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Error formulas for multivariate rational interpolation and Padé approximation

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Abstract: The univariate error formulas for Padé approximants and rational interpolants, which are repeated in Section 2, are generalized to the multivariate case in Section 4. We deal with "general order" multivariate Padé approximants and rational interpolants, where the numerator and denominator polynomials as well as the equations expressing the approximation order, can be chosen by the user of these multivariate rational functions.

Keywords: Error in multivariate Padé approximation, Newton–Padé approximation, rational interpolation, rational Hermite interpolation.

1. The multivariate rational interpolation problem

We shall often restrict our description to the bivariate case for the sake of notational simplicity although we use the term multivariate. Let a bivariate function f be known in the points $(x_i, y_j) \in \mathbb{C}^2$ with $(i, j) \in I$, a finite subset of $\mathbb{N}^2$, playing the role of index set. If none of the points in $\{(x_i, y_j)\}_{(i,j)\in I}$ coincides, then we are dealing with a rational interpolation problem and the values in $\{f_{ij}\}_{(i,j)\in I}$ are function values. If all the interpolation points coincide, then the problem is one of Padé approximation and it is well known that the given data are not function values but Taylor coefficients. If some of the points coincide and some do not, then the problem is of a mixed type and it is called a Hermite interpolation problem or a Newton–Padé approximation problem. In [3] is indicated how one should interpret the data f_{ij}: some of them are partial derivatives and some of them are function values. In the sequel of the text we shall distinguish, when necessary, between the Padé approximation case, where all the interpolation points coincide, and the Newton–Padé approximation case, where this is not so.

With our data points (x_i, y_j) we construct the polynomial basis functions

$$B_{ij}(x, y) = \prod_{k=0}^{i-1}(x - x_k)\prod_{l=1}^{j-1}(y - y_l).$$

The problem of interpolating the data f_{ij} by a bivariate rational function was formulated in [3] as follows. Choose finite subsets N (from "Numerator") and D (from "Denominator") of $\mathbb{N}^2$ with $N \subset I$ and compute bivariate polynomials

$$p(x, y) = \sum_{(i,j)\in N} a_{ij} B_{ij}(x, y), \quad \#N = n+1,$$
$$q(x, y) = \sum_{(i,j)\in D} b_{ij} B_{ij}(x, y), \quad \#D = m+1, \tag{1a}$$

such that

$$(fq - p)(x_i, y_j) = 0, \quad (i, j) \in I, \quad \#I = n + m + 1. \tag{1b}$$

If $q(x_i, y_j) \neq 0$, then this last condition implies that

$$f(x_i, y_j) = \frac{p}{q}(x_i, y_j), \quad (i, j) \in I.$$

If some of the x_i and y_j coincide, then also higher partial derivatives of $(fq - p)$ will cancel at (x_i, y_j) and higher partial derivatives of f will agree with those of p/q at (x_i, y_j) [3]. The following two conditions for the polynomials given in (1a) are sufficient to satisfy (1b), both in the Padé and Newton–Padé approximation case [3]:

$$(fq - p)(x, y) = \sum_{(i,j)\in \mathbb{N}^2 \setminus I} d_{ij} B_{ij}(x, y), \tag{2a}$$

I satisfies the inclusion property, (2b)

where the series development (2a) is still formal and where (2b) means that, when a point (i, j) belongs to I, all the points in the rectangle emanating from the origin with (i, j) as its furthermost corner belong to I. How this can be achieved in a lot of situations is explained in [3]. From now on we restrict ourselves to finite interpolation sets I satisfying this inclusion property. The series development (2a) is not only formal, but also involves undefined interpolation points. More precisely, it has to be interpreted as an accuracy-through-order condition. The rest of the paper is devoted to obtain usable formulas for $(fq - p)(x, y)$.

