\subset\mathbb{Z}^2}} a(k_1,k_2)x(n_1-k_1,n_2-k_2) - \sum_{\substack{(k_1,k_2)\in D^\circ \\ D^\circ\subset\mathbb{Z}^2\setminus\{(0,0)\}}} b(k_1,k_2)y(n_1-k_1,n_2-k_2) \quad (1)$$

where $D^\circ \neq \phi$. The sets N and $D = D^\circ \cup \{(0, 0)\}$ are the regions of support of the arrays $a(n_1, n_2)$ and $b(n_1, n_2)$, respectively, with $b(0, 0) = 1$. For $x(n_1, n_2) = \delta(n_1, n_2)$ the above difference equation becomes

$$h(n_1,n_2) = a(n_1,n_2) - \sum_{(k_1,k_2)\in D^\circ} b(k_1,k_2)h(n_1-k_1,n_2-k_2) \quad (2)$$

and since $D^\circ \neq \phi$ the signal $h(n_1, n_2)$, which is called the impulse response of the system, indeed has infinite extent. Taking the z-transform of both sides of equation (2) results in

$$\begin{aligned} H(z_1,z_2) &= A(z_1,z_2) - \sum_{(k_1,k_2)\in D^\circ} b(k_1,k_2)H(z_1,z_2)\, z_1^{-k_1} z_2^{-k_2} \\ &= \sum_{(k_1,k_2)\in N} a(k_1,k_2)z_1^{-k_1}z_2^{-k_2} - H(z_1,z_2)\sum_{(k_1,k_2)\in D^\circ} b(k_1,k_2)z_1^{-k_1}z_2^{-k_2}. \end{aligned} \quad (3)$$

Here, $H(z_1, z_2)$ is the z-transform of the impulse response $h(n_1, n_2)$ and is called the transfer function of the system. So the transfer function of a recursive system is the ratio of the z-transforms $A(z_1, z_2)$ and $B(z_1, z_2)$ of the coefficient arrays $a(n_1, n_2)$ and $b(n_1, n_2)$, with $b(0, 0) = 1$:

$$H(z_1,z_2) = \frac{\sum_{(k_1,k_2)\in N} a(k_1,k_2)z_1^{-k_1}z_2^{-k_2}}{1+\sum_{(k_1,k_2)\in D^\circ} b(k_1,k_2)z_1^{-k_1}z_2^{-k_2}} = \frac{A(z_1,z_2)}{B(z_1,z_2)}. \quad (4)$$

Without loss of generality (Dudgeon and Mersereau, 1984, pp.176–180) we restrict ourselves here to systems whose impulse response has support on the first quadrant. This restriction is, for instance, commonly imposed upon IIR filters to guarantee a recursive implementation. We can therefore write for the transfer function of the desired system

$$H(z_1,z_2)=\sum_{n_1=0}^{+\infty}\sum_{n_2=0}^{+\infty}h(n_1,n_2)z_1^{-n_1}z_2^{-n_2} \tag{5}$$

or equivalently, with $w_1=z_1^{-1}$ and $w_2=z_2^{-1}$,

$$H(1/w_1,1/w_2)=\sum_{n_1=0}^{+\infty}\sum_{n_2=0}^{+\infty}h(n_1,n_2)w_1^{n_1}w_2^{n_2} \tag{6}$$

where the equality sign is only formal. In the sequel the series representation in the right hand side of equations (5) or (6) is denoted by $\tilde{H}$ while the rational function (4) is still denoted by H. The output signal $y(n_1, n_2)$ of a LSI system (5) applied to an input which is a complex sinusoid of the form $x(n_1, n_2) = \exp \mathtt{i}(t_1 n_1 + t_2 n_2)$, is characterised by the system's frequency response which is given by

$$H(e^{\mathtt{i}t_1},e^{\mathtt{i}t_2})=\sum_{n_1=0}^{+\infty}\sum_{n_2=0}^{+\infty}h(n_1,n_2)\exp(-\mathtt{i}(t_1n_1+t_2n_2)).$$

An important issue is stability. If a system is unstable, any input, including computational noise, can cause the output to grow without bound. Thus, the condition referred to as Bounded-Input-Bounded-Output (BIBO) stability is generally imposed. As indicated in Theorem 1, the stability of a multi-dimensional LSI system with transfer function H is essentially related to the zero-set of the denominator polynomial B (O'Connor and Huang, 1978; Shanks et al., 1972; Strintzis, 1977).

