Numerical Algorithms **33:** 11–26, 2003.

Multivariate two-point Padé-type and two-point Padé approximants

J. Abouir [a], A. Cuyt [b] and R. Orive [c,*]

[a] *Departement de Mathématique, Faculté des Sciences et Techniques, Mohammedia, B.P. 146 Mohammedia, Morocco*
[b] *FWO, Department of Mathematics and Computer Science, University of Antwerp (UIA), Universiteitsplein 1, B-2610 Wilrijk, Belgium*
[c] *Department of Mathematical Analysis, La Laguna University, Tenerife, Canary Islands, Spain*

Received 5 December 2001; accepted 7 October 2002

In his paper the notions of two-point Padé-type and two-point Padé approximants are generalized for multivariate functions, with a generating denominator polynomial of general form. The multivariate two-point Padé approximant can be expressed as a ratio of two determinants and computed recursively using the E-algorithm. A comparison is made with previous definitions by other authors using particular generating denominator polynomials. The last section contains some convergence results.

Keywords: multivariate, two-point Padé approximant, Padé-type approximant, convergence

AMS subject classification: 41A21, 41A63, 65D15

1. Introduction

In [4], Brezinski introduces the notion of multivariate Padé-type approximant from the so-called rectangular form of a polynomial in two variables. In [18], Orive and Gonzalez-Vera extend the concept of two-point Padé approximants to functions of two variables, again following the rectangular approach, by means of certain linear functionals acting on the space of bivariate Laurent polynomials. For $(m_1, m_2) \in \mathbb{N}^2$, the numerator and denominator polynomials of these approximants respectively take the form

$$P(x,y) = \sum_{(i,j)\in N} a_{ij} x^i y^j, \qquad Q(x,y) = \sum_{(d,e)\in D} b_{de} x^d y^e,$$

where

$$N = \big([0, k_1 - 1] \times [0, k_2 - 1] \cup [k_1, m_1 - 1] \times [k_2, m_2 - 1]\big) \cap \mathbb{N}^2,$$

* The work of this author was performed as part of the project P1999/127 of Autónomo de Canarias.

$$D = [0, m_1] \times [0, m_2] \cap \mathbb{N}^2,$$
$$1 \leqslant k_1 \leqslant m_1 - 1, \qquad 1 \leqslant k_2 \leqslant m_2 - 1.$$

The aim of this paper is to define a new family of multivariate two-point Padé-type approximants, admitting a general form for the subsets N and D (see [1]). Next, we construct multivariate two-point Padé approximants satisfying higher order approximation conditions. Section 5 contains two examples illustrating the numerical advantage of the new approximants.

2. Univariate two-point Padé-type approximants

Univariate two-point Padé-type and two-point Padé approximants have been studied by Draux [11,12], Gonzalez-Vera [13], Jones and Thron [15], McCabe [17] and others. For results on the convergence of sequences of such approximants, see [10]. Let the (possibly) formal series expansions

$$f_0(x) = \sum_{i=0}^{\infty} c_i x^i, \qquad |x| \to 0, \tag{1}$$

$$f_\infty(x) = \sum_{i=1}^{\infty} c_{-i}^* x^{-i}, \quad |x| \to \infty, \tag{2}$$

and two integers m and k $(0 \leqslant k \leqslant m)$ be given.

Given an arbitrary polynomial $q(x) = \sum_{i=0}^{m} b_i x^i$ of degree m with $q(0) \neq 0$, called generating polynomial, we look for a polynomial $p(x) = \sum_{i=0}^{m-1} a_i x^i$ of degree $m-1$ satisfying

$$\left(f_0 - \frac{p}{q}\right)(x) = \mathrm{O}\left(x^k\right), \tag{3}$$

$$\left(f_\infty - \frac{p}{q}\right)(x) = \mathrm{O}\left(\left(x^{-1}\right)^{m-k+1}\right). \tag{4}$$

The coefficients a_i of $p(x)$ are then given by

$$a_i = \sum_{j=0}^{m} b_j c_{i-j}, \quad 0 \leqslant i \leqslant k-1, \tag{5}$$

$$a_i = \sum_{j=0}^{m} b_j c_{i-j}^*, \quad k \leqslant i \leqslant m-1, \tag{6}$$

where $c_r = 0$ if $r < 0$ and $c_s^* = 0$ if $s \geqslant 0$.

Remark that if $k = m$, the two-point Padé-type approximant is an ordinary Padé-type approximant (one-point case) [5].

3. Multivariate two-point Padé-type approximants

Without loss of generality we only write everything down for the bivariate case. Assume that f represents a function holomorphic in certain neighbourhoods V_0 and V_∞ of $(0,0)$ and (∞,∞), respectively. Let $f_0(x,y)$ and $f_\infty(x,y)$ be its respective Taylor and Laurent expansions,

$$f_0(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{i,j}x^i y^j, \tag{7}$$

$$f_\infty(x,y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c^*_{-i,-j}x^{-i} y^{-j}. \tag{8}$$

Let $Q(x,y)$ be an arbitrary polynomial of the form

$$Q(x,y) = \sum_{(d,e)\in D} b_{de}x^d y^e, \quad D \subset \mathbb{N}^2,\ b_{00} \neq 0$$

with the restriction that D satisfies the so-called inclusion property. This means that for every index point (d,e) in D, the whole rectangle of index points $[0,d] \times [0,e]$ is contained in D.