2. The univariate error formulas

Let the polynomials $p(z)$ and $q(z)$ solve the univariate (n, m) Padé approximation problem for a function $f(z)$ with $z \in \mathbb{C}$ and let the function $(fq)(z)$ be holomorphic in the disk $B(0; \rho)$ with center 0 and radius ρ. Then we know that for $|z| < \rho$

$$(fq - p)(z) = \sum_{i \geq n+m+1} d_i z^i. \tag{3}$$

The series $(fq - p)(z)$ can be expressed by Cauchy's integral formula as

$$(fq - p)(z) = \frac{1}{2\pi i} \int_{|w|=\rho} \frac{(fq - p)(w)}{w - z} \, dw,$$

with its Taylor coefficients given by

$$\frac{d^j (fq - p)(0)}{dz^j} = \frac{j!}{2\pi i} \int_{|w|=\rho} \frac{(fq - p)(w)}{w^{j+1}} \, dw.$$

With (3) in mind and knowing that the degree of p is at most n, Cauchy's integral formula for the rest series (3) reads [1, p.250]

$$(fq-p)(z)=\frac{1}{2\pi i}\int_{|w|=\rho}\left(\frac{z}{w}\right)^{n+m+1}\frac{(fq)(w)}{w-z}\mathrm{d}w. \tag{4}$$

From (3) we also know that for $|z|<\rho$

$$\begin{aligned}|(fq-p)(z)| &\leqslant \sup_{w\in[0,z]}|(fq-p)^{(n+m+1)}(w)|\frac{|z|^{n+m+1}}{(n+m+1)!}\\ &= \sup_{w\in[0,z]}|(fq)^{(n+m+1)}(w)|\frac{|z|^{n+m+1}}{(n+m+1)!}.\end{aligned} \tag{5}$$

Let the polynomials $p(z)$ and $q(z)$ solve the univariate (n, m) Newton–Padé approximation problem for a function $f(z)$ and for interpolation points $z_0,\dots, z_{n+m}$. Let the function $(fq)(z)$ be holomorphic in the disk $B(0;\ \rho)$ with center 0 and radius ρ, containing the interpolation points. Then for $|z|<\rho$

$$(fq-p)(z)=\sum_{i\geqslant n+m+1} d_iB_i(z),\quad B_i(z)=\prod_{k=0}^{i-1}(z-z_k). \tag{6}$$

The rest series can still be given by an integral formula [2, p.121]

$$(fq-p)(z)=\frac{1}{2\pi i}\int_{|w|=\rho}\frac{B_{n+m+1}(z)(fq)(w)}{B_{n+m+1}(w)(w-z)}\mathrm{d}w, \tag{7}$$

which is based on Hermite's formula for the divided differences d_i, namely

$$\begin{aligned}(fq-p)[z_0,\dots, z_{n+m}, z] &= (fq)[z_0,\dots, z_{n+m}, z]\\ &= \frac{1}{2\pi i}\int_{|w|=\rho}\frac{(fq)(w)}{B_{n+m+1}(w)(w-z)}\mathrm{d}w.\end{aligned}$$

The majorant for this error is given by [5, p.5]

$$|(fq-p)(z)|\leqslant \sup_{|w|\leqslant\rho}|(fq)^{(n+m+1)}(w)|\frac{|B_{n+m+1}(z)|}{(n+m+1)!}. \tag{8}$$

We shall now develop multivariate analogons of the formulas (4) and (7).

3. Tools from multivariate complex analysis

Let the multivariate function $f(z_1,\dots, z_p)$ be given in the polydisc $B(0;\ \rho_1,\dots, \rho_p)=\{(z_1,\dots, z_p)\in\mathbb{C}^p:\ |z_i|<\rho_i,\ i=1,\dots, p\}$ with center 0 and polyradius $(\rho_1,\dots, \rho_p)$. A multivariate function $f(z_1,\dots, z_p)$ holomorphic in the polydisc $B(0;\ \rho_1,\dots, \rho_p)$ is now given by the following Cauchy integral form [4]

$$f(z_1,\dots, z_p)=\left(\frac{1}{2\pi i}\right)^p\int_{|w_i|=\rho_i}\frac{f(w_1,\dots, w_p)}{(w_1-z_1)\cdots(w_p-z_p)}\ \mathrm{d}w_1\cdots\mathrm{d}w_p,$$

and its Taylor series coefficients are given by

$$\frac{\partial^{i_1+\cdots+i_p} f}{\partial x_1^{i_1} \cdots \partial x_p^{i_p}}(0,\ldots,0) = \frac{(i_1)! \cdots (i_p)!}{(2\pi i)^p} \int_{|w_i|=\rho_i} \frac{f(w_1,\ldots,w_p)}{w_1^{i_1+1} \cdots w_p^{i_p+1}} \mathrm{d}w_1 \cdots \mathrm{d}w_p.$$