Theorem 1: *Let the two-dimensional first-quadrant LSI system with rational transfer function given by* (4) *have no non-essential singularities of the second kind on the unit bicircle. Then the system is stable if and only if one of the following conditions numbered* (i), (ii), (iii) *or* (iv) *is fulfilled:*

(i) $B(z_1, z_2) \neq 0$ *for* $|z_1| \geq 1, |z_2| \geq 1$

(ii) (*a*) $B(z_1, z_2) \neq 0$ *for* $|z_1| \geq 1, |z_2| = 1$

(*b*) $B(z_1, z_2) \neq 0$ *for* $|z_1| = 1, |z_2| \geq 1$

(iii) (*a*) $B(z_1, z_2) \neq 0$ *for* $|z_1| = 1, |z_2| = 1$

(*b*) $B(a, z_2) \neq 0$ *for* $|z_2| \geq 1$ *and any* a *such that* $|a| = 1$

(*c*) $B(z_1, b) \neq 0$ *for* $|z_1| \geq 1$ *and any* b *such that* $|b| = 1$

(iv) (*a*) $B(z_1, z_2) \neq 0$ *for* $|z_1| \geq 1, |z_2| = 1$

(*b*) $B(a, z_2) \neq 0$ *for* $|z_2| \geq 1$ *and any* a *such that* $|a| \geq 1$

(*here the role of* z_1 *and* z_2 *can be interchanged*).

2 Multi-dimensional Padé-type approximation

The problem with unstable model reduction is with the location of the poles of the reduced system, which can usually not be predicted while performing the reduction of the given system. The stability of the reduced system can for instance be concluded from an inspection of its rootmap. The rootmap shows the loci of the roots of $B(\cos\theta + \mathtt{i}\sin\theta, z_2)$ and those of $B(z_1, \cos\theta + \mathtt{i}\sin\theta)$ as θ traverses the interval $[-\pi, \pi]$ as indicated in Theorem 1(ii). Applying a simple technique such as Padé approximation, which is also well-understood in the multivariate case (Cuyt et al., 1992; Cuyt, 1999), to a stable system such as

$$H(z_1, z_2) = 0.00895 \frac{\begin{array}{c} 1 - 1.62151(z_1^{-1} + z_2^{-1}) + 2.63704 z_1^{-1} z_2^{-1} + 0.99994(z_1^{-2} + z_2^{-2}) \\ -1.62129 z_1^{-1} z_2^{-1}(z_1^{-1} + z_2^{-1}) + 1.00203 z_1^{-2} z_2^{-2} \end{array}}{\begin{array}{c} 1 - 1.78813(z_1^{-1} + z_2^{-1}) + 3.20640 z_1^{-1} z_2^{-1} + 0.82930(z_1^{-2} + z_2^{-2}) \\ -1.49271 z_1^{-1} z_2^{-1}(z_1^{-1} + z_2^{-1}) + 0.69823 z_1^{-2} z_2^{-2} \end{array}} \tag{7}$$

in an attempt to further reduce the degree, can generate an instable system. For instance, the general order Padé approximant to (7) of partial degree 1 in z_1^{-1} and z_2^{-1} in numerator and denominator,

$$\frac{p(w_1, w_2)}{q(w_1, w_2)} = \frac{\sum_{k_1,k_2=0}^{1} a(k_1, k_2) w_1^{k_1} w_2^{k_2}}{\sum_{k_1,k_2=0}^{1} b(k_1, k_2) w_1^{k_1} w_2^{k_2}} \quad w_1 = z_1^{-1}, w_2 = z_2^{-1}$$

satisfying

$$(\tilde{H}q - p)(w_1, w_2) = \sum_{\substack{(i,j)\in\mathbb{N}^2 \setminus \{(0,0),(1,0), \\ (0,1),(2,0),(1,1),(0,2),(2,1)\}}} g(i, j) w_1^i w_2^j$$

has a rootmap as shown in Figure 1, since it is given by

$$\frac{p(w_1, w_2)}{q(w_1, w_2)} = \frac{0.00895 - 0.04379 w_1 - 0.02368 w_2 + 0.05357 w_1 w_2}{1 - 5.0586 w_1 - 2.8226 w_2 + 7.2707 w_1 w_2}.$$