Choose finite subsets N_0 and N_∞ of $\mathbb{N}^2$ such that

$$N_0 \cap N_\infty = \emptyset \tag{9}$$

and

$$N_0 \text{ as well as } N = N_0 \cup N_\infty \text{ satisfy the inclusion property.} \tag{10}$$

Let us now define the product index set of $A \subset \mathbb{N}^2$ and $B \subset \mathbb{N}^2$ by

$$A * B = \{(i+k, j+l)\colon\ (i,j) \in A, (k,l) \in B\}$$

and impose the extra restriction

$$(N_0 \cup N_\infty) * \{(0,0),(1,0),(0,1),(1,1)\} = D. \tag{11}$$

This is illustrated in figure 1.

We introduce the standard notation $\mathbb{Z}_0^-$ for the strictly negative integer numbers, and the notation E for the subset $E = (\mathbb{Z}_0^-)^2 * D$ of $\mathbb{Z}^2$. Note that, since D satisfies the inclusion property, so does the set $E \cap \mathbb{N}^2$.

To construct bivariate two-point Padé-type approximants, we look for a bivariate polynomial $P(x,y)$ of the form

$$P(x,y) = \sum_{(i,j)\in N_0} a_{ij}x^i y^j + \sum_{(i,j)\in N_\infty} a_{ij}x^i y^j, \tag{12}$$

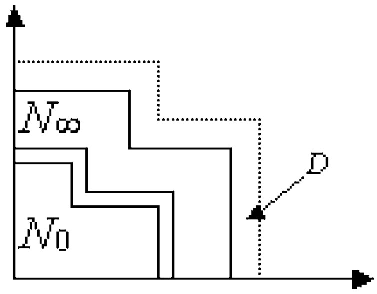

Figure 1.

satisfying

$$(f_0Q - P)(x, y) = \sum_{(i,j)\in\mathbb{N}^2\setminus N_0} d_{ij}x^i y^j, \tag{13}$$

$$(f_\infty Q - P)(x, y) = \sum_{(i,j)\in E\setminus N_\infty} d^*_{ij}x^i y^j. \tag{14}$$

The rational function $(P/Q)(x, y)$ will be called a multivariate two-point Padé-type approximant to the pair (f_0, f_∞), M2PTA for short, and we denote it by

$$(N_0, N_\infty/D)_{(f_0,f_\infty)} \quad \text{or} \quad (N_0, N_\infty/D)_f.$$

Remarks. (a) The construction of the M2PTA is similar to that of the multivariate Padé-type approximants in [2].

(b) We obtain the univariate situation as a special case by choosing for $f(x, 0)$

$$D = \{(i, 0) \in \mathbb{N}^2\colon\ 0 \leqslant i \leqslant m\},$$
$$N_0 = \{(i, 0) \in \mathbb{N}^2\colon\ 0 \leqslant i \leqslant k - 1\}, \quad 0 \leqslant k \leqslant m - 1,$$
$$N_\infty = \{(i, 0) \in \mathbb{N}^2\colon\ k \leqslant i \leqslant m - 1\},$$

while the set E equals

$$E = \{(i, 0) \in \mathbb{Z}^2\colon\ -\infty \leqslant i \leqslant m - 1\}.$$

(c) Condition (11) on N_0 and N_∞ is analogous to choosing, in the univariate case, the degree of the numerator one less than the degree of the generating denominator polynomial.

(d) Next, we deal with the following situation. Suppose that for two finite index sets $S_1 \subset \mathbb{N}^2$ and $S_2 \subset (\mathbb{Z}_0^-)^2$, the expansions $f_0(x, y)$ and $f_\infty(x, y)$ are given by

$$f_0(x, y) = \sum_{(i,j)\in S_2} c_{ij}x^i y^j + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}x^i y^j, \qquad (x, y) \in V_0,$$

$$f_\infty(x, y) = \sum_{(i,j)\in S_1} c^*_{ij}x^i y^j + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c^*_{-i,-j}x^{-i} y^{-j}, \quad (x, y) \in V_\infty.$$

If we define two other expansions

$$f_1(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}e_{ij}x^iy^j$$

with

$$e_{ij}=\begin{cases}c_{ij}-c^*_{ij}, & (i,j)\in S_1,\\ c_{ij}, & (i,j)\in\mathbb{N}^2\setminus S_1\end{cases}$$

and

$$f_2(x,y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}e^*_{ij}x^{-i}y^{-j}$$

with

$$e^*_{ij}=\begin{cases}c^*_{ij}-c_{ij}, & (i,j)\in S_2,\\ c^*_{ij}, & (i,j)\in\left[\left(\mathbb{Z}_0^-\right)^2\right]\setminus S_2,\end{cases}$$

then the M2PTA of (f_0, f_∞) is given by

$$(N_0,N_\infty/D)_{(f_0,f_\infty)}(x,y)=\sum_{(i,j)\in S_2}c_{ij}x^iy^j+\sum_{(i,j)\in S_1}c^*_{ij}x^iy^j+(N_0,N_\infty/D)_{(f_1,f_2)}(x,y).$$

This situation is studied by Draux in the univariate case [12], and is extended by Orive and Gonzalez-Vera [18] for the particular case

$$S_1=[0,\mu_1]\times[0,\mu_2]\cap\mathbb{N}^2,\qquad S_2=[-\nu_1,-1]\times[-\nu_2,-1]\cap\left(\mathbb{Z}_0^-\right)^2.$$

Theorem 1. Let the index sets N_0 and N_∞ be defined by (9)–(11). Then the numerator polynomial of the M2PTA is given by

$$P(x,y)=\sum_{(i,j)\in N_0}\left(\sum_{(d,e)\in D}b_{de}c_{i-d,j-e}\right)x^iy^j+\sum_{(i,j)\in N_\infty}\left(\sum_{(d,e)\in D}b_{de}c^*_{d-i,e-j}\right)x^iy^j.$$