For $f(z_1,\ldots,z_p)$ bounded in the polydisc by

$$|f(z_1,\ldots,z_p)| \leqslant M, \quad (z_1,\ldots,z_p) \in B(0;\rho_1,\ldots,\rho_p),$$

we also have [4]

$$\frac{\partial^{i_1+\cdots+i_p} f}{\partial x_1^{i_1} \cdots \partial x_p^{i_p}}(0,\ldots,0) \leqslant M \frac{(i_1)! \cdots (i_p)!}{\rho_1^{i_1} \cdots \rho_p^{i_p}}.$$

The Taylor coefficients of $f(z_1,\ldots,z_p)$ are limit values of its divided differences

$$f\left[z_1^{(0)},\ldots,z_1^{(i_1)}\right] \cdots \left[z_p^{(0)},\ldots,z_p^{(i_p)}\right],$$

when we let all the data points $(z_1^{(0)},\ldots,z_p^{(0)}), (z_1^{(1)},\ldots,z_p^{(1)}),\ldots$ coincide in the origin. The divided differences for a data set satisfying the inclusion property (2b), are recursively computed by [7]

$$\begin{aligned} & f\left[z_1^{(0)},\ldots,z_1^{(i_1)}\right] \cdots \left[z_p^{(0)},\ldots,z_p^{(i_p)}\right] \\ & \quad = \left\{ f\left[z_1^{(0)},\ldots,z_1^{(i_1)}\right] \cdots \left[z_k^{(1)},\ldots,z_k^{(i_k-1)}, z_k^{(i_k)}\right] \cdots \left[z_p^{(0)},\ldots,z_p^{(i_p)}\right] \right. \\ & \qquad \left. - f\left[z_1^{(0)},\ldots,z_1^{(i_1)}\right] \cdots \left[z_k^{(0)},\ldots,z_k^{(i_k-2)}, z_k^{(i_k-1)}\right] \cdots \left[z_p^{(0)},\ldots,z_p^{(i_p)}\right] \right\} \\ & \qquad \times \left\{ z_k^{(i_k)} - z_k^{(0)} \right\}^{-1}, \\ & \qquad k = 1,\ldots,p, \end{aligned}$$

with

$$f\left[z_1^{(k)}\right] \cdots \left[z_p^{(k)}\right] = f\left(z_1^{(k)},\ldots,z_p^{(k)}\right),$$

and they can still be given by Hermite's formula applied to each of the variables successively, namely [2, p.149]

$$\begin{aligned} & f\left[z_1^{(0)},\ldots,z_1^{(i_1)}\right] \cdots \left[z_p^{(0)},\ldots,z_p^{(i_p)}\right] \\ & \quad = \left(\frac{1}{2\pi i}\right)^p \int_{|w_i|=\rho_i} \frac{f(w_i,\ldots,w_p)}{B_{i_1+1,\ldots,i_p+1}(w_1,\ldots,w_p)} \mathrm{d}w_1 \cdots \mathrm{d}w_p, \end{aligned}$$

where the polydisc $B(0;\rho_1,\ldots,\rho_p)$ contains the interpolation points occurring in the divided difference. In what follows we shall treat, without loss of generality, the bivariate case. The reader is asked not to confuse the complex number i with the index i.

