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On the Convergence of General Order Multivariate Padé-Type Approximants

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In previous papers the convergence of sequences of "rectangular" multivariate Padé-type approximants was studied. In other publications definitions of "triangular" multivariate Padé-type approximants were given. We extend these results to the general order definition where the choice of the denominator polynomial is completely free. Also we develop convergence theorems and we distinguish between results obtained in polydiscs and in multivariate balls. The numerical examples section illustrates this difference and compares the obtained results with the approximation power of general order multivariate Padé approximants.

1. GENERAL ORDER MULTIVARIATE PADÉ-TYPE APPROXIMANTS

In the past 15 years different definitions were given for the notion of multivariate Padé-type approximant. We refer for instance to [Brez79, Ario87, Kida89, Sabl83, Beno91]. In [AbCu93] a general order definition was introduced that contained all the previous ones as special cases and was inspired on the definition of a general order multivariate Padé approximant as given in [Cuyt86]. The advantage of using Padé-type approximants instead of Padé approximants is that information on the poles of the given function can be used in order to obtain better numerical

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behaviour of the approximant. For the paper to be self-contained we briefly recall the general order multivariate definition.

Without loss of generality we write everything down for the bivariate case. Let the function $f(z)$ with $z=(x, y) \in \mathbb{C}^2$ be given by its Taylor series expansion

$$f(z)=\sum_{\gamma \in \mathbb{N}^2} c_\gamma z^\gamma \tag{1}$$

where we use the standard multi-index notation

$$\gamma=(\gamma_1, \gamma_2) \qquad z^\gamma = x^{\gamma_1} y^{\gamma_2}$$

and let the polynomial containing the pole information of f be given by

$$q(z)=\sum_{\beta \in D} b_\beta z^\beta \qquad b_{00} \neq 0$$

with D a finite subset of $\mathbb{N}^2$. Then it is straightforward that the polynomial $p(z)$ defined by

$$p(z)=\sum_{\alpha \in N} a_\alpha z^\alpha \qquad N \subset \mathbb{N}^2$$

with

$$a_\alpha = \sum_{\beta \in D} c_{\alpha-\beta} b_\beta$$

satisfies

$$(fq-p)(z)=\sum_{\gamma \in \mathbb{N}^2 \backslash N} e_\gamma z^\gamma \tag{2}$$

Here we make use of the convention that $c_\gamma = 0$ if $\gamma_1 < 0$ or $\gamma_2 < 0$. As usual in general order multivariate Padé approximation, we assume that N satisfies the so-called rectangle rule or inclusion property, meaning that when an index point belongs to N, then the rectangular subset of index points emanating from the origin with the given point as its furthermost corner, is contained in N. This restriction on N is a natural translation of the univariate accuracy-through-order condition for the remainder series $fq-p$. We call the rational function $p(z)/q(z)$ the general order multivariate Padé-type approximant to f and denote it by $(N/D)_f$. For more details on the special cases covered by this general definition we refer to [AbCu93].

For the definition introduced by Brezinski in [Brez79], convergence results can be found in [OrGo91] and [Dara89] with a new proof of

Daras' theorem in [Ruda94]. In those papers only rectangular choices for the index sets N and D are treated, while we now aim at obtaining general results.

Most other definitions are based on triangular-like choices for N and D and deal with homogeneous subexpressions in the series expansion of f by writing

$$f(z) = \sum_{i=0}^{\infty} \left(\sum_{|\gamma| = i} c_\gamma z^\gamma \right)$$

where $|\gamma| = \gamma_1 + \gamma_2$.