Proof. We write

$$P(x,y)=\sum_{(i,j)\in N_0}a_{ij}x^iy^j+\sum_{(i,j)\in N_\infty}a_{ij}x^iy^j.$$

Using Leibniz' rule, we have

$$\begin{aligned}(f_0Q)(x,y)&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\sum_{(d,e)\in D}b_{de}c_{i-d,j-e}\right)x^iy^j\\&=\sum_{(i,j)\in N_0}\left(\sum_{(d,e)\in D}b_{de}c_{i-d,j-e}\right)x^iy^j+\sum_{(i,j)\in\mathbb{N}^2\setminus N_0}\left(\sum_{(d,e)\in D}b_{de}c_{i-d,j-e}\right)x^iy^j,\end{aligned}$$

where $c_{i-d,j-e} = 0$ if $d > i$ or $e > j$. From condition (13), we obtain

$$a_{ij} = \sum_{(d,e)\in D} b_{de} c_{i-d,j-e}, \quad (i,j) \in N_0.$$

On the other hand, if $(x,y) \in V_\infty$ we have

$$(f_\infty Q)(x,y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c^*_{i,j} x^{-i} y^{-j} \times \sum_{(d,e)\in D} b_{de} x^d y^e = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{(d,e)\in D} b_{de} c^*_{i,j} x^{d-i} y^{e-j}.$$

Taking into account the definition of the set E, we can write

$$\begin{aligned}(f_\infty Q)(x,y) &= \sum_{(i,j)\in E}\Bigl(\sum_{(d,e)\in D} b_{de} c^*_{d-i,e-j}\Bigr)x^i y^j \\ &= \sum_{(i,j)\in N_\infty}\Bigl(\sum_{(d,e)\in D} b_{de} c^*_{d-i,e-j}\Bigr)x^i y^j + \sum_{(i,j)\in E\setminus N_\infty}\Bigl(\sum_{(d,e)\in D} b_{de} c^*_{d-i,e-j}\Bigr)x^i y^j,\end{aligned}$$

with $c^*_{d-i,e-j} = 0$ if $d \geqslant i$ or $e \geqslant j$. From (14) we obtain

$$a_{ij} = \sum_{(d,e)\in D} b_{de} c^*_{d-i,e-j}, \quad (i,j) \in N_\infty. \qquad \square$$

Next, we investigate the well-known consistency property. This property asserts that, given a rational function f, the approximation procedure under consideration reconstructs the given rational function, assuming that the numerator and denominator degree of the rational approximant are large enough. We are in a position to show the extension of this property to M2PTA.

Theorem 2. Let

$$f(x,y) = \frac{\overline{P}(x,y)}{Q(x,y)} = \frac{\sum_{(i,j)\in\overline{N}} \bar{a}_{ij} x^i y^j}{\sum_{(i,j)\in D} b_{ij} x^i y^j}$$

with $b_{00} \neq 0$. For all N_0, N_∞ satisfying (9)–(11) and also $N_0 \cup N_\infty \supseteq \overline{N}$, and with $Q(x,y)$ being the generating polynomial of the M2PTA, we have

$$(N_0, N_\infty/D)_f(x,y) = f(x,y).$$

Proof. It is clear that $f(x,y)Q(x,y) - \overline{P}(x,y) = 0$. Consider the M2PTA with denominator polynomial $Q(x,y)$: $(N_0, N_\infty/D)_f(x,y) = P(x,y)/Q(x,y)$. We have that

$$\bigl(\overline{P} - P\bigr)(x,y) = \bigl(\overline{P} - f_0 Q\bigr)(x,y) + (f_0 Q - P)(x,y) = \sum_{(i,j)\in\mathbb{N}^2\setminus N_0} d_{ij} x^i y^j.$$

On the other hand, we have

$$\bigl(\overline{P} - P\bigr)(x,y) = \bigl(\overline{P} - f_\infty Q\bigr)(x,y) + (f_\infty Q - P)(x,y) = \sum_{(i,j)\in E\setminus N_\infty} d^*_{ij} x^i y^j.$$

So we can write

$$\left(\overline{P} - P\right)(x, y) = \sum_{(i,j)\in\mathbb{N}^2\setminus(N_0\cup N_\infty)} \tilde{d}_{ij} x^i y^j.$$

Since $N_0 \cup N_\infty \supseteq \overline{N}$, we can immediately conclude that $\overline{P} = P$. □

In order to illustrate this property, consider the function

$$f(x, y) = \frac{1}{(1-x)(2-y)} + \frac{1}{(2-x)(1-y)}$$

which was used in [18] to show that for the multivariate two-point Padé approximants considered by these authors the consistency property does not hold. In contrast, in the present case, let

$$N_0 = \{(0,0), (1,0)\}, \qquad N_\infty = \{(0,1), (1,1)\},$$
$$D = \{(i, j) \in \mathbb{N}^2\colon\ 0 \leqslant i, j \leqslant 2\}.$$

The sets N_0 and N_∞ satisfy all conditions of theorem 2. If we take $Q(x, y) = (1 - x)(2 - x)(1 - y)(2 - y)$ as generating polynomial, we find that

$$(N_0, N_\infty/D)_f(x, y) = \frac{4 - 3x - 3y + 2xy}{Q(x, y)} = f(x, y).$$

4. Higher-order approximants

In [5], Brezinski introduces the notion of higher order approximants by adding orthogonality conditions to the generating polynomial. In the same way, let us construct higher order approximants and introduce multivariate two-point Padé approximants (M2PA). We shall see that these are an extension of the multivariate Padé approximants (MPA) defined in [8,16]. In order to do so, we impose additional conditions on $Q(x, y)$ or rather on its coefficients $\{b_{de}\}$.