4. The multivariate error formulas for Padé approximants: first approach

In order to generalize expression (4) for general interpolation sets $I \subset \mathbb{N}^2$, we divide the index set $\mathbb{N}^2 \backslash I$ of the rest series (2a) in three parts, called a "vertical" part, a "horizontal" part and a

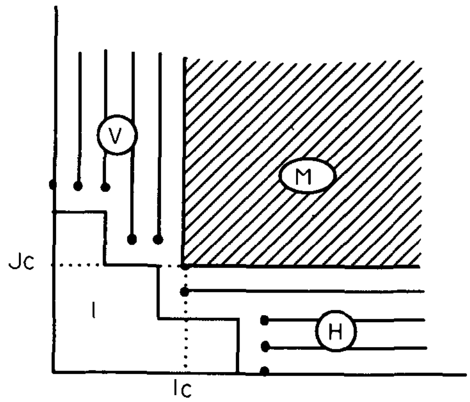

Fig. 1.

"mixed" part. For each of the three parts the infinite sums can be replaced by a finite number of contributions which can easily be majorized, thus establishing multivariate error formulas. Let us now first describe how to "divide and conquer". Choose a point (i_C, j_C) along the border of I which will keep the "horizontal" and "vertical" parts separated from each other by the "mixed" part, and write

$$V=\{(i, j) \mid 0 \leqslant i<i_C\} \cap (\mathbb{N}^2 \backslash I),$$
$$H=\{(i, j) \mid 0 \leqslant j<j_C\} \cap (\mathbb{N}^2 \backslash I),$$
$$M=\{(i_C+k, j_C+l) \mid k, l \geqslant 0\},$$
$$\begin{aligned}(fq-p)(x, y) &= \sum_{(i,j) \in \mathbb{N}^2 \backslash I} d_{ij} x^i y^j \\ &= \sum_{(i,j) \in V} d_{ij} x^i y^j + \sum_{(i,j) \in H} d_{ij} x^i y^j + \sum_{(i,j) \in M} d_{ij} x^i y^j. \end{aligned} \tag{9}$$

Remember that all interpolation points coincide when dealing with Padé approximants. Without loss of generality we let them coincide in the origin. It is clear that the set V consists of vertical half-lines containing indices from $\mathbb{N}^2 \backslash I$, that the set H consists of horizontal half-lines containing indices from $\mathbb{N}^2 \backslash I$ and that M is merely a translation of $\mathbb{N}^2$ over (i_C, j_C). Each of the vertical and horizontal half-lines have an origin along the border of I and the number of vertical and horizontal lines is finite (see Fig. 1).

For the indices (i, j) on a vertical half-line in V, the coefficient d_{ij} of the rest series is given by

$$\begin{aligned} d_{ij} &= \frac{1}{i!j!} \frac{\partial^{i+j}(fq-p)}{\partial x^i \partial y^j}(0, 0) \\ &= \frac{1}{j!} \frac{\partial^j}{\partial y^j}\left(\frac{1}{i!} \frac{\partial^i(fq-p)}{\partial x^i}\right)(0, 0) \\ &= \frac{1}{i!(2\pi \mathrm{i})} \int_{|v|=\rho_2} \frac{1}{v^{j+1}} \left.\frac{\partial^i(fq-p)(x, v)}{\partial x^i}\right|_{x=0} \mathrm{d}v, \end{aligned}$$

and analogously on a horizontal half-line in H,

$$\begin{aligned} d_{ij} &= \frac{1}{i!j!} \frac{\partial^{i+j}(fq-p)}{\partial x^i \partial y^j}(0, 0) \\ &= \frac{1}{i!} \frac{\partial^i}{\partial x^i}\left(\frac{1}{j!} \frac{\partial^j(fq-p)}{\partial y^j}\right)(0, 0) \\ &= \frac{1}{j!(2\pi i)} \int_{|u|=\rho_1} \frac{1}{u^{i+1}} \left.\frac{\partial^j(fq-p)(u, y)}{\partial y^j}\right|_{y=0} du. \end{aligned}$$