2. Error Formula and Convergence in Polydiscs

Throughout this section we denote by

$$B(0; r_1, r_2) = \{z \in \mathbb{C}^2 : |x| < r_1, |y| < r_2\}$$

the polydisc with polyradius (r_1, r_2) around the origin, and we consider functions holomorphic in the polydisc and continuous on its boundary. Then we can use Cauchy's integral formula on the polydisc [Rang86] for the general order multivariate Padé-type approximant $(N/D)_f$ to obtain

$$(fq - p)(z) = \frac{1}{(2\pi i)^2} \sum_{\gamma \in \mathbb{N}^2 \setminus N} z^\gamma \int_{|u| = r_1} \int_{|v| = r_2} \frac{(fq)(w)}{u^{\gamma_1 + 1} v^{\gamma_2 + 1}} \, dw \tag{3}$$

where we write $w = (u, v)$ for a point on the boundary. This representation together with the fact that N satisfies the inclusion property and hence has some staircase form as shown in Fig. 1, enables us to obtain a more detailed error formula for $f - (N/D)_f$.

Let

$$n_1 = \max\{\alpha_1 \mid (\alpha_1, \alpha_2) \in N\}$$

$$n_2 = \max\{\alpha_2 \mid (\alpha_1, \alpha_2) \in N\}$$

so that $\mathbb{N}^2 \setminus N$ can be subdivided in several parts like in [AbCu90], namely

$$V = \{(i, j) \mid 0 \leqslant i \leqslant n_1, n_2 + 1 \leqslant j\}$$

$$H = \{(i, j) \mid 0 \leqslant j \leqslant n_2, n_1 + 1 \leqslant i\}$$

$$M = \{(i, j) \mid n_1 + 1 \leqslant i, n_2 + 1 \leqslant j\}$$

$$T = [0, n_1] \times [0, n_2] \setminus N.$$

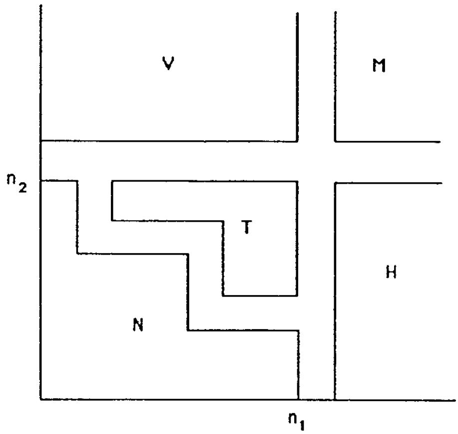

FIGURE 1

Then we treat the remainder series (2) as a sum of four sep
tions, namely

$$\sum_{\gamma\in H} e_\gamma z^\gamma = \sum_{i=n_1+1}^{\infty}\sum_{j=0}^{n_2} e_{ij}x^i y^i$$

$$= \frac{1}{(2\pi i)^2}\int_{|u|=r_1}\int_{|v|=r_2}\frac{(fq)(u,v)}{(u-x)(v-y)}\,(x/u)^{n_1+1}$$

$$\times\,(1-(y/v)^{n_2+1})\,du\,dv$$

$$\sum_{\gamma\in V} e_\gamma z^\gamma = \sum_{j=n_2+1}^{\infty}\sum_{i=0}^{n_1} e_{ij}x^i y^j$$

$$= \frac{1}{(2\pi i)^2}\int_{|u|=r_1}\int_{|v|=r_2}\frac{(fq)(u,v)}{(u-x)(v-y)}\,(y/v)^{n_2+1}$$

$$\times\,(1-(x/u)^{n_1+1})\,du\,dv$$

$$\sum_{\gamma\in M} e_\gamma z^\gamma = \sum_{i=n_1+1}^{\infty}\sum_{j=n_2+1}^{\infty} e_{ij}x^i y^j$$

$$= \frac{1}{(2\pi i)^2}\int_{|u|=r_1}\int_{|v|=r_2}\frac{(fq)(u,v)}{(u-x)(v-y)}\,(x/u)^{n_1+1}\,(y/$$

and

$$\sum_{\gamma\in T} e_\gamma z^\gamma = \frac{1}{(2\pi i)^2}\sum_{(i,j)\in[0,n_1]\times[0,n_2]\setminus N} x^i y^j \int_{|u|=r_1}\int_{|v|=r_2}\frac{(fq}{u^{i+}}$$