Let the finite subsets $I_0 \subset \mathbb{N}^2$ and $I_\infty \subset E$ be such that (see figures 2 and 3):

$N_0 \subset I_0$ and $N_\infty \subset I_\infty$,
I_0 satisfies the inclusion property,
I_∞ satisfies the inverse inclusion property, in other words,
if $(i, j) \in I_\infty$ then $(\mu, \nu) \in I_\infty$ for $(\mu, \nu) \in E$ and $\mu \geqslant i,\ \nu \geqslant j$,
$\#(I_0 \setminus N_0) + \#(I_\infty \setminus N_\infty) = \#(D) - 1.$

The additional conditions on $\{b_{de}\}$ are given by

$$\begin{cases} \displaystyle\sum_{(d,e)\in D} b_{de} c_{i-d,j-e} = 0, & (i, j) \in I_0 \setminus N_0, \\ \displaystyle\sum_{(d,e)\in D} b_{de} c^*_{d-i,e-j} = 0, & (i, j) \in I_\infty \setminus N_\infty, \end{cases} \tag{15}$$

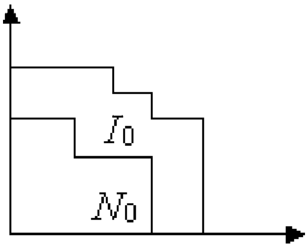

Figure 2.

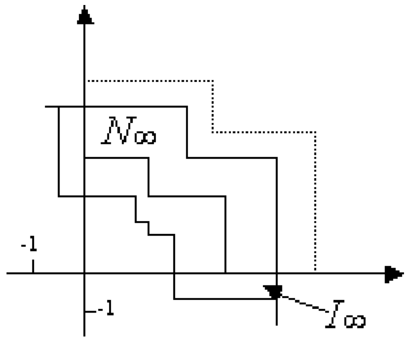

Figure 3.

where $c_{ij} = 0$ if $i < 0$ or $j < 0$ and $c^*_{ij} = 0$ if $i \leqslant 0$ or $j \leqslant 0$.

Therefore, the coefficients $\{a_{ij}\}$ are now given by

$$\begin{cases} a_{ij} = \sum\limits_{(d,e)\in D} b_{de} c_{i-d,j-e}, & (i,j) \in N_0, \\ a_{ij} = \sum\limits_{(d,e)\in D} b_{de} c^*_{d-i,e-j}, & (i,j) \in N_\infty. \end{cases} \tag{16}$$

Let $P_0(x,y)$ and $P_\infty(x,y)$ be given by

$$P_0(x,y) = \sum_{(i,j)\in N_0} a_{ij} x^i y^j, \qquad P_\infty(x,y) = \sum_{(i,j)\in N_\infty} a_{ij} x^i y^j,$$

where as before

$$P(x,y) = P_0(x,y) + P_\infty(x,y).$$

We call the rational function $P(x,y)/Q(x,y)$ the multivariate two-point Padé approximant and we denote it by

$$[N_0, N_\infty/D]^{(f_0,f_\infty)}_{(I_0,I_\infty)} \quad \text{or} \quad [N_0, N_\infty/D]^{f}_{(I_0,I_\infty)}.$$

Recall that the multivariate Padé approximant defined in [8] is denoted by $[N/D]^f_I$.

Theorem 3.

$$(f_0 Q - P)(x,y) = \sum_{(i,j)\in \mathbb{N}^2\setminus I_0} d_{ij} x^i y^j, \quad (x,y) \in V_0, \tag{17}$$

$$(f_\infty Q - P)(x,y) = \sum_{(i,j)\in E\setminus I_\infty} d^*_{ij} x^i y^j, \quad (x,y) \in V_\infty. \tag{18}$$

Proof. The result follows easily from (15) and (16). □

Corollary. Let

$$[N_0, N_\infty/D]^f_{(I_0,I_\infty)}(x, y) = \frac{P(x, y)}{Q(x, y)}.$$

If $I_\infty \setminus N_\infty = \phi$ then

$$\frac{P_0(x, y)}{Q(x, y)} = [N_0/D]^f_{I_0}(x, y).$$

Proof. The proof of this result immediately follows from the definition of multivariate Padé approximants. □

Following the same ideas as in [9] for MPA, the multivariate two-point Padé approximant introduced here can be expressed as a ratio of two determinants and computed recursively using the E-algorithm [6].

Brezinski also proved that Chisholm's [7] C-approximants in two variables can be seen as higher-order multivariate Padé-type approximants (see [4]). In this sense, by a particular choice of N_0, N_∞, I_0 and I_∞, our approximants can be viewed as an extension of the bivariate one-point C-approximants.

5. Numerical examples

In this section, we list some numerical experiments comparing our M2PTA $(N_0, N_\infty/D)_f$ with the multivariate two-point Padé approximants $((k_1, k_2)/(m_1, m_2))_f$ defined in [18]. We complement this with numerical results for the M2PA.