The original rootmap of $H(z_1, z_2)$, which remains inside the unit disc for both z_1 and z_2, is given in Figure 2. In both figures the unit circle is shown as a reference. The frequency response magnitude of the original system (7), which implements a lowpass filter of the form (17), is shown in Figure 3. Its $|H(e^{\mathrm{i}t_1}, e^{\mathrm{i}t_2})| = 0.1$ and $|H(e^{\mathrm{i}t_1}, e^{\mathrm{i}t_2})| = 0.5$ contour lines are shown in Figure 4.

The problem of the undesirable poles can be overcome by using the Padé-type approximation (Abouir and Cuyt, 1993), where the denominator polynomial of the reduced model is prechosen on the basis of other information, while still in some Padé approximation sense

$$H(1/w_1, 1/w_2) \approx \frac{p(w_1, w_2)}{q(w_1, w_2)}.$$

Figure 1 Rootmap of the unstable general order Padé approximant to $H(z_1, z_2)$ of partial degree 1 in z_1^{-1} and z_1^{-2} in numerator and denominator

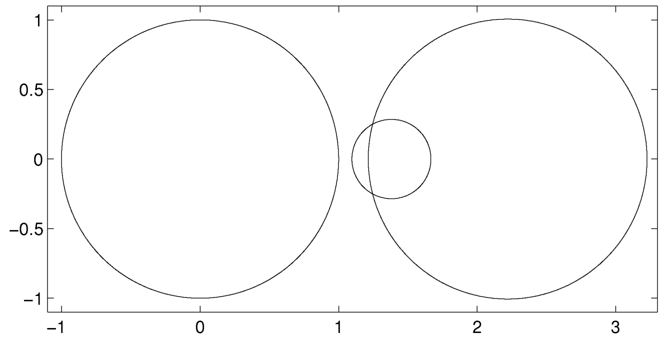


Figure 2 The original rootmap of $H(z_1, z_2)$, which remains inside the unit disc for both z_1 and z_2

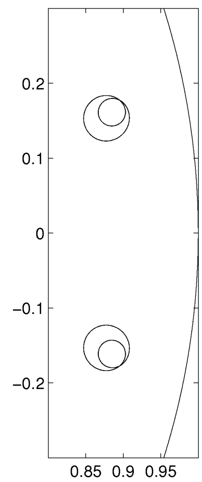


Figure 3 The frequency response magnitude of the system $H(z_1, z_2)$

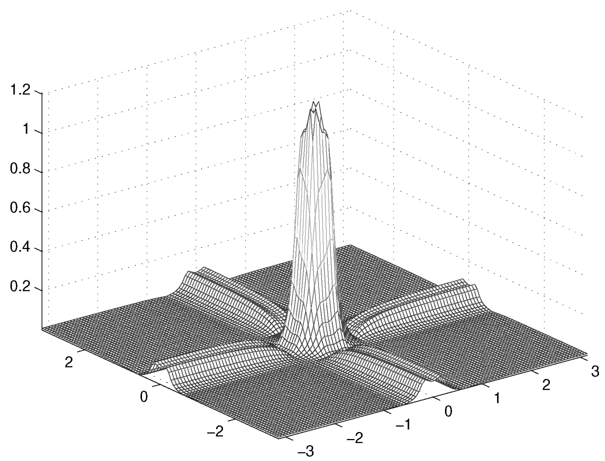
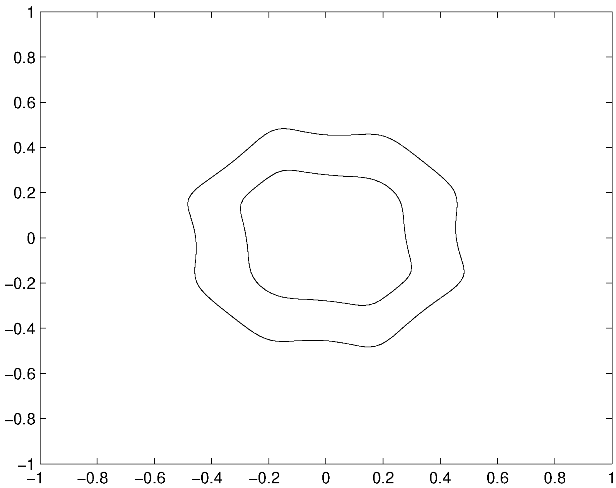