Since $N \subset I$ and (i, j) in $V \cup H$ lie outside I, we can replace $(fq-p)$ in the above expressions by (fq) since no terms of p will survive the differentiation to contribute to d_{ij}. Let us denote the origin of a vertical half-line by (i, j_i) with $0 \leqslant i < i_C$ and the origin of a horizontal half-line by (i_j, j) with $0 \leqslant j < j_C$. Finally we obtain from (9)

$$\begin{aligned} (fq-p)(x, y) = & \sum_{i=0}^{i_C-1} \frac{1}{i!(2\pi i)} \int_{|v|=\rho_2} \frac{x^i y^{j_i}}{v^{j_i}(v-y)} \left.\frac{\partial^i(fq)(x, v)}{\partial x^i}\right|_{x=0} dv \\ & + \sum_{j=0}^{j_C-1} \frac{1}{j!(2\pi i)} \int_{|u|=\rho_1} \frac{x^{i_j} y^j}{u^{i_j}(u-x)} \left.\frac{\partial^j(fq)(u, y)}{\partial y^j}\right|_{y=0} du \\ & + \left(\frac{1}{2\pi i}\right)^2 \int_{|u|=\rho_1} \int_{|v|=\rho_2} \left(\frac{x}{u}\right)^{i_C} \left(\frac{y}{v}\right)^{j_C} \frac{(fq)(u, v)}{(u-x)(v-y)} du\, dv. \end{aligned} \tag{10}$$

Using Fubini's and Taylor's theorem, formula (10) can be rewritten as

$$\begin{aligned} (fq-p)(x, y) = & \sum_{i=0}^{i_C-1} \frac{1}{i!j_i!} x^i y^{j_i} \frac{\partial^{j_i}}{\partial y^{j_i}} \frac{\partial^i(fq)}{\partial x^i}(0, \eta_i) \\ & + \sum_{j=0}^{j_C-1} \frac{1}{i_j!j!} x^{i_j} y^j \frac{\partial^{i_j}}{\partial x^{i_j}} \frac{\partial^j(fq)}{\partial y^j}(\xi_j, 0) \\ & + \frac{1}{2\pi i} \int_{|u|=\rho_1} \left(\frac{x}{u}\right)^{i_C} \frac{1}{(u-x)} \left(\frac{1}{2\pi i} \int_{|v|=\rho_2} \left(\frac{y}{v}\right)^{j_C} \frac{(fq)(u, v)}{(v-y)} dv\right) du \\ = & \sum_{i=0}^{i_C-1} \frac{1}{i!j_i!} x^i y^{j_i} \frac{\partial^{j_i}}{\partial y^{j_i}} \frac{\partial^i(fq)}{\partial x^i}(0, \eta_i) \\ & + \sum_{j=0}^{j_C-1} \frac{1}{i_j!j!} x^{i_j} y^j \frac{\partial^{i_j}}{\partial x^{i_j}} \frac{\partial^j(fq)}{\partial y^j}(\xi_j, 0) \\ & + \frac{1}{2\pi i} \int_{|u|=\rho_1} \left(\frac{x}{u}\right)^{i_C} \frac{1}{(u-x)} \frac{1}{j_C!} y^{j_C} \frac{\partial^{j_C}(fq)}{\partial y^{j_C}}(u, \eta)\, du \\ = & \sum_{i=0}^{i_C-1} \frac{1}{i!j_i!} x^i y^{j_i} \frac{\partial^{j_i}}{\partial y^{j_i}} \frac{\partial^i(fq)}{\partial x^i}(0, \eta_i) \\ & + \sum_{j=0}^{j_C-1} \frac{1}{i_j!j!} x^{i_j} y^j \frac{\partial^{i_j}}{\partial x^{i_j}} \frac{\partial^j(fq)}{\partial y^j}(\xi_j, 0) \\ & + \frac{1}{i_C!j_C!} x^{i_C} y^{j_C} \frac{\partial^{i_C+j_C}(fq)}{\partial x^{i_C} \partial y^{j_C}}(\xi, \eta), \end{aligned}$$

with $\xi, \xi_j \in [0, x]$ and $\eta, \eta_i \in [0, y]$ for $i = 0, \ldots, i_C$ and $j = 0, \ldots, j_C$.