Example 1. Consider again the function

$$f(x, y) = \frac{1}{(1-x)(2-y)} + \frac{1}{(2-x)(1-y)} = \frac{4-3x-3y+2xy}{Q(x, y)},$$

which admits the expansions

$$f_0(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(2^{-i-1} + 2^{-j-1})x^i y^j, \quad |x| < 1,\ |y| < 1,$$

$$f_\infty(x, y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(2^{i-1} + 2^{j-1})x^{-i} y^{-j}, \quad |x| > 2,\ |y| > 2.$$

For the computation of M2PA for $f(x, y)$, we choose the index sets

$$\begin{aligned}
N_0 &= \{(0,0), (1,0)\}, \qquad N_\infty = \{(0,1), (1,1)\},\\
D &= \{(i,j) \in \mathbb{N}^2\colon\ 0 \leqslant i, j \leqslant 2\},\\
I_0 &= N_0 \cup \{(0,1), (2,0), (1,1), (0,2)\},\\
I_\infty &= N_\infty \cup \{(1,0), (1,-1), (0,0), (-1,1)\}.
\end{aligned}$$

We then obtain

$$[N_0, N_\infty/D]^f_{(I_0,I_\infty)}(x,y) = \frac{16+8xy}{16-12(x+y)-x^2+10xy-y^2-6(x^2y+xy^2)+4x^2y^2}.$$

(x, y)	$[N_0, N_\infty/D]_f(x, y)$	$f(x, y)$
(0.10D−02, 0.20D−02)	0.1002255082D+01	0.1002254132D+01
(0.10D+00, 0.10D+00)	0.1176436186D+01	0.1169590643D+01
(−0.50D+01, −0.60D+01)	0.4290262982D−01	0.4124149660D−01
(0.75D+02, 0.75D+02)	0.3703703114D−03	0.3702332469D−03
(0.50D+03, 0.40D+03)	0.1006795992D−04	0.1006785800D−04

Example 2. Let us consider the function

$$f(x,y) = \ln\left(1+\frac{1}{(1-x)(1-y)}\right)$$

which admits the expansions

$$f_0(x,y) = \ln(2) + \frac{1}{2}x + \frac{1}{2}y + \frac{3}{8}x^2 + \frac{1}{4}xy + \frac{3}{8}y^2 + \frac{7}{24}x^3 + \frac{1}{8}x^2y + \frac{1}{8}xy^2 + \frac{7}{24}y^3 + \cdots, \quad |x|<1, |y|<1,$$

$$f_\infty(x,y) = x^{-1}y^{-1} + x^{-2}y^{-1} + x^{-1}y^{-2} + x^{-3}y^{-1} + \frac{1}{2}x^{-2}y^{-2} + x^{-1}y^{-3} + x^{-4}y^{-1} + x^{-1}y^{-4} + \cdots, \quad |x|>1,\ |y|>1.$$

First consider, for the construction of the M2PTA,

$$N_0 = \{(0,0),(1,0)\}, \qquad N_\infty = \{(0,1),(1,1)\}$$
$$D = \{(i,j)\in\mathbb{N}^2:\ 0\leqslant i,j\leqslant 2\}.$$

If we choose

$$Q(x,y) = 2-3(x+y)+x^2+5xy+y^2-2(x^2y+xy^2)+x^2y^2,$$

we obtain

$$(N_0, N_\infty/D)_f(x,y) = \frac{1.386-1.079x-y+xy}{2-3(x+y)+x^2+5xy+y^2-2(x^2y+xy^2)+x^2y^2}$$

and

$$((1,1)/(2,2))_f(x,y) = \frac{1.386+xy}{2-3(x+y)+x^2+5xy+y^2-2(x^2y+xy^2)+x^2y^2}.$$

Therefore

(x, y)	$(N_0, N_\infty/D)_f(x, y)$	$((1,1)/(2,2))_f(x, y)$	$f(x, y)$
(−0.50D−03, −0.50D−03)	0.6926276202D+00	0.6921085389D+00	0.6926474305D+00
(0.10D+00, 0.20D+00)	0.8869106994D+00	0.1135573615D+01	0.8708283578D+00
(0.30D+00, 0.30D+00)	0.1167596084D+01	0.2022044056D+01	0.1112126008D+01
(−0.40D+00, −0.60D+00)	0.3662465531D+00	0.2240815648D+00	0.3690974639D+00
(0.25D+01, 0.30D+01)	0.2656408747D+00	0.7405245304D+00	0.2876820725D+00
(0.10D+02, 0.10D+02)	0.1213367644D−01	0.1526442252D−01	0.1227009259D−01
(0.50D+03, 0.50D+03)	0.4015397591D−05	0.4032166808D−05	0.4016040064D−05

To compute the M2PA for $f(x, y)$, we choose the index sets

$$I_0 = N_0 \cup \{(0,1), (2,0), (1,1), (0,2)\},$$
$$I_\infty = N_\infty \cup \{(1,0), (1,-1), (0,0), (-1,1)\},$$

which yields

$$[N_0, N_\infty/D]^f_{(I_0,I_\infty)}(x, y) = \frac{24\ln(2) + (12 - 18\ln(2))x + 3y + 8xy}{24 - 18x - 12y + 9xy - 3y^2 - 8x^2y - 5xy^2 + 8x^2y^2}.$$

(x, y)	$[N_0, N_\infty/D]_f(x, y)$	$f(x, y)$
(0.40D−02, 0.30D−02)	0.6965790391D+00	0.6966595927D+00
(0.40D−01, 0.60D−01)	0.7450661659D+00	0.7458136452D+00
(0.10D+00, 0.10D+00)	0.8061587811D+00	0.8040478766D+00
(0.75D+02, 0.75D+02)	0.1825231011D−03	0.1825983754D−03
(0.50D+03, 0.40D+03)	0.5021868219D−05	0.5022563868D−05

6. Error formulas

Integral expressions for the error involved in multivariate two-point Padé-type approximation, are based on the multivariate versions of Cauchy's and Laurent's theorems (see, e.g., [19,20]). In the sequel, we say that a function f belongs to the polydisc algebra $\mathcal{A}(\mathcal{P})$ if f is holomorphic on $\mathcal{P}$ and continuous on its closure.