Figure 4 $|H(e^{\mathrm{i}t_1}, e^{\mathrm{i}t_2})| = 0.1$ and $|H(e^{\mathrm{i}t_1}, e^{\mathrm{i}t_2})| = 0.5$ contour lines for $H(z_1, z_2)$ of Figure 3



In the context of multi-dimensional model reduction, the Padé-type approximation goes as follows. Let the polynomial $q(w_1, w_2)$ be chosen. For instance, q can be a lower degree approximant to the denominator polynomial B of H that satisfies the same stability conditions as B. Or the coefficients of the polynomial q can be obtained as the solution of a least squares approximation subject to linear constraints such as the ones obtained in Theorem 2. The latter is illustrated in Section 3. Given the power series expansion (6) for the transfer function, we compute a Padé-type approximant $p(w_1, w_2)/q(w_1, w_2)$ to (6) by determining $p(w_1, w_2)$, for a given $q(w_1, w_2)$, from the Padé-like accuracy-through-order principle (8). Let the polynomials $p(w_1, w_2)$ and $q(w_1, w_2)$ be of the general form

$$p(w_1, w_2) = \sum_{(i,j)\in\tilde{N}} a(i,j) w_1^i w_2^j$$

$$q(w_1, w_2) = \sum_{(i,j)\in\tilde{D}} b(i,j) w_1^i w_2^j$$

where $\tilde{N}$ (Numerator) and $\tilde{D}$ (Denominator) are finite subsets of $\mathbb{N}^2$. When approximating $H(z_1, z_2)$ in (4), one usually chooses $\tilde{N} \subset N$ and $\tilde{D} \subset D$. It is now possible to let the free polynomial $p(w_1, w_2)$ satisfy

$$(\tilde{H}q - p)(w_1, w_2) = \sum_{(i,j)\in\mathbb{N}^2\setminus\tilde{N}} g(i,j) w_1^i w_2^j. \tag{8}$$

Equation (8) says nothing more than that $p(w_1, w_2)$ is the appropriate partial sum of the series development $(\tilde{H}q)(w_1, w_2)$. Its coefficients can be computed from

$$a(i,j) = \sum_{(k,\ell)\in\tilde{D}} b(k,\ell) h(i-k, j-\ell) \quad (i,j)\in\tilde{N} \tag{9}$$

where $h(i-k, j-\ell) = 0$ for $i < k$ or $j < \ell$. Despite the simplicity of the approximation procedure, multi-dimensional Padé-type approximants enjoy a number of nice properties (Abouir and Cuyt, 1993, pp.307–310). For a Padé-type approximant to the transfer function $H(z_1, z_2)$ with denominator $q(w_1, w_2)$ of the form (10), it is now easy to guarantee a stable reduced system. We use formulation (*ii*) of Theorem 1 to derive the a priori stability of a denominator $q(w_1, w_2)$ of the form

$$q(z_1, z_2) = \prod_{k=1}^{M_1} (1 - \alpha_{1k} z_1^{-1} - \beta_{1k} z_2^{-1}) \ \prod_{\ell=1}^{M_2} (1 - \alpha_{2\ell} z_1^{-1} - \beta_{2\ell} z_2^{-1} - \gamma_\ell z_1^{-1} z_2^{-1}) \tag{10}$$

with real coefficients α_{1k}, β_{1k}, $\alpha_{2\ell}$, $\beta_{2\ell}$ and γ_ℓ. It suffices to obtain the stability for each factor in q.