5. The multivariate error formulas for Newton–Padé approximants

In order to generalize expression (7) for general interpolation sets $I \subset \mathbb{N}^2$ we again divide the index set $\mathbb{N}^2 \setminus I$ of the rest series (2a) in three parts, called a "vertical" part, a "horizontal" part and a "mixed" part. From (2a) we can write

$$\begin{aligned}(fq-p)(x, y) &= \sum_{(i,j)\in \mathbb{N}^2\setminus I} d_{ij}B_{ij}(x, y)\\ &= \sum_{(i,j)\in V} d_{ij}B_{ij}(x, y) + \sum_{(i,j)\in H} d_{ij}B_{ij}(x, y) + \sum_{(i,j)\in M} d_{ij}B_{ij}(x, y).\end{aligned} \tag{11}$$

Each vertical half-line from V originates in an index (i, j_i) with $0 \leqslant i < i_C$ and each horizontal half-line from H originates in an index (i_j, j) with $0 \leqslant j < j_C$. Hence the contribution to the rest series from V can be rewritten as

$$\begin{aligned}\sum_{(i,j)\in V} d_{ij}B_{ij}(x, y) &= \sum_{i=0}^{i_C-1} B_i(x) \sum_{j=j_i}^{\infty} d_{ij}B_j(y)\\ &= \sum_{i=0}^{i_C-1} B_i(x)(fq-p)[x_0,\dots, x_i][y_0,\dots, y_{j_i-1}, y]B_{j_i}(y)\\ &= \sum_{i=0}^{i_C-1} \frac{1}{2\pi \mathrm{i}} \int_{|v|=\rho_2} \frac{B_{i,j_i}(x, y)}{B_{j_i}(v)} \frac{(fq)[x_0,\dots, x_i][v]}{v-y} \mathrm{d}v.\end{aligned}$$

Analogously for the contribution from H

$$\begin{aligned}\sum_{(i,j)\in H} d_{ij}B_{ij}(x, y) &= \sum_{j=0}^{j_C-1} B_j(y) \sum_{i=i_j}^{\infty} d_{ij}B_i(x)\\ &= \sum_{j=0}^{j_C-1} B_j(y)(fq-p)[x_0,\dots, x_{i_j-1}, x][y_0,\dots, y_j]B_{i_j}(x)\\ &= \sum_{j=0}^{j_C-1} \frac{1}{2\pi \mathrm{i}} \int_{|u|=\rho_1} \frac{B_{i_j,j}(x, y)}{B_{i_j}(u)} \frac{(fq)[u][y_0,\dots, y_j]}{u-x} \mathrm{d}u.\end{aligned}$$

For the "mixed part" we have

$$\begin{aligned}\sum_{(i,j)\in M} d_{ij}B_{ij}(x, y) &= (fq-p)[x_0,\dots, x_{i_C-1}, x][y_0,\dots, y_{j_C-1}, y]B_{i_C,j_C}(x, y)\\ &= (fq)[x_0,\dots, x_{i_C-1}, x][y_0,\dots, y_{j_C-1}, y]B_{i_C,j_C}(x, y)\\ &= \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_{|u|=\rho_1}\int_{|v|=\rho_2} \frac{B_{i_C,j_C}(x, y)}{B_{i_C,j_C}(u, v)} \frac{(fq)(u, v)}{(u-x)(v-y)} \mathrm{d}u\, \mathrm{d}v.\end{aligned}$$