We restrict our attention to functions being holomorphic in a particular class of domains containing $(0, 0)$ and (∞, ∞), the so-called polydisc-type domains (see [19]). These are domains of the form $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_\infty$, with

$$\mathcal{P}_0 = \{(x, y) \in \mathbb{C}^2\colon |x| < r_1, |y| < r_2\} \quad \text{and} \qquad (19)$$
$$\mathcal{P}_\infty = \{(x, y) \in \mathbb{C}^2\colon |x| > R_1, |y| > R_2\}$$

with $R_i \geqslant r_i$, $i = 1, 2$. Let us also consider the respective distinguished boundaries of the above domains

$$S_0 = \{(x, y) \in \mathbb{C}^2\colon |x| = r_1, |y| = r_2\} \quad \text{and} \qquad (20)$$
$$S_\infty = \{(x, y) \in \mathbb{C}^2\colon |x| = R_1, |y| = R_2\}.$$

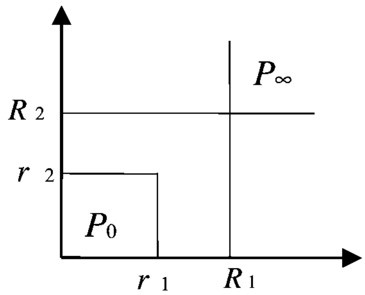

Figure 4.

If $f \in \mathcal{A}(\mathcal{P})$, then from Cauchy's and Laurent's theorems, the following integral representation for the numerator P holds where $w = (u, v)$:

$$P(z) = \frac{1}{(2\pi \mathrm{i})^2}\left\{ \int_{S_0} \frac{(f_0 Q)(w)}{uv}\left(\sum_{\gamma\in N_0}\left(\frac{z}{w}\right)^{\gamma}\right) \mathrm{d}u\,\mathrm{d}v + \int_{S_\infty} \frac{(f_\infty Q)(w)}{uv}\left(\sum_{\gamma\in N_\infty}\left(\frac{z}{w}\right)^{\gamma}\right) \mathrm{d}u\,\mathrm{d}v\right\}. \tag{21}$$

On the other hand, since $(f_0 Q) \in \mathcal{A}(\mathcal{P})$, by Cauchy's theorem we have for $z \in \mathcal{P}_0$:

$$(f_0 Q)(z) = \frac{1}{(2\pi \mathrm{i})^2}\int_{S_0} \frac{(f_0 Q)(w)}{(u-x)(v-y)}\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{(2\pi \mathrm{i})^2}\int_{S_0} \frac{(f_0 Q)(w)}{uv}\left(\sum_{\gamma\in \mathbb{N}^2}\left(\frac{z}{w}\right)^{\gamma}\right)\mathrm{d}u\,\mathrm{d}v. \tag{22}$$

In the same way, since $(f_\infty Q) \in \mathcal{A}(\mathcal{K})$, with $\mathcal{K} = \{z \in \mathbb{C}^2\colon R_i < |z_i| < R_i',\ i = 1, 2\}$ (R' arbitrarily large), we can write

$$(f_\infty Q)(z) = \frac{1}{(2\pi \mathrm{i})^2}\int_{S_\infty} \frac{(f_\infty Q)(w)}{(u-x)(v-y)}\left(\sum_{\gamma\in E}\left(\frac{z}{w}\right)^{\gamma}\right) \mathrm{d}u\,\mathrm{d}v. \tag{23}$$

As a consequence the following result holds.

Theorem 4. If $f \in \mathcal{A}(\mathcal{P})$, where $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_\infty$, with $\mathcal{P}_0$ and $\mathcal{P}_\infty$ as in (19) and $w = (u, v)$, the following integral representations for the error hold: for $z \in \mathcal{P}_0$,

$$\begin{aligned}\mathcal{E}(N_0, N_\infty/D)_f(z) &= (f - (N_0, N_\infty/D)_f)(z)\\ &= \frac{1}{(2\pi \mathrm{i})^2 Q(z)}\left\{ \int_{S_0} \frac{(f_0 Q)(w)}{uv}\left(\sum_{\gamma\in \mathbb{N}^2\setminus N_0}\left(\frac{z}{w}\right)^{\gamma}\right) \mathrm{d}u\,\mathrm{d}v - \int_{S_\infty} \frac{(f_\infty Q)(w)}{uv}\left(\sum_{\gamma\in N_\infty}\left(\frac{z}{w}\right)^{\gamma}\right) \mathrm{d}u\,\mathrm{d}v\right\},\end{aligned} \tag{24}$$

and for $z \in \mathcal{P}_\infty$,

$$\begin{aligned}\mathcal{E}(N_0, N_\infty/D)_f(z) &= (f - (N_0, N_\infty/D)_f)(z) \\ &= \frac{1}{(2\pi \mathrm{i})^2 Q(z)} \Biggl\{ -\int_{S_0} \frac{(f_0 Q)(w)}{uv} \Biggl(\sum_{\gamma \in N_0} \left(\frac{z}{w}\right)^\gamma \Biggr) \mathrm{d}u\, \mathrm{d}v \\ &\quad + \int_{S_\infty} \frac{(f_\infty Q)(w)}{(u-x)(v-y)} \Biggl(\sum_{\gamma \in E \setminus N_\infty} \left(\frac{z}{w}\right)^\gamma \Biggr) \mathrm{d}u\, \mathrm{d}v \Biggr\}. \qquad (25)\end{aligned}$$

Note that for the special rectangular case treated in [18], the expressions above take a simpler form.