Theorem 2: *The polynomial q with α_{1k}, β_{1k}, $\alpha_{2\ell}$, $\beta_{2\ell}$ and $\gamma_\ell \in \mathbb{R}$ satisfies* (ii) *of Theorem 1 if:*

(*a*) *for all $1 \le k \le M_1$ the coefficients α_{1k} and β_{1k} satisfy*

$|\alpha_{1k}| < 1 \quad |\beta_{1k}| < 1 \quad |\alpha_{1k} + \beta_{1k}| < 1 \quad |\alpha_{1k} - \beta_{1k}| < 1$

(*b*) *for all $1 < \ell < M_2$ the coefficients $\alpha_{2\ell}$, $\beta_{2\ell}$ and γ_ℓ satisfy*

$|\alpha_{2\ell}| < 1 \quad |\beta_{2\ell}| < 1 \quad |\gamma_\ell| < 1$

$\alpha_{2\ell} + \beta_{2\ell} + \gamma_\ell < 1 \quad \alpha_{2\ell} - \beta_{2\ell} - \gamma_\ell < 1$

$-\alpha_{2\ell} + \beta_{2\ell} - \gamma_\ell < 1 \quad -\alpha_{2\ell} - \beta_{2\ell} + \gamma_\ell < 1$

Proof: The proofs of (a) and (b) are very similar. For each of the M_1 factors of the form $1 - \alpha z_1^{-1} - \beta z_2^{-1}$ we find that:

- for $|z_1| = 1$, in other words $z_1 = \cos\theta + \mathrm{i}\sin\theta$ with $|\theta| \le \pi$, the zero z_2 is given by

$$z_2 = \frac{\beta - \alpha\beta\cos\theta}{1 - 2\alpha\cos\theta + \alpha^2} + \mathrm{i}\frac{-\alpha\beta\sin\theta}{1 - 2\alpha\cos\theta + \alpha^2} \tag{11}$$

- and analogously, for $z_2 = \cos\phi + \mathrm{i}\sin\phi$ with $|\phi| \le \pi$ the zero z_1 equals

$$z_1 = \frac{\alpha - \alpha\beta\cos\phi}{1 - 2\beta\cos\phi + \beta^2} + \mathrm{i}\frac{-\alpha\beta\sin\phi}{1 - 2\beta\cos\phi + \beta^2}. \tag{12}$$

For equations (11) and (12) to satisfy $|z_2| < 1$ and $|z_1| < 1$ we need

$$\begin{cases} |\beta| < \sqrt{1 - 2\alpha\cos\theta + \alpha^2} \\ |\alpha| < \sqrt{1 - 2\beta\cos\phi + \beta^2} \end{cases} \tag{13}$$

which automatically implies $|\alpha| < 1$ and $|\beta| < 1$ because $\alpha\cos\theta + \beta\cos\phi < 1$ for all θ and ϕ. The conditions (13) simplify to

$|\alpha| < 1 \quad |\beta| < 1 \quad |\alpha + \beta| < 1 \quad |\alpha - \beta| < 1.$

For each of the M_2 factors $1-\alpha z_1^{-1}-\beta z_1^{-1}-\gamma z_1^{-1} z_2^{-1}$ in (b) the expressions (11) and (12) are replaced by

$$z_2=\frac{\beta-\alpha\gamma+(\gamma-\alpha\beta)\cos\theta}{1-2\alpha\cos\theta+\alpha^2}+\mathrm{i}\frac{-(\alpha\beta+\gamma)\sin\theta}{1-2\alpha\cos\theta+\alpha^2} \tag{14}$$

and

$$z_1=\frac{\alpha-\beta\gamma+(\gamma-\alpha\beta)\cos\phi}{1-2\beta\cos\phi+\beta^2}+\mathrm{i}\frac{-(\alpha\beta+\gamma)\sin\phi}{1-2\beta\cos\phi+\beta^2}. \tag{15}$$