In this way we obtain an error formula containing only a finite number of terms analogous to the univariate error formula:

$$(f-(N/D)_f)(z)=\frac{1}{(2\pi i)^2\,q(z)}\int_{|u|=r_1}\int_{|v|=r_2}\frac{(fq)(u,v)}{(u-x)(v-y)}$$

$$\times\left[\left(\frac{x}{u}\right)^{n_1+1}+\left(\frac{y}{v}\right)^{n_2+1}-\left(\frac{x}{u}\right)^{n_1+1}\left(\frac{y}{v}\right)^{n_2+1}\right]du\,dv$$

$$+\frac{1}{(2\pi i)^2\,q(z)}\sum_{(i,j)\in[0,n_1]\times[0,n_2]\backslash N}x^iy^j$$

$$\times\int_{|u|=r_1}\int_{|v|=r_2}\frac{(fq)(u,v)}{u^{i+1}v^{j+1}}\,du\,dv$$

Furthermore let

$$s_N=\max\{k\mid\forall i,\ 0\leqslant i\leqslant k\colon (i,k-i)\in N\}$$

denote the size of the largest isosceles triangle that can be inscribed in N with top in $(0, 0)$ and base along the antidiagonal. It is the largest value for which the index set

$$S_N=\{\alpha\in\mathbb{N}^2\mid\alpha=(\alpha_1,\alpha_2),\ \alpha_1+\alpha_2\leqslant s_N\}$$

is entirely contained in N. For a similar construction see [Cuyt90]. Then we can prove the following.

THEOREM 1. *Let $f(z)$ be a function holomorphic on the polydisc $B=B(0;r_1,r_2)$ and continuous on its boundary, and let N_m and D_m be two sequences of index sets satisfying the following conditions*:

(a) $\liminf_{m\to\infty} s_{N_m}=\infty$

(b) *the sequence of polynomials $q_m(z)=\sum_{\beta\in D_m}b_\beta z^\beta$ is such that there exist strictly positive constants C_1 and C_2 for which*

$$\sup_{|u|=r_1,\ |v|=r_2}|q_m(u,v)|\leqslant C_1\qquad m=0,1,2,...$$

$$\inf_{(x,y)\in B}|q_m(x,y)|\geqslant C_2\qquad m=0,1,2,...$$

Then the sequence of general order multivariate Padé-type approximants converges to f uniformly on compact subsets of B. If $\liminf_{m\to\infty} s_{N_m}/m>0$ *then the rate of convergence is geometrical.*

Proof. From (3) we have for $z=(x, y)\in B$

$$
\begin{aligned}
&|(f-(N_m/D_m)_f)(z)| \\
&\quad \leqslant C(f; r_1, r_2)\,\frac{\sup_{|u|=r_1,\,|v|=r_2} |q_m(w)|}{\inf_{(x,y)\in B} |q_m(z)|} \sum_{\gamma\in\mathbb{N}^2\setminus N_m} (|x|/r_1)^{\gamma_1}\,(|y|/r_2)^{\gamma_2}
\end{aligned}
$$

Now consider the largest cube C_{N_m} that is contained in the index set S_{N_m} with side $c_{N_m}=\lfloor s_{N_m}/2\rfloor$. Then we can write in a fixed compact subset of B

$$
\begin{aligned}
|(f-(N_m/D_m)_f)(z)| &\leqslant C(f; r_1, r_2)\,\frac{C_1}{C_2} \sum_{\gamma\in\mathbb{N}^2\setminus C_{N_m}} (|x|/r_1)^{\gamma_1}\,(|y|/r_2)^{\gamma_2} \\
&\leqslant C(f; r_1, r_2)\,\frac{C_1}{C_2}\,[(|x|/r_1)^{c_{N_m}} + (|y|/r_2)^{c_{N_m}} \\
&\quad -(|x|/r_1)^{c_{N_m}}\,(|y|/r_2)^{c_{N_m}}]
\end{aligned}
$$