7. Convergence

Throughout this section, let us assume that f is holomorphic in a domain $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_\infty$ as above. In order to study the convergence of sequences of multivariate two-point Padé type approximants, we introduce the sequences of lattices

$$D(m) = [0, n_1(m)] \times [0, n_2(m)], \qquad N(m) = [0, n_1(m) - 1] \times [0, n_2(m) - 1],$$
$$N_0(m) \cup N_\infty(m) = N(m), \qquad N_0(m) \cap N_\infty(m) = \emptyset,$$

satisfying the initial assumptions

$$\lim_{m\to\infty} n_1(m) = \lim_{m\to\infty} n_2(m) = \lim_{m\to\infty} k(m) = \lim_{m\to\infty} h(m) = \infty, \qquad (26)$$

where $k(m) + h(m) \leqslant n_1(m) + n_2(m) - 2$ and $k(m) - 1$ and $h(m) - 1$ are the respective sides of the largest isosceles triangles $T_0(m)$ and $T_\infty(m)$ inscribed in $N_0(m)$ and $N_\infty(m)$ (see figure 5).

We suppose in addition that the zeros of the sequence of polynomials $\{Q_m\}$ lie outside $\mathcal{P}$. Suitable choices for $\{Q_m\}$ ensure that the error decays uniformly (and if possible geometrically) to zero. Let us solve this problem for some particular cases.

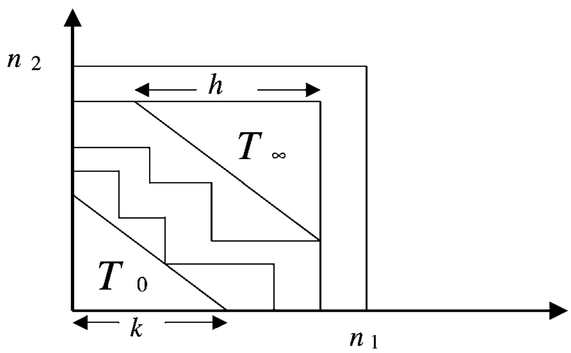

Figure 5.

Example 1. Let us suppose that $f \in H(\mathcal{P}_0 \cup \mathcal{P}_\infty)$ with $r_i = R_i$ for $i = 1, 2$. Then the sequence of denominators

$$Q_m(x, y) = \left(x^{n_1} - r_1^{n_1}\right)\left(y^{n_2} - r_2^{n_2}\right) \tag{27}$$

with $n_1 = n_1(m)$ and $n_2 = n_2(m)$ will provide the desired convergence of the approximants. Indeed, we can apply the error formulas to

$$\mathcal{P}_0' = \left\{(x, y) \in \mathbb{C}^2\colon\ |x| < r_1',\ |y| < r_2'\right\} \quad \text{and}$$
$$\mathcal{P}_\infty' = \left\{(x, y) \in \mathbb{C}^2\colon\ |x| > R_1', |y| > R_2'\right\},$$

with $r_i' = r_i - \delta_i$, $R_i' = r_i + \varepsilon_i$ and δ_i and ε_i arbitrarily small. Under these conditions, the following result can be proved.

Theorem 5. Let f be holomorphic in the domain

$$\mathcal{A} = \left\{(x, y) \in \mathbb{C}^2\colon\ |x| < r_1, |y| < r_2\right\} \cup \left\{(x, y) \in \mathbb{C}^2\colon\ |x| > r_1, |y| > r_2\right\}$$

and let the sequence of multivariate two-point Padé-type approximants $\{(N_0(m), N_\infty(m)/D(m))_f\}_{m\in\mathbb{N}}$ satisfy (26) with the denominators given by (27). Then, this sequence converges uniformly to f in compact subsets of $\mathcal{A}$. Moreover, if

$$\lim_{m\to\infty} \frac{k(m)}{m} = s > 0 \quad \text{and} \quad \lim_{m\to\infty} \frac{h(m)}{m} = t > 0,$$

then the convergence is geometrical, with respective asymptotic degrees of convergence given by

$$\mathrm{O}\left\{\max\left(\frac{|x|}{r_1}, \frac{|y|}{r_2}\right)^s\right\} \quad \text{for } z \in \mathcal{P}_0, \qquad \mathrm{O}\left\{\max\left(\frac{r_1}{|x|}, \frac{r_2}{|y|}\right)^t\right\} \quad \text{for } z \in \mathcal{P}_\infty.$$

Observe that the restriction on the radius is compensated by a total freedom in the choice of the lattices N_0 and N_∞. Otherwise, one must impose some restriction on the lattices of contact. In this respect, recall that in [19], the sequence of denominators

$$Q_m(x, y) = \left(x^k - r_1^k\right)\left(x^h - R_1^h\right)\left(y^k - r_2^k\right)\left(y^h - R_2^h\right)$$

where $k + h = m$, delivers optimal results in the case of lattices as in figure 6. The following example extends this result.