For equations (14) and (15) to satisfy $|z_2| < 1$ and $|z_1| < 1$ we need

$$\begin{cases}\beta^2+\gamma^2+2\beta\gamma\cos\theta<1-2\alpha\cos\theta+\alpha^2\\ \alpha^2+\gamma^2+2\alpha\gamma\cos\phi<1-2\beta\cos\phi+\beta^2\end{cases} \tag{16}$$

which leads to

$$|\alpha| < 1 \quad |\beta| < 1 \quad |\gamma| < 1$$

$$\alpha+\beta+\gamma < 1 \quad \alpha-\beta-\gamma < 1$$

$$-\alpha+\beta-\gamma < 1 \quad -\alpha-\beta+\gamma < 1.$$

A stable Padé-type approximant to (5) with $h(n_1, n_2)$ given by (18) is for instance

$$\frac{p(w_1,w_2)}{q(w_1,w_2)}=\frac{0.00895-0.01443(w_1+w_2)+0.00862(w_1^2+w_2^2)+0.02336w_1w_2}{(1-0.88949(w_1+w_2)+0.79691w_1w_2)^2}.$$

3 Stable filter design

In IIR filter design (Hasegawa et al., 2000) the issue is to obtain a stable filter p/q such that

$$H(e^{\mathrm{i}t_1},e^{\mathrm{i}t_2})\approx\frac{p(e^{-\mathrm{i}t_1},e^{-\mathrm{i}t_2})}{q(e^{-\mathrm{i}t_1},e^{-\mathrm{i}t_2})}.$$

An ideal lowpass filter can for instance be specified by the frequency response

$$\left|H(e^{\mathrm{i}t_1},e^{\mathrm{i}t_2})\right| \text{ or } H(e^{\mathrm{i}t_1},e^{\mathrm{i}t_2})=\begin{cases}1 & (t_1,t_2)\in T\subset[-\pi,\pi]\times[-\pi,\pi]\\ 0 & (t_1,t_2)\notin T\end{cases} \tag{17}$$

where the domain T can be as simple as a square, disk or diamond, which are all 2D balls with radius r in the ℓ_∞, ℓ_2 or ℓ_1-norm respectively. For $T = [-\pi/8, \pi/8] \times [-\pi/8, \pi/8]$ we have for instance

$$h(n_1,n_2)=\frac{\sin((\pi/8)n_1)}{\pi n_1}\frac{\sin((\pi/8)n_2)}{\pi n_2}. \tag{18}$$

In practical situations it is known to be a problem how to determine a suitable, preferably separable, denominator polynomial. Now let q take one of the forms specified in Theorem 2 and let p be determined by (8) with $h(n_1, n_2)$ specified by (17). In addition, let S and U be finite grids of points covering respectively T and its complement $[-\pi, \pi] \times [-\pi, \pi] \backslash T$ and let $\lambda_{1,2}$ be positive weights with $\lambda_1 + \lambda_2 = 1$. Then, optimal values for the parameters α_{1k}, β_{1k}, $\alpha_{2\ell}$, $\beta_{2\ell}$ and γ_ℓ can be obtained from the solution of the discretised optimisation problem

$$\min_{\substack{\alpha_{1k},\beta_{1k},\\ \alpha_{2\ell},\beta_{2\ell},\gamma_\ell}} \left(\lambda_1 \sum_{(t_1,t_2)\in S} (Hq-p)^2 (e^{-\mathrm{i}t_1}, e^{-\mathrm{i}t_2}) + \lambda_2 \sum_{(t_1,t_2)\in U} (Hq-p)^2 (e^{-\mathrm{i}t_1}, e^{-\mathrm{i}t_2}) \right)$$

which simplifies to

$$\min_{\alpha_{1k},\beta_{1k},\alpha_{2\ell},\beta_{2\ell},\gamma_\ell} \left(\lambda_1 \sum_{(t_1,t_2)\in S} (q-p)^2 (e^{-\mathrm{i}t_1}, e^{-\mathrm{i}t_2}) + \lambda_2 \sum_{(t_1,t_2)\in U} p^2 (e^{-\mathrm{i}t_1}, e^{-\mathrm{i}t_2}) \right). \tag{19}$$