Since $\liminf_{m\to\infty} s_{N_m}=\infty$ we obtain the uniform convergence of the sequence $(N_m/D_m)_f$ in compact subsets of B. As for the geometrical convergence, if $\liminf_{m\to\infty} s_{N_m}/m=t>0$ then we have

$$
\limsup_{m\to\infty} |(f-(N_m/D_m)_f)(z)|^{1/m} \leqslant \max(|x|/r_1, |y|/r_2)^{t/2} < 1
$$

which completes the proof. ▮

The conditions on $q_m(x, y)$ can be weakened to for instance

$$
\lim_{m\to\infty} \Big[\sup_{|u|=r_1,\,|v|=r_2} |q_m(u, v)|\Big]^{1/m} = 1
$$

$$
\lim_{m\to\infty} \Big[\inf_{(x,y)\in B} |q_m(x, y)|\Big]^{1/m} = 1
$$

and a similar remark applies to the next theorem.

3. Error Formula and Convergence in Balls

In this section we concentrate on functions holomorphic in a ball $B(0, R)=\{z\in\mathbb{C}^2\colon \|z\|<R\}$ and continuous on its boundary. The boundary of the ball is the sphere $S(0, R)=\{z\in\mathbb{C}^2\colon \|z\|=R\}$. Cauchy's integral formula states that for any $z\in B$ [Rudi80]

$$
f(z) = C\int_S \frac{f(w)}{(R^2-\langle z, w\rangle)^2}\,d\sigma(w)
$$

where $\langle z, w\rangle$ denotes the usual inner product, the constant C depends only on the dimension of the space (here the dimension is two because everything is detailed for the bivariate case) and σ is the rotation-invariant positive Borel measure on $S=S(0, R)$ which can be normalized so that $C=1$. The Taylor coefficients c_γ of f admit the integral representation

$$c_\gamma=\frac{(1+|\gamma|)!}{\gamma!}\frac{1}{R^{2(2+|\gamma|)}}\int_S f(w)\,\bar{w}^\gamma\,d\sigma(w)$$

where $\bar{w}$ is the complex conjugate of w and $\gamma!=\gamma_1!\,\gamma_2!$. From the expression (2) for the remainder series $fq-p$ we obtain the error formula

$$(f-(N/D)_f)(z)=\frac{1}{R^4q(z)}\int_S (fq)(w)\sum_{\gamma\in\mathbb{N}^2\backslash N}\frac{(1+|\gamma|)!}{\gamma!}\frac{1}{R^{2|\gamma|}}z^\gamma\bar{w}^\gamma\,d\sigma(w) \tag{4}$$

THEOREM 2. *Let $f(z)$ be a function holomorphic on the ball $B=B(0, R)$ and continuous on its boundary, and let N_m and D_m be two sequences of index sets satisfying the following conditions*:

(a) $\liminf_{m\to\infty} s_{N_m}=\infty$

(b) *the sequence of polynomials $q_m(z)=\sum_{\beta\in D_m} b_\beta z^\beta$ is such that there exist strictly positive constants C_1 and C_2 for which*

$$\sup_{\|w\|=R}|q_m(w)|\leqslant C_1 \qquad m=0, 1, 2, ...$$

$$\inf_{z\in B}|q_m(z)|\geqslant C_2 \qquad m=0, 1, 2, ...$$

Then the sequence of general order multivariate Padé-type approximants converges to f uniformly on compact subsets of B. If $\liminf_{m\to\infty} s_{N_m}/m>0$ *then the rate of convergence is geometrical.*

Proof. From (4) we obtain for $z\in B$, with the notations $|z|=(|x|, |y|)$ and $|w|=(|u|, |v|)$:

$$|(f-(N_m/D_m)_f)(z)|\leqslant\frac{1}{R^4\,|q_m(z)|}\int_S |(fq_m)(w)|$$

$$\times\sum_{\gamma\in\mathbb{N}^2\backslash N_m}\frac{(1+|\gamma|)!}{\gamma!}\frac{1}{R^{2|\gamma|}}|z|^\gamma\,|w|^\gamma\,d\sigma(w)$$