Example 2. Suppose that for each $m \in \mathbb{N}$, there exist six nonnegative integers $k_i = k_i(m)$, $k_i' = k_i'(m)$ and $k_i'' = k_i''(m)$ with $i = 1, 2$, such that

$$\begin{aligned} &k_i'' \leqslant k_i \leqslant k_i', \quad i = 1, 2,\\ &[0, k_1'' - 1] \times [0, k_2'' - 1] \subseteq N_0(m) \subseteq [0, k_1' - 1] \times [0, k_2' - 1],\\ &[k_1, n_1 - 1] \times [k_2, n_2 - 1] \subseteq N_\infty(m) \subseteq [k_1'', n_1 - 1] \times [k_2'', n_2 - 1]. \end{aligned} \tag{28}$$

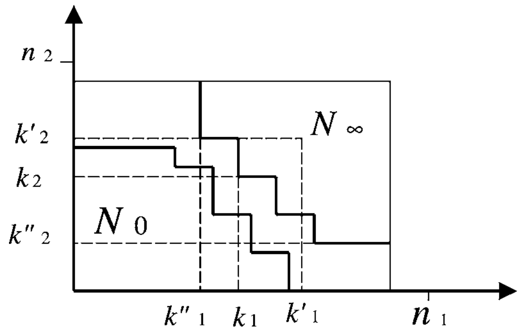

Figure 6.

Then, for the sequence of denominators

$$Q_m(x, y) = \left(x^{k_1} - r_1^{k_1}\right)\left(x^{h_1} - R_1^{h_1}\right)\left(y^{k_2} - r_2^{k_2}\right)\left(y^{h_2} - R_2^{h_2}\right), \tag{29}$$

with $h_i = n_i - k_i$, $i = 1, 2$, the following result holds.

Theorem 6. If (28) is satisfied and

$$\lim_{m\to\infty} \frac{k_i(m)}{m} = \lim_{m\to\infty} \frac{k'_i(m)}{m} = \lim_{m\to\infty} \frac{k''_i(m)}{m} = s_i, \quad 0 < s_i < 1,\ i = 1, 2,$$

then the sequence $\{(N_0(m), N_\infty(m)/D(m))_f\}_{m\in\mathbb{N}}$ of multivariate two-point Padé-type approximants with the denominators given by (29), converges geometrically to f in compact subsets of $\mathcal{P}_0 \cup \mathcal{P}_\infty$. The respective asymptotic degrees of convergence are

$$\mathrm{O}\left\{\max\left(\left(\frac{|x|}{r_1}\right)^{s_1}, \left(\frac{|y|}{r_2}\right)^{s_2}\right)\right\} \qquad \text{for } z \in \mathcal{P}_0,$$
$$\mathrm{O}\left\{\max\left(\left(\frac{R_1}{|x|}\right)^{1-s_1}, \left(\frac{R_2}{|y|}\right)^{1-s_2}\right)\right\} \qquad \text{for } z \in \mathcal{P}_\infty.$$

References

[1] J. Abouir, Erreur et convergence dans l'approximation rationnelle à plusieurs variables, Thesis, University of La Laguna, Spain (1994).

[2] J. Abouir and A. Cuyt, Multivariate partial Newton–Padé and Newton–Padé-type approximants, J. Approx. Theory 72 (1993) 301–316.

[3] J. Abouir, A. Cuyt, R. Orive and P. Gonzalez-Vera, On the convergence of general order multivariate Padé-type approximants, J. Approx. Theory 86 (1996) 216–128.

[4] C. Brezinski, Padé-type pproximants for double power series, J. Indian Math. Soc. 42 (1979) 267–282.

[5] C. Brezinski, *Padé-Type Approximants and General Orthogonal Polynomials* (Birkäuser, Basel, 1980).

[6] C. Brezinski, A general extrapolation algorithm, Numer. Math. 35 (1980) 175–187.

[7] J.S.R. Chisholm, N-variable rational approximants, in: *Padé and Rational Approximation*, eds. E.B. Saff and R.S. Varga (Academic Press, New York, 1977) pp. 23–42.

[8] A. Cuyt, General order multivariate rational Hermite interpolants, Habilitation, University of Antwerp (1986).

[9] A. Cuyt, A recursive computation scheme for multivariate rational interpolants, SIAM J. Numer. Anal. 24 (1987) 228–238.

[10] C. Diaz Mendoza, P. Gonzalez-Vera and R. Orive, On the convergence of two-point Padé approximants, Numer. Math. 72 (1996) 295–312.

[11] A. Draux, *Approximants de Type-Padé et Padé en Deux Points*, Publication A.N.O., Vol. 110 (Lille, 1983).

[12] A. Draux, On two-point Padé-type and two-point Padé approximants.

[13] P. Gonzalez-Vera and L. Casasus, Two-point Padé-type approximants for Stieljes functions, in: *Polynômes Orthogonaux et Applications, Proceedings*, Bar-le-Duc, 1984, eds. C. Brezinski et al., Lecture Notes in Mathematics, Vol. 1171 (Springer, Berlin, 1985) pp. 408–418.

[14] T. Havie, Generalized Neville type extrapolation schemes, BIT 19 (1979) 204–213.

[15] W.B. Jones, O. Njastad and W.J. Thron, Two-point Padé expansions for a family of analytic functions, J. Comput. Appl. Math. 9 (1983) 105–123.

[16] D. Levin, General order Padé-type rational approximants defined from double power series, J. Inst. Math. Appl. 18 (1976) 1–8.

[17] J.H. McCabe, A formal extension of the Padé table to include two-point Padé quotients, J. Inst. Math. Appl. 15 (1975) 363–376.

[18] R. Orive and P. Gonzalez-Vera, On a family of multivariate rational approximants, Appl. Numer. Math. 9 (1992) 493–510.

[19] R. Orive and P. Gonzalez-Vera, On the convergence of bivariate two-point Padé-type approximants, Appl. Numer. Math. 10 (1992) 497–508.

[20] R.M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables* (Springer, New York, 1986).