The difference with a classical (weighted) least-squares approach as in Hasegawa et al. (2002) is that in (19):

- the denominator coefficients in $q(w_1, w_2)$ are expressed in terms of some parameters α, β and γ
- the numerator coefficients in $p(w_1, w_2)$ depend through (8), or equivalently (9), on the same parameters
- only then is the optimisation problem written down, guaranteeing stability through a number of linear constraints for the α, β and γ, expressed in Theorem 2.

As an example, we take $T = [-\pi/8, \pi/8] \times [-\pi/8, \pi/8]$, $\lambda_1 = 0.75$, $\lambda_2 = 0.25$ and

$$q(w_1, w_2) = (1 - \alpha_{21}(z_1^{-1} + z_2^{-1}) - \gamma_1 z_1^{-1} z_2^{-1})(1 - \alpha_{22}(z_1^{-1} + z_2^{-1}) - \gamma_2 z_1^{-1} z_2^{-1})$$

with $p(w_1, w_2)$ in (8) also of partial degree 2 in both w_1 and w_2. So

$$\tilde{N} = \{(i, j) \mid 0 \le i, j \le 2\}.$$

Covering $[-\pi, \pi] \times [-\pi, \pi]$ in (19) by a grid of only 33×33 equidistant points (t_1, t_2) leads to

$$\alpha_{21} = 0.7939 \quad \alpha_{22} = 0.1436 \quad \gamma_1 = -0.7096 \quad \gamma_2 = 0.2501$$

for the denominator polynomial. The numerator polynomial of the filter is then given by

$$\begin{aligned} p(w_1, w_2) = {} & 0.04350 + 0.001609(z_1^{-1} + z_2^{-1}) + 0.004381(z_1^{-2} + z_1^{-2}) - 0.009265 z_1^{-1} z_2^{-1} \\ & + 0.000902(z_1^{-2} z_2^{-1} + z_1^{-1} z_2^{-2}) + 0.001505 z_1^{-2} z_2^{-2}. \end{aligned}$$

The frequency response magnitude of the latter lowpass filter implementation p/q, shown on $[-\pi, \pi] \times [-\pi, \pi]$ and more closely on $[-\pi/4, \pi/4] \times [-\pi/4, \pi/4]$ can be found in the Figures 5 and 6, respectively. Its contour lines $|H(e^{\mathrm{i}t_1}, e^{\mathrm{i}t_2})| = 0.1$ and $|H(e^{\mathrm{i}t_1}, e^{\mathrm{i}t_2})| = 0.5$ are shown in Figure 7.

Figure 5 The frequency response magnitude of the approximating lowpass filter p/q, shown on $[-\pi, \pi] \times [-\pi, \pi]$