Since the index set $S_{N_m} \subset N_m$ we can write

$$
\begin{aligned}
&|(f-(N_m/D_m)_f)(z)| \\
&\quad \leqslant \frac{1}{R^4\, |q_m(z)|} \int_S |(fq_m)(w)| \sum_{\gamma \in \mathbb{N}^2 \setminus S_{N_m}} \frac{(1+|\gamma|)!}{\gamma!} \frac{1}{R^{2\,|\gamma|}} |z|^\gamma\, |w|^\gamma\, d\sigma(w) \\
&\quad \leqslant C(f, R) \frac{C_1}{|q_m(z)|} \max_{w \in S} \sum_{i = s_{N_m}+1}^{\infty} \sum_{|\gamma| = i} \frac{(1+|\gamma|)!}{\gamma!} \frac{1}{R^{2\,|\gamma|}} |z|^\gamma\, |w|^\gamma \\
&\quad = C(f, R) \frac{C_1}{|q_m(z)|} \max_{w \in S} \left[\frac{(\langle |z|, |w| \rangle / R^2)^{s_{N_m}+2}}{(1-\langle |z|, |w| \rangle / R^2)^2} \right. \\
&\qquad \left. + (s_{N_m}+2) \frac{(\langle |z|, |w| \rangle / R^2)^{s_{N_m}+1}}{(1-\langle |z|, |w| \rangle / R^2)} \right]
\end{aligned}
$$

By the inequality of Cauchy–Schwarz we have

$$
|\langle |z|, |w| \rangle| \leqslant \|z\| \cdot \|w\| = R\, \|z\|
$$

and consequently for z in a fixed compact subset of B

$$
|(f-(N_m/D_m)_f)(z)| \leqslant \frac{C_1\, C(f, R)}{C_2} \left[\frac{(\|z\|/R)^{s_{N_m}+2}}{(1-\|z\|/R)^2} + (s_{N_m}+2) \frac{(\|z\|/R)^{s_{N_m}+1}}{(1-\|z\|/R)} \right]
$$

which finishes the proof. If $\liminf_{m \to \infty} s_{N_m}/m = t > 0$ then

$$
\limsup_{m \to \infty} |(f-(N_m/D_m)_f)(z)|^{1/m} \leqslant (\|z\|/R)^t < 1
$$

and the rate of convergence is geometrical. ∎

4. Discussion of Special Cases

4.1. *Polydiscs*

A first special case is again the definition in [Brez79] where N and D are chosen to be rectangles. In this case the index set T is empty and the error formula simplifies to

$$
\begin{aligned}
(f-(N/D)_f)(z) = {} & \frac{1}{(2\pi i)^2\, q(z)} \int_{|u| = r_1} \int_{|v| = r_2} \frac{(fq)(u, v)}{(u-x)(v-y)} \\
& \times \left[\left(\frac{x}{u}\right)^{n_1+1} + \left(\frac{y}{v}\right)^{n_2+1} - \left(\frac{x}{u}\right)^{n_1+1} \left(\frac{y}{v}\right)^{n_2+1} \right] du\, dv
\end{aligned}
$$

When the set N is triangular as in [Ario87], the set T is also triangular and the theorem proves the convergence for appropriate sequences $q_m(z)$. When the set N is band-structured as in [Kida89], the condition for $q_m(z)$ to be bounded away from zero in the polydisc and its boundary should be replaced by a similar condition on the polydisc excluding a thin set of zeros. The convergence results then also apply to the polydisc excluding that thin set.

In any case it is not difficult to construct sequences $q_m(z)$ that satisfy the conditions of the theorem. Consider for instance

$$q_m^{(1)}(z) = \rho_1^m p_2^m - x^m y^m \qquad \rho_i > r_i$$

or

$$q_m^{(2)}(z) = (\rho_1^m - x^m)(\rho_2^m - y^m) \qquad \rho_i > r_i$$

For the band-structured choice the $q_m^{(i)}(z)$ can be multiplied by a suitable $x^\mu y^\mu$.