Grouping our results we get the following integral formula for the error given by (11)

$$(fq-p)(x, y) = \sum_{i=0}^{i_C-1} \frac{1}{2\pi i} \int_{|v|=\rho_2} \frac{B_{i,j_i}(x, y)}{B_{j_i}(v)} \frac{(fq)[x_0,\ldots, x_i][v]}{v-y} \mathrm{d}v$$
$$+ \sum_{j=0}^{j_C-1} \frac{1}{2\pi i} \int_{|u|=\rho_1} \frac{B_{i_j,j}(x, y)}{B_{i_j}(u)} \frac{(fq)[u][y_0,\ldots, y_j]}{u-x} \mathrm{d}u$$
$$+ \left(\frac{1}{2\pi i}\right)^2 \int_{|u|=\rho_1} \int_{|v|=\rho_2} \frac{B_{i_C,j_C}(x, y)}{B_{i_C,j_C}(u, v)} \frac{(fq)(u, v)}{(u-x)(v-y)} \mathrm{d}u\, \mathrm{d}v, \tag{12}$$

which transforms to (10) when we let all the interpolation points coincide in the origin. Using Hermite's integral formula for divided differences, (12) can be rewritten as

$$(fq-p)(x, y) = \sum_{i=0}^{i_C-1} B_{i,j_i}(x, y)(fq)[x_0,\ldots, x_i][y_0,\ldots, y_{j_i-1}, y]$$
$$+ \sum_{j=0}^{j_C-1} B_{i_j,j}(x, y)(fq)\left[x_0,\ldots, x_{i_j-1}, x\right]\left[y_0,\ldots, y_j\right]$$
$$+ B_{i_C,j_C}(x, y)(fq)\left[x_0,\ldots, x_{i_C}, x\right]\left[y_0,\ldots, y_{j_C}, y\right].$$

6. The multivariate error formulas for Padé approximants: second approach

Instead of dividing $\mathbb{N}^2\backslash I$ in V, H and M we can also include I in a circumscribing triangle as follows. Let

$$k_D = \max_{(i,j)\in I} (i+j).$$

This integer indicates the diagonal furthermost from the origin that contains index points from I. Then define

$$T = \{(i, j) \mid 0 \leqslant i+j \leqslant k_D\}$$

(see Fig. 2).

We rewrite (9) as

$$(fq-p)(x, y) = \sum_{(i,j)\in\mathbb{N}^2\backslash I} d_{ij}x^iy^j$$
$$= \sum_{(i,j)\in T\backslash I} d_{ij}x^iy^j + \sum_{(i,j)\in\mathbb{N}^2\backslash T} d_{ij}x^iy^i. \tag{13}$$

Using a multivariate Taylor's formula [6] and knowing that the index set N defining $p(x, y)$ is a subset of I and hence of T, (13) can be rewritten as

$$(fq-p)(x, y) = \sum_{(i,j)\in T\backslash I} \frac{1}{i!j!} \frac{\partial^{i+j}(fq)}{\partial x^i y^j}(0, 0)x^iy^j$$
$$+ \frac{1}{(k_D+1)!} \sum_{i=0}^{k_D+1} \binom{k_D+1}{i} \frac{\partial^{k_D+1}(fq)}{\partial x^i y^{k_D+1-i}}(\xi, \eta)x^iy^{k_D+1-i},$$

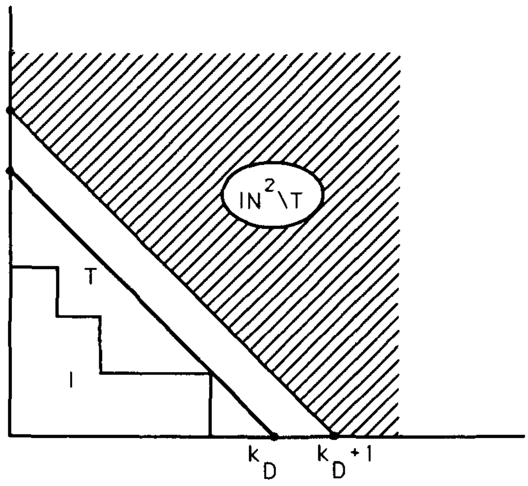

Fig. 2.

with (ξ, η) on the line segment joining (0, 0) and (x, y). Sometimes the set $T \backslash I$ may be large and then this approach involves more partial derivatives of the function $(fq)(x, y)$ than the approach of Section 4. Moreover, even if $T \backslash I$ is small, its use involves information in undefined interpolation points, because the indices in $T \backslash I$ lie outside I. The choice of the error formula depends on the configuration of the data set I.

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