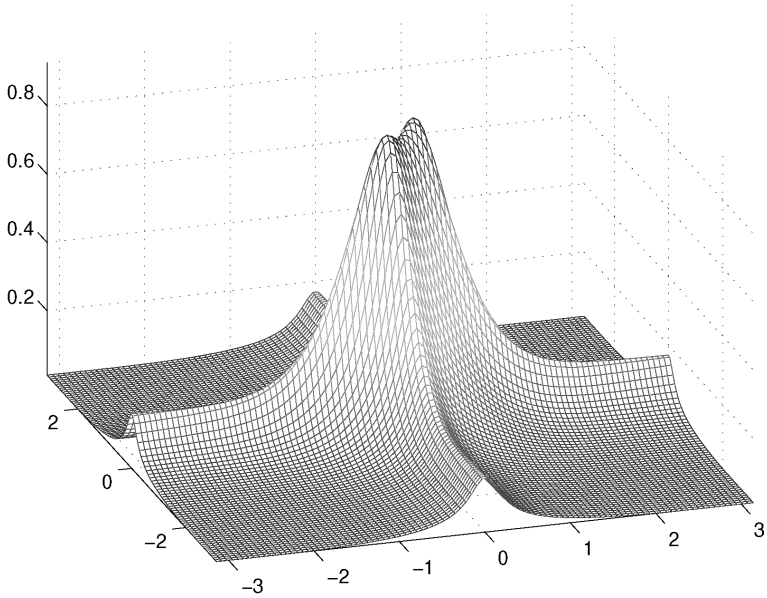


Figure 6 The frequency response magnitude of the approximating lowpass filter p/q, shown on $[-\pi/4, \pi/4] \times [-\pi/4, \pi/4]$

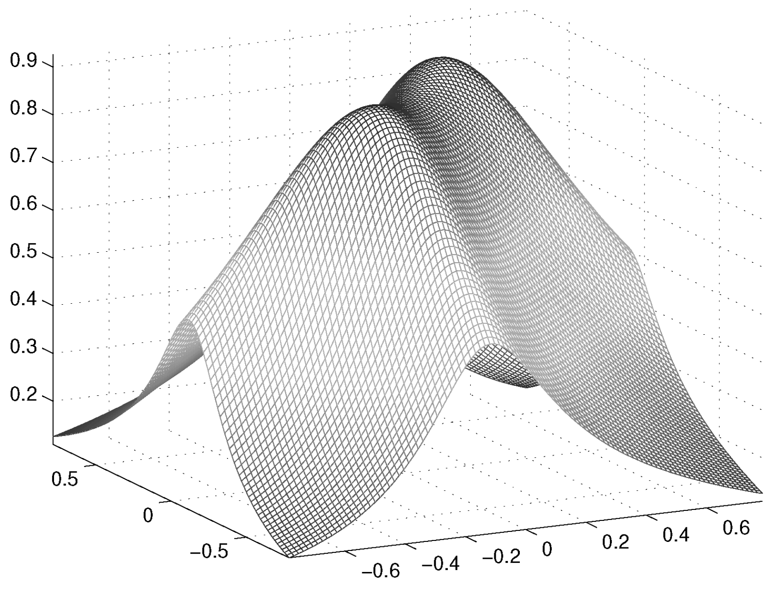
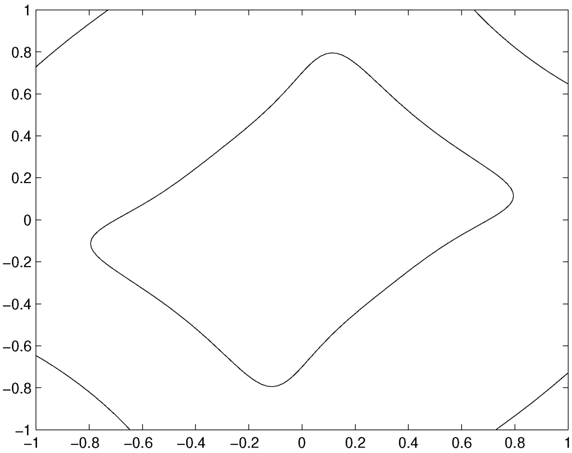


Figure 7 $|H(e^{\mathrm{i}t_1}, e^{\mathrm{i}t_2})| = 0.1$ and $|H(e^{\mathrm{i}t_1}, e^{\mathrm{i}t_2})| = 0.5$ contour lines for $H(z_1, z_2)$ of Figure 5



More complex optimisation schemes, where weights are used not only for the pass- and stopband, but also for the transition band and where different discretisations are applied to the different regions, can be considered. A study of the effect of these variations is outside the scope of this paper where we want to illustrate the usefulness of the concept of Padé-type approximation.

4 Future work

The possibility to determine the numerator and denominator coefficients in a multi-dimensional rational digital filter with an a priori stability guarantee, opens new perspectives. The main drawback of the technique is still the least-squares problem (19) which does not lead to a unique global optimum. The authors plan to investigate the applicability of quadratic programming techniques as in Salazar Celis et al. (2007) to the subject of digital filter design.

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