4.2. *Balls in Multivariate Space*

The simplest case to which theorem 2 applies is the homogeneous approach with $N_m = S_{N_m}$. In this case the error formula (4) takes a simpler form. Indeed,

$$(fq-p)(z) = \frac{1}{R^4} \int_S (fq)(w) \sum_{i=s_{N_m}+1}^{\infty} \left[\sum_{|\gamma|=i} \frac{(1+|\gamma|)!}{\gamma!} \frac{1}{R^{2|\gamma|}} z^\gamma \bar{w}^\gamma \right] d\sigma(w)$$

Taking into account the fact that

$$\sum_{|\gamma|=i} \frac{(1+|\gamma|)!}{\gamma!} \frac{1}{R^{2|\gamma|}} z^\gamma \bar{w}^\gamma = (i+1) \left(\frac{\langle z, w \rangle}{R^2} \right)^i$$

one obtains

$$(fq-p)(z) = \frac{1}{R^4} \int_S (fq)(w) \left[\sum_{i=s_{N_m}+1}^{\infty} (i+1) \left(\frac{\langle z, w \rangle}{R^2} \right)^i \right] d\sigma(w)$$

and the proof is more straightforward.

Choices for denominators $q_m(z)$ satisfying the conditions of theorem 2 are for instance

$$q_m^{(1)}(z) = R^{2m} - (x^2 + y^2)^m$$

or

$$q_m^{(2)}(z) = R^{2m} - (xy)^m$$

5. Numerical Illustration

5.1. *Polydiscs*

Let us consider the function

$$f(x, y) = \frac{\exp(xy)}{(1-x)(1-y)}$$

which is holomorphic in polydiscs around 0 with polyradius (r_1, r_2) componentwise less than 1. Our choice for the index sets N_m and the polynomials q_m is the following:

$$m \geqslant 10 \qquad q_m(z) = (1 - x^m)(1 - y^m)$$

$$D_m = \{(0, 0), (m, 0), (0, m), (m, m)\}$$

$$N_m = [0, m-9] \times [0, m-1] \cup [m-8, m-7] \times [0, m-4]$$
$$\cup [m-6, m-4] \times [0, m-7] \cup [m-3, m-1] \times [0, m-9]$$

We display the value of the approximant $(N_m/D_m)_f$ and the expression

$$\varepsilon_m = |(f - (N_m/D_m)_f)(x, y)|^{1/m}$$

which converges towards

$$\max(|x|/r_1, |y|/r_2)$$

in the points $z_I = (0.3, -0.4)$ and $z_{II} = (0.85, 0.85)$ and we compare the value of the general order multivariate Padé-type approximant with that of the general order multivariate Padé approximant $[N_m/D_1]_f$ as computed in [Cuyt87]. The extra conditions necessary for the computation of the denominator in the Padé case (because now it cannot be fixed in advance) are chosen to be

$$(fq_1 - p_m)(z) = \sum_{(i,j) \in \mathbb{N}^2 \setminus I_m} d_{ij} x^i y^j$$

$$I_m = N_m \cup \{(m, 0), (m-6, m-6), (0, m)\}$$

For the sake of completeness we also compare with the partial sum

$$[I_m/D_0]_f = \sum_{(i,j) \in I_m} c_{ij} x^i y^j$$

(Tables I and II).

TABLE I

	$(x, y) = (0.3, -0.4)$	$f(x, y) = 0.9050208537930180...$		
m	$[I_m/D_0]_f$	$(N_m/D_m)_f$	ε_m	$[N_m/D_1]_f$
10	0.9052747627000001	0.9047026067623132	0.447	0.9050177140738382
14	0.9050199196685386	0.9050195432833030	0.380	0.9050208538026430
20	0.9050208496545943	0.9050208486066260	0.385	0.9050208537945944
25	0.9050208538355379	0.9050208538462221	0.388	0.9050208536565296
30	0.9050208537925841	0.9050208537924730	0.390	
35	0.9050208537930243	0.9050208537930243	0.393	

5.2. *Multivariate Balls*

Let us take the function

$$f(x, y) = \frac{\exp(xy)}{1 - (x^2 + y^2)}$$

which is holomorphic in the ball centered at 0 with radius 1. We now consider a more homogeneous approach and choose N_m and q_m as follows:

$$m \geqslant 5 \qquad q_m(z) = 1 - (x^2 + y^2)^m$$

$$D_m = \{(0, 0)\} \cup \{(2k, 2m - k) \mid 0 \leqslant k \leqslant m\}$$

$$N_m = \{(i, j) \mid 0 \leqslant i + j \leqslant 2m - 1\}$$

We display the value of the approximant $(N_m/D_m)_f$ and the expression

$$\varepsilon_m = |(f - (N_m/D_m)_f)(x, y)|^{1/2m}$$

which converges towards $\|z\|$, in the points $z_I = (0.3, -0.4)$ and $z_{II} = (0.6, -0.6)$ and now compare with the homogeneous Padé approximants

TABLE II

	$(x, y) = (0.85, 0.85)$	$f(x, y) = 91.53669862789831...$		
m	$[I_m/D_0]_f$	$(N_m/D_m)_f$	ε_m	$[N_m/D_1]_f$
10	37.97794294950821	57.12494023598670	1.42	156.8244449193780
20	83.03637933882605	89.75287359137341	1.03	91.53669862928274
30	89.94496019720368	91.31690334150360	0.951	91.53669862789741
40	91.22649493028024	91.49815628366130	0.922	91.53669862793585
50	91.47575028314775	91.52929133557080	0.907	
60	91.52470421940957	91.53524730610090	0.897	
70	91.53433742039013	91.53641317034410	0.890	

TABLE III

	$(x, y) = (0.3, -0.4)$	$f(x, y) = 1.18256058228954...$		$\|z_I\| = 0.5$
m	$\|f-[2m, 0]_f\|$	$\|f-(N_m/D_m)_f\|$	ε_m	$\|f-[2m-1, 1]_f\|$
5	$0.201E-03$	$0.349E-03$	0.451	$0.806E-03$
8	$0.315E-05$	$0.546E-05$	0.469	$0.126E-04$
13	$0.307E-08$	$0.533E-08$	0.481	$0.123E-07$
16	$0.480E-10$	$0.832E-10$	0.484	$0.192E-09$
20	$0.187E-12$	$0.325E-12$	0.487	$0.750E-12$
23	$0.244E-14$	$0.527E-14$	0.489	$0.113E-13$

$[2m-1, 1]_f$ that can be computed using the multivariate ε-algorithm as detailed in [Cuyt82]. Input for the multivariate ε-algorithm are the Taylor series coefficients c_{ij} of f with $0 \leqslant i+j \leqslant 2m$. The choices for m, z_I and z_{II} are such that the number of coefficients to be determined in the approximant and the position of the points with respect to the boundary, respectively of the poly-disc and the unit ball, are comparable. Again for the sake of completeness we also compare with the partial sum

$$[2m, 0]_f = \sum_{0 \leqslant i+j \leqslant 2m} c_{ij} x^i y^j$$

(Tables III and IV).

For the entries that were not filled in Tables I–IV, the rounding error became larger than the truncation error. It is clear from the two tables above that near a singularity the use of a series is restricted and the use of a rational function is recommended. For multivariate Padé approximants a convergence theorem was proved in [Cuyt90] which applies when approximating a meromorphic function as is the case here.

TABLE IV

	$(x, y) = (0.6, -0.6)$	$f(x, y) = 2.49170116453939...$		$\|z_{II}\| = 0.8485281$
m	$\|f-[2m, 0]_f\|$	$\|f-(N_m/D_m)_f\|$	ε_m	$\|f-[2m-1, 1]_f\|$
5	$0.302E+00$	$0.781E-01$	0.775	$0.419E+00$
13	$0.218E-01$	$0.461E-02$	0.813	$0.303E-01$
20	$0.219E-02$	$0.457E-03$	0.825	$0.304E-02$
35	$0.158E-04$	$0.331E-05$	0.835	$0.220E-04$
50	$0.115E-06$	$0.239E-07$	0.839	$0.159E-06$
70		$0.336E-10$	0.842	$0.223E-09$

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